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ON STRONG CONVERGENCE THEOREMS FOR A VISCOSITY-TYPE TSENG'S EXTRAGRADIENT METHODS SOLVING QUASIMONOTONE VARIATIONAL INEQUALITIES

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Abstract. The main goal of this research is to solve variational inequalities involving quasimonotone operators in infinite-dimensional real Hilbert spaces numerically. The main advantage of these iterative schemes is the ease with which step size rules can be designed based on an operator explanation rather than the Lipschitz constant or another line search method. The proposed iterative schemes use a monotone and non-monotone step size strategy based on mapping (operator) knowledge as a replacement for the Lipschitz constant or another line search method. The strong convergences have been demonstrated to correspond well to the proposed methods and to settle certain control specification conditions. Finally, we propose some numerical experiments to assess the effectiveness and influence of iterative methods.

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1. Introduction

The main objective of this study is to examine the iterative methods employed to determine the solution to variational inequality problem (shortly, VIP) [34] involving quasimonotone operators in any real Hilbert space. Consider that Σ is a real Hilbert space and Δ is a nonempty, closed and convex subset of Σ. Look at the operator $\Gamma : \Sigma \to \Sigma$. The variational inequality problem for Γ on Δ is described in the subsequent fashion:

$$\text{Find } \omega^* \in \Delta \quad \text{such that} \quad \big\langle \Gamma(\omega^*), y - \omega^* \big\rangle \geq 0, \ \forall\, y \in \Delta. \tag{VIP}$$

The mathematical model of the variational inequality problem is a key problem in nonlinear analysis. It is a remarkable mathematical design that consolidates a lot of essential notions in applied mathematics, such as a nonlinear system of equation, optimization conditions for problems with the optimization process, the complementarity problems, network equilibrium problems and finance (see for more details [8, 11, 12, 13, 14, 15, 23]). As a consequence, this notion has various applications in the fields of mathematical programming, engineering, transport analysis, network economics, game theory and computer science. The regularized method and the projection method are two prominent and general procedures for finding a solution to variational inequalities. It is also noted that the first approach is most commonly used to deal with the variational inequalities accompanied by the class of monotone operators. The regularized sub-problem in this method is strongly monotone, and its unique solution is found more conveniently than the initial problem.

In this study, we discuss the projection methods that are well known for their simpler numerical computing. Many authors have dedicated themselves to studying not only the theory of existence and stability of solutions but also iterative methods for solving variational inequality problems. In addition, projection methods are useful to approximate the numerical solution of variational inequalities. Many researchers have established different variants of projection methods to solve such problems (see for more details [3, 5, 6, 7, 10, 16, 20, 22, 24, 33, 35, 38, 39, 41]) and others in [4, 9, 21, 25, 26, 27, 28, 29, 30, 31, 32, 36, 40, 42]. Almost all methods for solving the problem (VIP) are based on the computation of a projection on the feasible set Δ. Korpelevich [16] and Antipin [1] introduced the following extragradient method. Their method takes the following form:

$$\begin{cases} u_1 \in \Delta, \\ y_n = P_\Delta[u_n - \varkappa \Gamma(u_n)], \\ u_{n+1} = P_\Delta[u_n - \varkappa \Gamma(y_n)], \end{cases} \tag{1.1}$$

where $0 < \varkappa < \frac{1}{L}$. In view of the above method, we have used two projections on the underlying set Δ for each iteration. This, of course, can affect the computational effectiveness of the method if the feasible set Δ has a complicated structure.

Here, we present some methods which can remove this drawback. The first is the following subgradient extragradient method introduced by Censor et al. [6]. This method takes the following form:

$$\begin{cases} u_1 \in \Delta, \\ y_n = P_\Delta[u_n - \varkappa\Gamma(u_n)], \\ u_{n+1} = P_{\Sigma_n}[u_n - \varkappa\Gamma(y_n)], \end{cases} \tag{1.2}$$

where $0 < \varkappa < \frac{1}{L}$ and

$$\Sigma_n = \{z \in \Sigma : \langle u_n - \varkappa\Gamma(u_n) - y_n, z - y_n\rangle \leq 0\}.$$

In this article, our main focus on the Tseng's extragradient method [35] that uses only one projection for each iteration. This method takes the following form:

$$\begin{cases} u_1 \in \Delta, \\ y_n = P_\Delta[u_n - \varkappa\Gamma(u_n)], \\ u_{n+1} = y_n + \varkappa\big[\Gamma(u_n) - \Gamma(y_n)\big], \end{cases} \tag{1.3}$$

where $0 < \varkappa < \frac{1}{L}$. It is important to note that the above mentioned methods have two major flaws: a fixed constant step size rule that is dependent on the Lipschitz constant of mapping and generates a weakly convergent iterative sequence. The Lipschitz constant is generally unknown or difficult to compute. From a computational point of view, it can be difficult to consider a fixed step size constraint that affects the method's efficiency and rate of convergence. In addition, the study of a strongly convergent iterative sequence is important in the context of an infinite-dimensional Hilbert space.

The main objective of this paper is to introduce a new strongly convergent method by using viscosity and Tseng's extragradient-type method, including a monotonic and non-monotonic variable step size rule to solve variational inequalities involving the quasimonotone operator. Furthermore, to show that the iterative sequences generated by all four subgradient extragradient algorithms strongly converge to a solution. Both the monotone and non-monotone variable step size rules are used in subgradient and extragradient algorithms.

The paper is arranged in the following way: In Sect. 2, preliminary results were presented. Sect. 3 gives all new algorithms and their convergence analysis. Finally, Sect. 4 gives some numerical results to explain the practical efficiency of the proposed methods.

2. Preliminaries

This section contains a number of important identities and relevant lemmas. For any $u, y \in \Sigma$, we have

$$\|u+y\|^2 = \|u\|^2 + 2\langle u, y\rangle + \|y\|^2.$$

Lemma 2.1. ([2]) *For any $y_1, y_2 \in \Sigma$ and $\ell \in \mathbb{R}$, Tthe following inequalities are hold.*

(i) $\|\ell y_1 + (1-\ell)y_2\|^2 = \ell\|y_1\|^2 + (1-\ell)\|y_2\|^2 - \ell(1-\ell)\|y_1 - y_2\|^2$.
(ii) $\|y_1 + y_2\|^2 \leq \|y_1\|^2 + 2\langle y_2, y_1 + y_2\rangle$.

A metric projection $P_\Delta(y_1)$ of $y_1 \in \Sigma$ is defined by

$$P_\Delta(y_1) = \arg\min\{\|y_1 - y_2\| : y_2 \in \Delta\}.$$

First, we list some of the important features of projection mapping.

Lemma 2.2. ([2]) *Let $P_\Delta : \Sigma \to \Delta$ be a metric projection. Then, the following conditions are satisfied.*

(i) $y_3 = P_\Delta(y_1)$ *if and only if* $\langle y_1 - y_3, y_2 - y_3\rangle \leq 0, \ \forall\, y_2 \in \Delta$,
(ii) $\|y_1 - P_\Delta(y_2)\|^2 + \|P_\Delta(y_2) - y_2\|^2 \leq \|y_1 - y_2\|^2, \ y_1 \in \Delta, y_2 \in \Sigma$,
(iii) $\|y_1 - P_\Delta(y_1)\| \leq \|y_1 - y_2\|, \ y_2 \in \Delta, y_1 \in \Sigma$.

Lemma 2.3. ([37]) *Let $\{e_n\} \subset [0, +\infty)$ be a sequence satisfies the following condition*

$$e_{n+1} \leq (1 - f_n)e_n + f_n g_n, \ \forall\, n \in \mathbb{N}.$$

In addition, two sequences $\{f_n\} \subset (0,1)$ and $\{g_n\} \subset \mathbb{R}$ satisfies the following conditions:

$$\lim_{n\to+\infty} f_n = 0, \ \sum_{n=1}^{+\infty} f_n = +\infty \ \ \textit{and} \ \ \limsup_{n\to+\infty} g_n \leq 0.$$

Then, $\lim_{n\to+\infty} e_n = 0$.

Lemma 2.4. ([19]) *Let $\{e_n\} \subset \mathbb{R}$ be a sequence and there exists a subsequence $\{n_i\}$ of $\{n\}$ such that*

$$e_{n_i} < e_{n_{i+1}}, \ \forall\ i \in \mathbb{N}.$$

Then, there exists a nondecreasing sequence $m_k \subset \mathbb{N}$ such that $m_k \to +\infty$ as $k \to +\infty$ with

$$e_{m_k} \leq e_{m_{k+1}} \quad \textit{and} \quad e_k \leq e_{m_{k+1}}, \forall\, k \in \mathbb{N}.$$

Indeed, $m_k = \max\{j \leq k : e_j \leq e_{j+1}\}$.

3. Main results

In this section, we propose a class of iterative algorithms for solving quasimonotone variational inequalities based on Tseng's extragradient method. In the following text, all methods are described in detail. The following conditions are assumed to be met in order to verify the strong convergence:

(Γ1) the solution set for problem (VIP) is denoted by $VI(\Delta, \Gamma)$ and it is nonempty;

(Γ2) an operator $\Gamma : \Sigma \to \Sigma$ is said to be quasimonotone if

$$\big\langle \Gamma(u), y - u \big\rangle > 0 \Longrightarrow \big\langle \Gamma(y), y - u \big\rangle \geq 0, \ \forall\, u, y \in \Delta; \tag{QM}$$

(Γ3) an operator $\Gamma : \Sigma \to \Sigma$ is said to be Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|\Gamma(u) - \Gamma(y)\| \leq L\|u - y\|, \ \forall\, u, y \in \Delta; \tag{LC}$$

(Γ4) an operator $\Gamma : \Sigma \to \Sigma$ is weakly sequentially continuous if $\{\Gamma(u_n)\}$ weakly converges to $\Gamma(u)$ for every sequence $\{u_n\}$ weakly converges to u.

In this part, we present an iterative scheme for solving quasimonotone variational inequality problems that is based on Tseng's extragradient method [35] and viscosity scheme [20]. It is important to note that the proposed method has a straightforward structure for achieving strong convergence. Suppose that $\mathrm{g} : \Sigma \to \Sigma$ is a strict contraction function with constant $\xi \in [0, 1)$. The main algorithm has been presented as follows:

Algorithm 3.1. (Viscosity Extragradient Method With Fixed Step Size Rule)

Step 0. Let $u_1 \in \Delta$, $0 < \varkappa < \frac{1}{L}$ and $\{\vartheta_n\} \subset (0, 1)$ meet the following conditions:

$$\lim_{n \to +\infty} \vartheta_n = 0 \quad \text{and} \quad \sum_{n=1}^{+\infty} \vartheta_n = +\infty.$$

Step 1. Compute

$$y_n = P_{\Delta}(u_n - \varkappa \Gamma(u_n)).$$

If $u_n = y_n$, Stop. Otherwise, go to Step 2.

Step 2. Compute

$$z_n = y_n + \varkappa\big[\Gamma(u_n) - \Gamma(y_n)\big].$$

Step 3. Compute

$$u_{n+1} = \vartheta_n \mathrm{g}(u_n) + (1 - \vartheta_n) z_n.$$

Set $n := n + 1$ and go back to Step 1.

Algorithm 3.2. (Monotonic Explicit Viscosity Extragradient Method)

Step 0. Let $u_1 \in \Delta$, $\varkappa_1 > 0$, $\chi \in (0,1)$ and $\{\vartheta_n\} \subset (0,1)$ meet the following conditions:

$$\lim_{n\to+\infty} \vartheta_n = 0 \quad \text{and} \quad \sum_{n=1}^{+\infty} \vartheta_n = +\infty.$$

Step 1. Compute

$$y_n = P_\Delta(u_n - \varkappa_n \Gamma(u_n)).$$

If $u_n = y_n$, STOP. Otherwise, go to Step 2.

Step 2. Compute

$$z_n = y_n + \varkappa_n\big[\Gamma(u_n) - \Gamma(y_n)\big].$$

Step 3. Compute $u_{n+1} = \vartheta_n \mathrm{g}(u_n) + (1-\vartheta_n) z_n$.

Step 4. Compute

$$\varkappa_{n+1} = \begin{cases} \min\left\{\varkappa_n, \frac{\chi\|u_n - y_n\|}{\|\Gamma(u_n)-\Gamma(y_n)\|}\right\} & \text{if} \quad \Gamma(u_n) - \Gamma(y_n) \neq 0, \\ \varkappa_n & \quad\text{otherwise.} \end{cases} \tag{3.1}$$

Set $n := n+1$ and go back to Step 1.

Lemma 3.3. *The sequence $\{\varkappa_n\}$ generated by* (3.1) *is decreasing monotonically and converges to $\varkappa > 0$.*

Proof. It is given that Γ is Lipschitz-continuous with constant $L > 0$. Let $\Gamma(u_n) \neq \Gamma(y_n)$ such that

$$\frac{\chi\|u_n - y_n\|}{\|\Gamma(u_n) - \Gamma(y_n)\|} \geq \frac{\chi\|u_n - y_n\|}{L\|u_n - y_n\|} \geq \frac{\chi}{L}. \tag{3.2}$$

The above expression implies that $\lim_{n\to+\infty} \varkappa_n = \varkappa$. □

Algorithm 3.4. (Non-Monotonic Explicit Viscosity Extragradient Method)

Step 0. Let $u_1 \in \Delta$, $\varkappa_1 > 0$, $\chi \in (0,1)$ and sequence $\{\varphi_n\}$ satisfying $\sum_{n=1}^{+\infty} \varphi_n < +\infty$. Moreover, $\{\vartheta_n\} \subset (0,1)$ satisfying the following conditions:

$$\lim_{n\to+\infty} \vartheta_n = 0 \quad \text{and} \quad \sum_{n=1}^{+\infty} \vartheta_n = +\infty.$$

Step 1. Compute

$$y_n = P_\Delta(u_n - \varkappa_n \Gamma(u_n)).$$

If $u_n = y_n$, STOP. Otherwise, go to Step 2.

Step 2. Compute

$$z_n = y_n + \varkappa_n\big[\Gamma(u_n) - \Gamma(y_n)\big].$$

Step 3. Compute $u_{n+1} = \vartheta_n \mathrm{g}(u_n) + (1-\vartheta_n) z_n$.

Step 4. Compute

$$\varkappa_{n+1} = \begin{cases} \min\left\{\varkappa_n + \varphi_n, \frac{\chi\|u_n - y_n\|}{\|\Gamma(u_n)-\Gamma(y_n)\|}\right\} & \text{if} \quad \Gamma(u_n) - \Gamma(y_n) \neq 0, \\ \varkappa_n + \varphi_n & \text{otherwise.} \end{cases} \tag{3.3}$$

Set $n := n + 1$ and go back to Step 1.

Lemma 3.5. *A sequence* $\{\varkappa_n\}$ *generated by* (3.3) *is convergent to* $\varkappa$ *and satisfying the following inequality*

$$\min\left\{\frac{\chi}{L}, \varkappa_1\right\} \leq \varkappa_n \leq \varkappa_1 + P \quad \textit{where} \quad P = \sum_{n=1}^{+\infty} \varphi_n.$$

Proof. It is given that Γ is Lipschitz-continuous with constant $L > 0$. Let $\Gamma(u_n) \neq \Gamma(y_n)$ such that

$$\frac{\chi\|u_n - y_n\|}{\|\Gamma(u_n) - \Gamma(y_n)\|} \geq \frac{\chi\|u_n - y_n\|}{L\|u_n - y_n\|} \geq \frac{\chi}{L}. \tag{3.4}$$

By using mathematical induction on the definition of $\varkappa_{n+1}$, we have

$$\min\left\{\frac{\chi}{L}, \varkappa_1\right\} \leq \varkappa_n \leq \varkappa_1 + P.$$

Let

$$[\varkappa_{n+1} - \varkappa_n]^+ = \max\left\{0, \varkappa_{n+1} - \varkappa_n\right\}$$

and

$$[\varkappa_{n+1} - \varkappa_n]^- = \max\left\{0, -(\varkappa_{n+1} - \varkappa_n)\right\}.$$

Then, from the definition of $\{\varkappa_n\}$, we have

$$\sum_{n=1}^{+\infty}[\varkappa_{n+1} - \varkappa_n]^+ = \sum_{n=1}^{+\infty} \max\left\{0, \varkappa_{n+1} - \varkappa_n\right\} \leq P < +\infty. \tag{3.5}$$

That means that the series $\sum_{n=1}^{+\infty}[\varkappa_{n+1} - \varkappa_n]^+$ is convergent.

Next, we need to prove the convergence of $\sum_{n=1}^{+\infty}[\varkappa_{n+1} - \varkappa_n]^-$.

Let $\sum_{n=1}^{+\infty}[\varkappa_{n+1} - \varkappa_n]^- = +\infty$. Due to the reason that $\varkappa_{n+1} - \varkappa_n = (\varkappa_{n+1} - \varkappa_n)^+ - (\varkappa_{n+1} - \varkappa_n)^-$, we have

$$\varkappa_{k+1} - \varkappa_1 = \sum_{n=0}^{k}(\varkappa_{n+1} - \varkappa_n) = \sum_{n=0}^{k}[\varkappa_{n+1} - \varkappa_n]^+ - \sum_{n=0}^{k}[\varkappa_{n+1} - \varkappa_n]^-. \tag{3.6}$$

By allowing $k \to +\infty$ in (3.6), we have $\varkappa_k \to -\infty$ as $k \to +\infty$. This is a contradiction. Due to the convergence of the series $\sum_{n=0}^{k}[\varkappa_{n+1} - \varkappa_n]^+$ and $\sum_{n=0}^{k}[\varkappa_{n+1} - \varkappa_n]^-$ taking $k \to +\infty$ in (3.6), we obtain $\lim_{n\to+\infty} \varkappa_n = \varkappa$. This completes the proof. □

Lemma 3.6. *Suppose that* $\Gamma : \Sigma \to \Sigma$ *satisfies the conditions* (Γ1)-(Γ4) *and sequence* $\{u_n\}$ *generated by Algorithm* 3.1. *Then, we have*

$$\|u_{n+1} - \omega^*\|^2 \leq \left\|u_n - \omega^*\right\|^2 - \left(1 - \varkappa^2 L^2\right)\left\|u_n - y_n\right\|^2.$$

Proof. Since $\omega^* \in VI(\Delta, \Gamma)$, we have

$$\begin{aligned}
&\left\|u_{n+1} - \omega^*\right\|^2 \\
&= \left\|y_n + \varkappa[\Gamma(u_n) - \Gamma(y_n)] - \omega^*\right\|^2 \\
&= \left\|y_n - \omega^*\right\|^2 + \varkappa^2\left\|\Gamma(u_n) - \Gamma(y_n)\right\|^2 + 2\varkappa\langle y_n - \omega^*, \Gamma(u_n) - \Gamma(y_n)\rangle \\
&= \left\|y_n + u_n - u_n - \omega^*\right\|^2 + \varkappa^2\left\|\Gamma(u_n) - \Gamma(y_n)\right\|^2 + 2\varkappa\langle y_n - \omega^*, \Gamma(u_n) - \Gamma(y_n)\rangle \\
&= \left\|y_n - u_n\right\|^2 + \left\|u_n - \omega^*\right\|^2 + 2\langle y_n - u_n, u_n - \omega^*\rangle \\
&\quad + \varkappa^2\left\|\Gamma(u_n) - \Gamma(y_n)\right\|^2 + 2\varkappa\langle y_n - \omega^*, \Gamma(u_n) - \Gamma(y_n)\rangle \\
&= \left\|u_n - \omega^*\right\|^2 + \left\|y_n - u_n\right\|^2 + 2\langle y_n - u_n, y_n - \omega^*\rangle + 2\langle y_n - u_n, u_n - y_n\rangle \\
&\quad + \varkappa^2\left\|\Gamma(u_n) - \Gamma(y_n)\right\|^2 + 2\varkappa\langle y_n - \omega^*, \Gamma(u_n) - \Gamma(y_n)\rangle. \qquad (3.7)
\end{aligned}$$

It is given that $y_n = P_\Delta[u_n - \varkappa\Gamma(u_n)]$ and it gives that

$$\langle u_n - \varkappa\Gamma(u_n) - y_n, y - y_n\rangle \leq 0, \ \forall\, y \in \Delta. \qquad (3.8)$$

Thus, we have

$$\langle u_n - y_n, \omega^* - y_n\rangle \leq \varkappa\langle\Gamma(u_n), \omega^* - y_n\rangle. \qquad (3.9)$$

Combining expressions (3.7) and (3.9), we have

$$\begin{aligned}
&\left\|u_{n+1} - \omega^*\right\|^2 \\
&\leq \left\|u_n - \omega^*\right\|^2 + \left\|y_n - u_n\right\|^2 + 2\varkappa\langle\Gamma(u_n), \omega^* - y_n\rangle - 2\langle u_n - y_n, u_n - y_n\rangle \\
&\quad + \varkappa^2\left\|\Gamma(u_n) - \Gamma(y_n)\right\|^2 - 2\varkappa\langle\Gamma(u_n) - \Gamma(y_n), \omega^* - y_n\rangle \\
&= \left\|u_n - \omega^*\right\|^2 - \left\|u_n - y_n\right\|^2 + \varkappa^2\left\|\Gamma(u_n) - \Gamma(y_n)\right\|^2 - 2\varkappa\langle\Gamma(y_n), y_n - \omega^*\rangle. \qquad (3.10)
\end{aligned}$$

It is given that ω^* is the solution of the problem (VIP) implies that

$$\langle\Gamma(\omega^*), y - \omega^*\rangle \geq 0, \ \forall\, y \in \Delta.$$

It implies that

$$\langle \Gamma(y), y - \omega^* \rangle \geq 0, \ \forall\, y \in \Delta.$$

By substituting $y = y_n \in \Delta$, we have

$$\langle \Gamma(y_n), y_n - \omega^* \rangle \geq 0. \tag{3.11}$$

From expressions (3.10) and (3.11), we obtain

$$\begin{aligned} \|u_{n+1} - \omega^*\|^2 &\leq \|u_n - \omega^*\|^2 - \|u_n - y_n\|^2 + \varkappa^2 L^2 \|u_n - y_n\|^2 \\ &= \|u_n - \omega^*\|^2 - (1 - \varkappa^2 L^2)\|u_n - y_n\|^2. \end{aligned} \tag{3.12}$$

□

Lemma 3.7. *Assume that* $\Gamma : \Sigma \to \Sigma$ *satisfies the conditions* (Γ1)-(Γ4). *Let* $\{u_n\}$ *be a sequence is generated by Algorithm* 3.2 *and* 3.4. *Then, for each* $\omega^* \in VI(\Delta, \Gamma)$, *we have*

$$\|u_{n+1} - \omega^*\|^2 \leq \|u_n - \omega^*\|^2 - \left(1 - \chi^2 \frac{\varkappa_n^2}{\varkappa_{n+1}^2}\right)\|u_n - y_n\|^2.$$

Proof. Let $\omega^* \in VI(\Delta, \Gamma)$ and by definition of u_{n+1}, we have

$$\begin{aligned} &\|u_{n+1} - \omega^*\|^2 \\ &= \|y_n + \varkappa_n[\Gamma(u_n) - \Gamma(y_n)] - \omega^*\|^2 \\ &= \|y_n - \omega^*\|^2 + \varkappa_n^2\|\Gamma(u_n) - \Gamma(y_n)\|^2 + 2\varkappa_n\langle y_n - \omega^*, \Gamma(u_n) - \Gamma(y_n)\rangle \\ &= \|y_n + u_n - u_n - \omega^*\|^2 + \varkappa_n^2\|\Gamma(u_n) - \Gamma(y_n)\|^2 \\ &\quad + 2\varkappa_n\langle y_n - \omega^*, \Gamma(u_n) - \Gamma(y_n)\rangle \\ &= \|y_n - u_n\|^2 + \|u_n - \omega^*\|^2 + 2\langle y_n - u_n, u_n - \omega^*\rangle \\ &\quad + \varkappa_n^2\|\Gamma(u_n) - \Gamma(y_n)\|^2 + 2\varkappa_n\langle y_n - \omega^*, \Gamma(u_n) - \Gamma(y_n)\rangle \\ &= \|u_n - \omega^*\|^2 + \|y_n - u_n\|^2 + 2\langle y_n - u_n, y_n - \omega^*\rangle + 2\langle y_n - u_n, u_n - y_n\rangle \\ &\quad + \varkappa_n^2\|\Gamma(u_n) - \Gamma(y_n)\|^2 + 2\varkappa_n\langle y_n - \omega^*, \Gamma(u_n) - \Gamma(y_n)\rangle. \end{aligned} \tag{3.13}$$

It is given that $y_n = P_\Delta[u_n - \varkappa_n\Gamma(u_n)]$ and it further implies that

$$\langle u_n - \varkappa_n\Gamma(u_n) - y_n, y - y_n\rangle \leq 0, \ \forall\, y \in \Delta \tag{3.14}$$

or equivalently for some $\omega^* \in VI(\Delta, \Gamma)$, we can write

$$\langle u_n - y_n, \omega^* - y_n\rangle \leq \varkappa_n\langle \Gamma(u_n), \omega^* - y_n\rangle. \tag{3.15}$$

Combining expressions (3.13) and (3.15), we have

$$\begin{aligned}
&\|u_{n+1} - \omega^*\|^2 \\
&\le \|u_n - \omega^*\|^2 + \|y_n - u_n\|^2 + 2\varkappa_n\langle\Gamma(u_n), \omega^* - y_n\rangle - 2\langle u_n - y_n, u_n - y_n\rangle \\
&\quad + \varkappa_n^2\|\Gamma(u_n) - \Gamma(y_n)\|^2 - 2\varkappa_n\langle\Gamma(u_n) - \Gamma(y_n), \omega^* - y_n\rangle \\
&= \|u_n - \omega^*\|^2 - \|u_n - y_n\|^2 + \varkappa_n^2\|\Gamma(u_n) - \Gamma(y_n)\|^2 - 2\varkappa_n\langle\Gamma(y_n), y_n - \omega^*\rangle.
\end{aligned} \tag{3.16}$$

It is given that ω^* is the solution of the problem (VIP), implies that

$$\langle\Gamma(\omega^*), y - \omega^*\rangle > 0, \ \forall\, y \in \Delta.$$

Due to the property of Γ on Δ, we obtain

$$\langle\Gamma(y), y - \omega^*\rangle \ge 0, \ \forall\, y \in \Delta.$$

Substituting $y = y_n \in \Delta$, we have

$$\langle\Gamma(y_n), y_n - \omega^*\rangle \ge 0. \tag{3.17}$$

Combining expressions (3.16) and (3.17), we obtain

$$\begin{aligned}
\|u_{n+1} - \omega^*\|^2 &\le \|u_n - \omega^*\|^2 - \|u_n - y_n\|^2 + \chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\|u_n - y_n\|^2 \\
&= \|u_n - \omega^*\|^2 - \left(1 - \chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\right)\|u_n - y_n\|^2.
\end{aligned} \tag{3.18}$$

□

Lemma 3.8. *Let* $\Gamma : \Sigma \to \Sigma$ *be an operator satisfies the conditions* (Γ1)–(Γ4). *If there exists a weakly convergent subsequence* $\{u_{n_k}\}$ *to* $\hat{u}$ *and* $\lim_{k\to+\infty}\|u_{n_k} - y_{n_k}\| = 0$. *Then,* $\hat{u} \in VI(\Delta, \Gamma)$.

Proof. Since $\{u_{n_k}\}$ weakly convergent to $\hat{u}$ and due to $\lim_{k\to+\infty}\|u_{n_k} - y_{n_k}\| = 0$, the sequence $\{y_{n_k}\}$ also weakly convergent to $\hat{u}$. Next, we need to prove that $\hat{u} \in VI(\Delta, \Gamma)$. By value of y_n, we have

$$y_{n_k} = P_\Delta[u_{n_k} - \varkappa_{n_k}\Gamma(u_{n_k})],$$

that is equivalent to

$$\langle u_{n_k} - \varkappa_{n_k}\Gamma(u_{n_k}) - y_{n_k}, y - y_{n_k}\rangle \le 0, \ \forall\, y \in \Delta. \tag{3.19}$$

The above inequality implies that

$$\langle u_{n_k} - y_{n_k}, y - y_{n_k}\rangle \le \varkappa_{n_k}\langle\Gamma(u_{n_k}), y - y_{n_k}\rangle, \ \forall\, y \in \Delta. \tag{3.20}$$

Thus, we obtain

$$\frac{1}{\varkappa_{n_k}}\langle u_{n_k} - y_{n_k}, y - y_{n_k}\rangle + \langle \Gamma(u_{n_k}), y_{n_k} - u_{n_k}\rangle \leq \langle \Gamma(u_{n_k}), y - u_{n_k}\rangle,\ \forall\, y \in \Delta. \tag{3.21}$$

By use of $\lim_{k\to+\infty}\|u_{n_k} - y_{n_k}\| = 0$ and $k \to +\infty$ in (3.21), we have

$$\liminf_{k\to+\infty}\langle \Gamma(u_{n_k}), y - u_{n_k}\rangle \geq 0,\ \forall\, y \in \Delta. \tag{3.22}$$

Furthermore, it implies that

$$\begin{aligned}\langle \Gamma(y_{n_k}), y - y_{n_k}\rangle &= \langle \Gamma(y_{n_k}) - \Gamma(u_{n_k}), y - u_{n_k}\rangle \\ &\quad + \langle \Gamma(u_{n_k}), y - u_{n_k}\rangle + \langle \Gamma(y_{n_k}), u_{n_k} - y_{n_k}\rangle.\end{aligned} \tag{3.23}$$

Since $\lim_{k\to+\infty}\|u_{n_k} - y_{n_k}\| = 0$, we have

$$\lim_{k\to+\infty}\|\Gamma(u_{n_k}) - \Gamma(y_{n_k})\| = 0, \tag{3.24}$$

which together with (3.23) and (3.24), we obtain

$$\liminf_{k\to+\infty}\langle \Gamma(y_{n_k}), y - y_{n_k}\rangle \geq 0,\ \forall\, y \in \Delta. \tag{3.25}$$

Moreover, let us take a positive sequence $\{\epsilon_k\}$ that is decreasing and convergent to zero. For each $\{\epsilon_k\}$ there exists a least positive integer denoted by m_k such that

$$\langle \Gamma(u_{n_i}), y - u_{n_i}\rangle + \epsilon_k > 0,\ \forall\, i \geq m_k. \tag{3.26}$$

Since $\{\epsilon_k\}$ is decreasing sequence, it is easy to see that the sequence $\{m_k\}$ is increasing. If there exists a natural number $N_0 \in \mathbb{N}$ such that for all $\Gamma(u_{n_{m_k}}) \neq 0$, $n_{m_k} \geq N_0$. Consider that

$$\aleph_{n_{m_k}} = \frac{\Gamma(u_{n_{m_k}})}{\|\Gamma(u_{n_{m_k}})\|^2},\ \forall\, n_{m_k} \geq N_0. \tag{3.27}$$

Due to the above definition, we have

$$\langle \Gamma(u_{n_{m_k}}), \aleph_{n_{m_k}}\rangle = 1,\ \forall\, n_{m_k} \geq N_0. \tag{3.28}$$

Moreover, from expressions (3.26) and (3.28) for all $n_{m_k} \geq N_0$, we have

$$\langle \Gamma(u_{n_{m_k}}),\ y + \epsilon_k \aleph_{n_{m_k}} - u_{n_{m_k}}\rangle > 0. \tag{3.29}$$

By the definition of quasimonotone, we have

$$\langle \Gamma(y + \epsilon_k \aleph_{n_{m_k}}),\ y + \epsilon_k \aleph_{n_{m_k}} - u_{n_{m_k}}\rangle > 0. \tag{3.30}$$

For all $n_{m_k} \geq N_0$, we have

$$\langle \Gamma(y), y - u_{n_{m_k}}\rangle \geq \langle \Gamma(y) - \Gamma(y + \epsilon_k \aleph_{n_{m_k}}),\ y + \epsilon_k \aleph_{n_{m_k}} - u_{n_{m_k}}\rangle - \epsilon_k \langle \Gamma(y), \aleph_{n_{m_k}}\rangle. \tag{3.31}$$

Due to $\{u_{n_k}\}$ converges weakly to $\hat{u} \in \Delta$ with Γ is weakly sequentially continuous on the set Δ, we obtain $\{\Gamma(u_{n_k})\}$ converges weakly to $\Gamma(\hat{u})$. Let $\Gamma(\hat{u}) \neq 0$. Then

$$\|\Gamma(\hat{u})\| \leq \liminf_{k\to+\infty} \|\Gamma(u_{n_k})\|. \tag{3.32}$$

Since $\{u_{n_{m_k}}\} \subset \{u_{n_k}\}$ and $\lim_{k\to+\infty} \epsilon_k = 0$, we have

$$0 \leq \lim_{k\to+\infty} \|\epsilon_k \aleph_{n_{m_k}}\| = \lim_{k\to+\infty} \frac{\epsilon_k}{\|\Gamma(u_{n_{m_k}})\|} \leq \frac{0}{\|\Gamma(\hat{u})\|} = 0. \tag{3.33}$$

By letting $k \to +\infty$ in (3.31), we obtain

$$\langle \Gamma(y), y - \hat{u} \rangle \geq 0, \ \forall\, y \in \Delta. \tag{3.34}$$

Let $u \in \Delta$ be arbitrary element and for $0 < \varkappa \leq 1$, let

$$\hat{u}_\varkappa = \varkappa u + (1 - \varkappa)\hat{u}. \tag{3.35}$$

Then $\hat{u}_\varkappa \in \Delta$ and from (3.34) we have

$$\varkappa \big\langle \Gamma(\hat{u}_\varkappa), u - \hat{u} \big\rangle \geq 0. \tag{3.36}$$

Hence

$$\big\langle \Gamma(\hat{u}_\varkappa), u - \hat{u} \big\rangle \geq 0. \tag{3.37}$$

Let $\varkappa \to 0$. Then $\hat{u}_\varkappa \to \hat{u}$ along a line segment. By the continuity of an operator, $\Gamma(\hat{u}_\varkappa)$ converges to $\Gamma(\hat{u})$ as $\varkappa \to 0$. It follows from (3.37) that

$$\big\langle \Gamma(\hat{u}), u - \hat{u} \big\rangle \geq 0. \tag{3.38}$$

Therefore $\hat{u}$ is a solution of problem (VIP). □

Theorem 3.9. *Assume that an operator $\Gamma : \Delta \to \Sigma$ satisfies the conditions* (Γ1)–(Γ4) *and ω^* belongs to the solution set $VI(\Delta, \Gamma)$. Moreover, sequence $\{\vartheta_n\} \subset (0, 1)$ satisfying the following conditions:*

$$\lim_{n\to+\infty} \vartheta_n = 0 \quad \textit{and} \quad \sum_{n=1}^{+\infty} \vartheta_n = +\infty.$$

Then, the sequences $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by Algorithm 3.4 *converge strongly to $\omega^* = P_{VI(\Delta,\Gamma)} \circ \mathrm{g}(\omega^*)$.*

Proof. By using Lemma 3.7, we have

$$\|z_n - \omega^*\|^2 \leq \big\|u_n - \omega^*\big\|^2 - \left(1 - \chi^2 \frac{\varkappa_n^2}{\varkappa_{n+1}^2}\right) \big\|u_n - y_n\big\|^2. \tag{3.39}$$

Given that $\varkappa_n \to \varkappa$, so there exists a fixed number $\epsilon \in (0, 1 - \chi^2)$ such that

$$\lim_{n\to+\infty} \left(1 - \chi^2 \frac{\varkappa_n^2}{\varkappa_{n+1}^2}\right) = 1 - \chi^2 > \epsilon > 0.$$

Thus, there exists a fixed number $N_1 \in \mathbb{N}$ such that

$$\left(1 - \chi^2 \frac{\varkappa_n^2}{\varkappa_{n+1}^2}\right) > \epsilon > 0, \ \forall\, n \geq N_1. \tag{3.40}$$

Therefore, we obtain

$$\|z_n - \omega^*\|^2 \leq \|u_n - \omega^*\|^2, \ \forall\, n \geq N_1. \tag{3.41}$$

It is given that $\omega^* \in VI(\Delta, \Gamma)$. From sequence $\{u_{n+1}\}$ and the reason that g is a contraction with constant $\xi \in [0, 1)$ and $n \geq N_1$, we have

$$\begin{aligned}
\big\|u_{n+1} - \omega^*\big\| &= \big\|\vartheta_n \mathrm{g}(u_n) + (1-\vartheta_n) z_n - \omega^*\big\| \\
&= \big\|\vartheta_n[\mathrm{g}(u_n) - \omega^*] + (1-\vartheta_n)[z_n - \omega^*]\big\| \\
&= \big\|\vartheta_n[\mathrm{g}(u_n) + \mathrm{g}(\omega^*) - \mathrm{g}(\omega^*) - \omega^*] + (1-\vartheta_n)[z_n - \omega^*]\big\| \\
&\leq \vartheta_n \big\|\mathrm{g}(u_n) - \mathrm{g}(\omega^*)\big\| + \vartheta_n \big\|\mathrm{g}(\omega^*) - \omega^*\big\| + (1-\vartheta_n)\big\|z_n - \omega^*\big\| \\
&\leq \vartheta_n \xi \big\|u_n - \omega^*\big\| + \vartheta_n \big\|\mathrm{g}(\omega^*) - \omega^*\big\| + (1-\vartheta_n)\big\|z_n - \omega^*\big\|.
\end{aligned} \tag{3.42}$$

Combining expressions (3.41) with (3.42) and $\vartheta_n \in (0, 1)$, we deduce that

$$\begin{aligned}
\big\|u_{n+1} - \omega^*\big\| &\leq \vartheta_n \xi \big\|u_n - \omega^*\big\| + \vartheta_n \big\|\mathrm{g}(\omega^*) - \omega^*\big\| + (1-\vartheta_n)\big\|u_n - \omega^*\big\| \\
&= [1 - \vartheta_n + \xi\vartheta_n]\big\|u_n - \omega^*\big\| + \vartheta_n(1-\xi)\frac{\big\|\mathrm{g}(\omega^*) - \omega^*\big\|}{(1-\xi)} \\
&\leq \max\left\{\big\|u_n - \omega^*\big\|, \frac{\big\|\mathrm{g}(\omega^*) - \omega^*\big\|}{(1-\xi)}\right\} \\
&\leq \max\left\{\big\|u_{N_1} - \omega^*\big\|, \frac{\big\|\mathrm{g}(\omega^*) - \omega^*\big\|}{(1-\xi)}\right\}.
\end{aligned} \tag{3.43}$$

Therefore, we deduce that $\{u_n\}$ is a bounded sequence. Due to the continuity and monotonicity of the operator Γ implies that the solution set $VI(\Delta, \Gamma)$ is a closed and convex set (for more details see [17, 18]). Since the mapping is a contraction and so does $P_{VI(\Delta,\Gamma)} \circ \mathrm{g}$.

Now, we are in position to use the Banach contraction theorem for the existence of a fixed point of $\omega^* \in VI(\Delta, \Gamma)$ such that

$$\omega^* = P_{VI(\Delta,\Gamma)}(\mathrm{g}(\omega^*)).$$

By using Lemma 2.2 (ii), we have

$$\langle \mathrm{g}(\omega^*) - \omega^*, y - \omega^* \rangle \leq 0, \ \forall\, y \in VI(\Delta, \Gamma). \tag{3.44}$$

It is given that $u_{n+1} = \vartheta_n \mathrm{g}(u_n) + (1-\vartheta_n)z_n$, and using Lemma 2.1 (i) and Lemma 3.7, we have

$$\begin{aligned}
&\big\|u_{n+1} - \omega^*\big\|^2 \\
&= \big\|\vartheta_n \mathrm{g}(u_n) + (1-\vartheta_n)z_n - \omega^*\big\|^2 \\
&= \big\|\vartheta_n[\mathrm{g}(u_n) - \omega^*] + (1-\vartheta_n)[z_n - \omega^*]\big\|^2 \\
&= \vartheta_n\|\mathrm{g}(u_n) - \omega^*\|^2 + (1-\vartheta_n)\|z_n - \omega^*\|^2 - \vartheta_n(1-\vartheta_n)\|\mathrm{g}(u_n) - z_n\|^2 \\
&\le \vartheta_n\|\mathrm{g}(u_n) - \omega^*\|^2 + (1-\vartheta_n)\Big[\big\|u_n - \omega^*\big\|^2 - \Big(1 - \chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\Big)\big\|u_n - y_n\big\|^2\Big] \\
&\quad - \vartheta_n(1-\vartheta_n)\|\mathrm{g}(u_n) - z_n\|^2 \\
&\le \vartheta_n\|\mathrm{g}(u_n) - \omega^*\|^2 + \|u_n - \omega^*\|^2 - (1-\vartheta_n)\Big(1 - \chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\Big)\big\|u_n - y_n\big\|^2.
\end{aligned} \tag{3.45}$$

The remainder of the proof shall be divided into the following two parts:

Case 1: Assume that there is a fixed number $N_2 \in \mathbb{N}$ $(N_2 \ge N_1)$ such that

$$\|u_{n+1} - \omega^*\| \le \|u_n - \omega^*\|, \ \forall\, n \ge N_2. \tag{3.46}$$

Then, $\lim_{n\to+\infty}\|u_n - \omega^*\|$ exists and let $\lim_{n\to+\infty}\|u_n - \omega^*\| = l$. From expression (3.45), we have

$$\begin{aligned}
(1-\vartheta_n)\Big(1 - \chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\Big)\big\|u_n - y_n\big\|^2 \le\ & \vartheta_n\|\mathrm{g}(u_n) - \omega^*\|^2 \\
& + \|u_n - \omega^*\|^2 - \|u_{n+1} - \omega^*\|^2.
\end{aligned} \tag{3.47}$$

From the existence of $\lim_{n\to+\infty}\|u_n - \omega^*\| = l$, and $\vartheta_n \to 0$, we infer that

$$\lim_{n\to+\infty}\|u_n - y_n\| = 0. \tag{3.48}$$

It follows that

$$\|z_n - y_n\| = \|y_n + \varkappa_n[\Gamma(u_n) - \Gamma(y_n)] - y_n\| \le \varkappa_0 L\|u_n - y_n\|.$$

This implies that

$$\lim_{n\to+\infty}\|z_n - y_n\| = 0. \tag{3.49}$$

It follows that

$$\lim_{n\to+\infty}\|u_n - z_n\| \le \lim_{n\to+\infty}\|u_n - y_n\| + \lim_{n\to+\infty}\|y_n - z_n\| = 0. \tag{3.50}$$

We can also obtain

$$\begin{aligned}\big\|u_{n+1}-u_n\big\| &= \big\|\vartheta_n \mathrm{g}(u_n)+(1-\vartheta_n)z_n-u_n\big\| \\ &= \big\|\vartheta_n[\mathrm{g}(u_n)-u_n]+(1-\vartheta_n)[z_n-u_n]\big\| \\ &\le \vartheta_n\big\|\mathrm{g}(u_n)-u_n\big\|+(1-\vartheta_n)\big\|z_n-u_n\big\| \\ &\longrightarrow 0, \quad \text{as} \quad n\to+\infty. \end{aligned} \tag{3.51}$$

Hence we have

$$\lim_{n\to+\infty}\|u_{n+1}-u_n\|=0. \tag{3.52}$$

Since the sequence $\{u_n\}$ is bounded, the sequences $\{y_n\}$ and $\{z_n\}$ are also bounded. Thus, we can take a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ weakly converges to some $\hat{u}\in\Sigma$. Moreover, from $\|u_n-y_n\|\to 0$, we have $\hat{u}\in VI(\Delta,\Gamma)$. It follows that

$$\begin{aligned}\limsup_{n\to+\infty}\langle \mathrm{g}(\omega^*)-\omega^*,u_n-\omega^*\rangle &= \limsup_{k\to+\infty}\langle \mathrm{g}(\omega^*)-\omega^*,u_{n_k}-\omega^*\rangle \\ &= \langle \mathrm{g}(\omega^*)-\omega^*,\hat{u}-\omega^*\rangle \\ &\le 0. \end{aligned} \tag{3.53}$$

Since $\lim_{n\to+\infty}\big\|u_{n+1}-u_n\big\|=0$. It follows that

$$\begin{aligned}\limsup_{n\to+\infty}\langle \mathrm{g}(\omega^*)-\omega^*,u_{n+1}-\omega^*\rangle &\le \limsup_{n\to+\infty}\langle \mathrm{g}(\omega^*)-\omega^*,u_{n+1}-u_n\rangle \\ &\quad +\limsup_{n\to+\infty}\langle \mathrm{g}(\omega^*)-\omega^*,u_n-\omega^*\rangle \\ &\le 0. \end{aligned} \tag{3.54}$$

From Lemma 2.1 (ii) and Lemma 3.7 for all $n\ge N_2$, we obtain

$$\begin{aligned}\big\|u_{n+1}-\omega^*\big\|^2 &= \big\|\vartheta_n \mathrm{g}(u_n)+(1-\vartheta_n)z_n-\omega^*\big\|^2 \\ &= \big\|\vartheta_n[\mathrm{g}(u_n)-\omega^*]+(1-\vartheta_n)[z_n-\omega^*]\big\|^2 \\ &\le (1-\vartheta_n)^2\big\|z_n-\omega^*\big\|^2 \\ &\quad +2\vartheta_n\langle \mathrm{g}(u_n)-\omega^*,(1-\vartheta_n)[z_n-\omega^*]+\vartheta_n[\mathrm{g}(u_n)-\omega^*]\rangle \\ &= (1-\vartheta_n)^2\big\|z_n-\omega^*\big\|^2 \\ &\quad +2\vartheta_n\langle \mathrm{g}(u_n)-\mathrm{g}(\omega^*)+\mathrm{g}(\omega^*)-\omega^*,u_{n+1}-\omega^*\rangle \end{aligned}$$

$$
\begin{aligned}
&= (1-\vartheta_n)^2 \big\|z_n - \omega^*\big\|^2 + 2\vartheta_n \langle \mathrm{g}(u_n) - \mathrm{g}(\omega^*), u_{n+1} - \omega^* \rangle \\
&\quad + 2\vartheta_n \langle \mathrm{g}(\omega^*) - \omega^*, u_{n+1} - \omega^* \rangle \\
&\le (1-\vartheta_n)^2 \big\|z_n - \omega^*\big\|^2 + 2\vartheta_n \xi \big\|u_n - \omega^*\big\| \big\|u_{n+1} - \omega^*\big\| \\
&\quad + 2\vartheta_n \langle \mathrm{g}(\omega^*) - \omega^*, u_{n+1} - \omega^* \rangle \\
&\le (1+\vartheta_n^2 - 2\vartheta_n) \big\|u_n - \omega^*\big\|^2 + 2\vartheta_n \xi \big\|u_n - \omega^*\big\|^2 \\
&\quad + 2\vartheta_n \langle \mathrm{g}(\omega^*) - \omega^*, u_{n+1} - \omega^* \rangle \\
&= (1-2\vartheta_n) \big\|u_n - \omega^*\big\|^2 + \vartheta_n^2 \big\|u_n - \omega^*\big\|^2 + 2\vartheta_n \xi \big\|u_n - \omega^*\big\|^2 \\
&\quad + 2\vartheta_n \langle \mathrm{g}(\omega^*) - \omega^*, u_{n+1} - \omega^* \rangle \\
&= \big[1 - 2\vartheta_n(1-\xi)\big] \big\|u_n - \omega^*\big\|^2 \\
&\quad + 2\vartheta_n(1-\xi) \left[\frac{\vartheta_n \big\|u_n - \omega^*\big\|^2}{2(1-\xi)} + \frac{\langle \mathrm{g}(\omega^*) - \omega^*, u_{n+1} - \omega^* \rangle}{1-\xi} \right].
\end{aligned} \tag{3.55}
$$

It follows from expressions (3.54) and (3.55) that

$$
\limsup_{n \to +\infty} \left[\frac{\vartheta_n \big\|u_n - \omega^*\big\|^2}{2(1-\xi)} + \frac{\langle \mathrm{g}(\omega^*) - \omega^*, u_{n+1} - \omega^* \rangle}{1-\xi} \right] \le 0. \tag{3.56}
$$

Let choose $n \ge N_3 \in \mathbb{N}$ ($N_3 \ge N_2$) large enough such that

$$
2\vartheta_n(1-\xi) < 1.
$$

By using (3.55) and (3.56), and applying Lemma 2.3, we conclude that $\big\|u_n - \omega^*\big\| \to 0$, as $n \to +\infty$.

Case 2: Assume there is a subsequence $\{n_i\}$ of $\{n\}$ such that

$$
\|u_{n_i} - \omega^*\| \le \|u_{n_{i+1}} - \omega^*\|, \ \forall i \in \mathbb{N}.
$$

Then, by Lemma 2.4, there is a sequence $\{m_k\} \subset \mathbb{N}$ as $m_k \to +\infty$, such that

$$
\|u_{m_k} - \omega^*\| \le \|u_{m_{k+1}} - \omega^*\| \quad \text{and} \quad \|u_k - \omega^*\| \le \|u_{m_{k+1}} - \omega^*\|, \ \text{for all } k \in \mathbb{N}. \tag{3.57}
$$

As similar to Case 1, from (3.45), we have

$$
\begin{aligned}
&(1-\vartheta_{m_k}) \left(1 - \chi^2 \frac{\varkappa_{m_k}^2}{\varkappa_{m_k+1}^2} \right) \big\|u_{m_k} - y_{m_k}\big\|^2 \\
&\le \vartheta_{m_k} \|\mathrm{g}(u_{m_k}) - \omega^*\|^2 + \|u_{m_k} - \omega^*\|^2 - \|u_{m_k+1} - \omega^*\|^2.
\end{aligned} \tag{3.58}
$$

Due to $\vartheta_{m_k} \to 0$, we deduce the following:

$$
\lim_{k \to +\infty} \|u_{m_k} - y_{m_k}\| = 0. \tag{3.59}
$$

Similar to above case we can prove that

$$\lim_{k\to+\infty} \|u_{m_k} - z_{m_k}\| = \lim_{k\to+\infty} \|y_{m_k} - z_{m_k}\| = 0. \tag{3.60}$$

Also, we obtain

$$\begin{aligned}\left\|u_{m_k+1} - u_{m_k}\right\| &= \left\|\vartheta_{m_k} \mathrm{g}(u_{m_k}) + (1-\vartheta_{m_k}) z_{m_k} - u_{m_k}\right\| \\ &= \left\|\vartheta_{m_k}[\mathrm{g}(u_{m_k}) - u_{m_k}] + (1-\vartheta_{m_k})[z_{m_k} - u_{m_k}]\right\| \\ &\leq \vartheta_{m_k}\left\|\mathrm{g}(u_{m_k}) - u_{m_k}\right\| + (1-\vartheta_{m_k})\left\|z_{m_k} - u_{m_k}\right\| \\ &\longrightarrow 0. \end{aligned} \tag{3.61}$$

We have to use the same justification as in the Case 1 such that

$$\limsup_{k\to+\infty} \langle \mathrm{g}(\omega^*) - \omega^*, u_{m_k+1} - \omega^* \rangle \leq 0. \tag{3.62}$$

By the use of expressions (3.55) and (3.57), we have

$$\begin{aligned}&\left\|u_{m_k+1} - \omega^*\right\|^2 \\ &\leq \left[1 - 2\vartheta_{m_k}(1-\xi)\right]\left\|u_{m_k} - \omega^*\right\|^2 \\ &\quad + 2\vartheta_{m_k}(1-\xi)\left[\frac{\vartheta_{m_k}\left\|u_{m_k} - \omega^*\right\|^2}{2(1-\xi)} + \frac{\langle \mathrm{g}(\omega^*) - \omega^*, u_{m_k+1} - \omega^*\rangle}{1-\xi}\right] \\ &\leq \left[1 - 2\vartheta_{m_k}(1-\xi)\right]\left\|u_{m_k+1} - \omega^*\right\|^2 \\ &\quad + 2\vartheta_{m_k}(1-\xi)\left[\frac{\vartheta_{m_k}\left\|u_{m_k} - \omega^*\right\|^2}{2(1-\xi)} + \frac{\langle \mathrm{g}(\omega^*) - \omega^*, u_{m_k+1} - \omega^*\rangle}{1-\xi}\right]. \end{aligned} \tag{3.63}$$

It follows that

$$\left\|u_{m_k+1} - \omega^*\right\|^2 \leq \frac{\vartheta_{m_k}\left\|u_{m_k} - \omega^*\right\|^2}{2(1-\xi)} + \frac{\langle \mathrm{g}(\omega^*) - \omega^*, u_{m_k+1} - \omega^*\rangle}{1-\xi}. \tag{3.64}$$

Since $\vartheta_{m_k} \to 0$, as $k \to +\infty$ and $\left\|u_{m_k} - \omega^*\right\|$ is a bounded sequence, it follows from (3.62) and (3.64) that

$$\|u_{m_k+1} - \omega^*\|^2 \to 0, \text{ as } k \to +\infty. \tag{3.65}$$

The above expression with (3.57) implies that

$$\lim_{k\to+\infty} \|u_k - \omega^*\|^2 \leq \lim_{k\to+\infty} \|u_{m_k+1} - \omega^*\|^2 \leq 0. \tag{3.66}$$

Consequently, we have $u_n \to \omega^*$ as $n \to +\infty$. This completes the proof of the theorem. □

4. Numerical illustration

This section describes the numerical performance of the proposed algorithms, in contrast to some related work in the literature, as well as the analysis of how variations in control parameters affect the numerical effectiveness of the proposed algorithms.

All computations are done in MATLAB R2018b and run on HP i-5 Core(TM) i5-6200 8.00 GB (7.78 GB usable) RAM laptop.

Example 4.1. Let $\Sigma = l_2$ be a real Hilbert space with the sequences of real numbers satisfying the following condition:

$$\|u_1\|^2 + \|u_2\|^2 + \cdots + \|u_n\|^2 + \cdots < +\infty. \tag{4.1}$$

Assume that a mapping $\Gamma : \Delta \to \Delta$ is defined by

$$\Gamma(u) = (5 - \|u\|)u, \ \forall\, u \in \Sigma,$$

where $\Delta = \{u \in \Sigma : \|u\| \leq 3\}$. Then, we can easily seen that Γ is weakly sequentially continuous on Σ and the solution set is $VI(\Delta, \Gamma) = \{0\}$.

For any $u, y \in \Sigma$, we have

$$\begin{aligned}\big\|\Gamma(u) - \Gamma(y)\big\| &= \big\|(5 - \|u\|)u - (5 - \|y\|)y\big\| \\ &= \big\|5(u - y) - \|u\|(u - y) - (\|u\| - \|y\|)y\big\| \\ &\leq 5\|u - y\| + \|u\|\|u - y\| + \big|\|u\| - \|y\|\big|\|y\| \\ &\leq 5\|u - y\| + 3\|u - y\| + 3\|u - y\| \\ &\leq 11\|u - y\|. \end{aligned} \tag{4.2}$$

Hence Γ is L-Lipschitz continuous with $L = 11$.

And, for any $u, y \in \Sigma$ and let $\big\langle \Gamma(u), y - u \big\rangle > 0$, such that

$$(5 - \|u\|)\big\langle u, y - u \big\rangle > 0.$$

Since $\|u\| \leq 3$ and it implies that

$$\big\langle u, y - u \big\rangle > 0.$$

Consider that

$$\begin{aligned}\big\langle \Gamma(y), y - u \big\rangle &= (5 - \|y\|)\big\langle y, y - u \big\rangle \\ &\geq (5 - \|y\|)\big\langle y, y - u \big\rangle - (5 - \|y\|)\big\langle u, y - u \big\rangle \\ &\geq 2\|u - y\|^2 \geq 0. \end{aligned} \tag{4.3}$$

Hence a mapping Γ is quasimonotone on Δ.

Let $u = (\frac{5}{2}, 0, 0, \cdots, 0, \cdots)$ and $y = (3, 0, 0, \cdots, 0, \cdots)$ such that

$$\big\langle \Gamma(u) - \Gamma(y), u - y \big\rangle = \Big(\frac{5}{2} - 3\Big)^3 < 0.$$

Let considered the following projection formula:

$$P_\Delta(u) = \begin{cases} u & \text{if} \quad \|u\| \leq 3, \\ \frac{3u}{\|u\|}, & \text{otherwise.} \end{cases}$$

Figure 1–2 and Table 1 show numerical results. The control conditions are taken in the following way:

(i) Algorithm 3.1 (shortly, **Alg11**): $\varkappa = \frac{0.7}{L}, \vartheta_n = \frac{1}{(n+2)}, \mathrm{g}(u) = \frac{u}{2}$, $D_n = \|u_n - u_y\|$;

(ii) Algorithm 3.2 (shortly, **Alg22**): $\varkappa_1 = 0.22, \chi = 0.44, \vartheta_n = \frac{1}{(n+2)}$, $\mathrm{g}(u) = \frac{u}{2}, D_n = \|u_n - y_n\|$;

(iii) Algorithm 3.4 (shortly, **Alg33**): $\varkappa_1 = 0.22, \chi = 0.44, \varphi_n = \frac{100}{(n+1)^2}$, $\vartheta_n = \frac{1}{(n+2)}, \mathrm{g}(u) = \frac{u}{2}, D_n = \|u_n - y_n\|$.

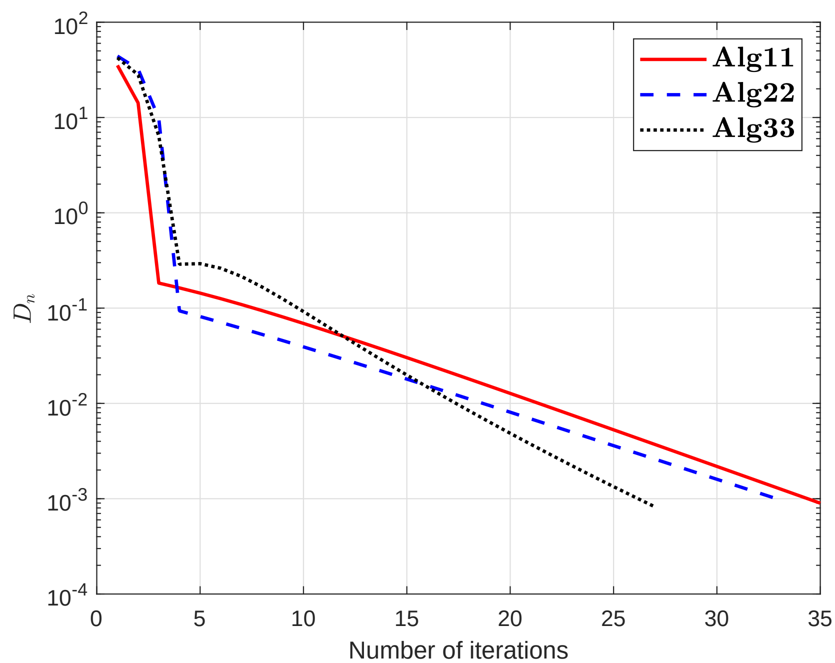

FIGURE 1. Numerical illustration of Algorithm 3.1 and Algorithm 3.2 with Algorithm 3.4 while $u_1 = (1, 1, \cdots, 1_{5000}, 0, 0, \cdots)$.

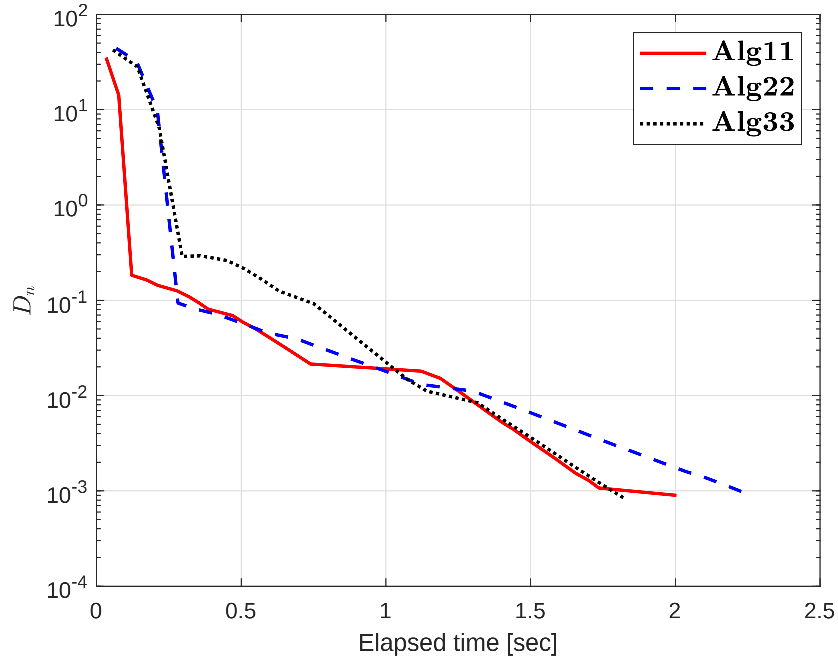

FIGURE 2. Numerical illustration of Algorithm 3.1 and Algorithm 3.2 with Algorithm 3.4 while $u_1 = (1, 1, \cdots, 1_{5000}, 0, 0, \cdots)$.

TABLE 1. Numerical explanation for Example 4.1.

	Number of iterations			Execution time in seconds		
u_1	**Alg11**	**Alg22**	**Alg33**	**Alg11**	**Alg22**	**Alg33**
$(1, 1, \cdots, 1_{5000}, 0, 0, \cdots)$	35	33	27	2.0040884	2.2295634	1.8277622
$(1, 2, \cdots, 10000, 0, 0, \cdots)$	37	34	27	2.2736534	2.5582612	1.9910394
$(10, 10, \cdots, 10_{10000}, 0, 0, \cdots)$	43	39	34	3.2163693	2.9575764	2.2758355
$(20, 20, \cdots, 20_{10000}, 0, 0, \cdots)$	49	43	38	4.5876948	3.8674743	3.1847685

CONCLUSION

We developed various modified extragradient type methods to provide a numerical solution to quasimonotone variational inequality problems in real

Hilbert space. Despite the fact that each sequence is generated by a different step size rule, all sequences generated by the proposed method are strongly convergent to the solution.The numerical results are given to demonstrate the numerical effectiveness of our algorithm. These numerical investigations have demonstrated that the variable step size impacts the efficiency of the iterative sequence in this context.

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