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TWIN OF POSITIVE SOLUTIONS FOR FOUR POINT SINGULAR BOUNDARY VALUE PROBLEMS WITH p-LAPLACIAN OPERATOR

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Abstract. In this paper, we study the multiplicity of positive solutions for the following singular four point boundary value problem with p-Laplacian:

$$\begin{cases} (\phi_p(u'(t)))' + a(t)f(t,u(t)) = 0, \ 0 < t < 1, \\ \alpha\phi_p(u(0)) - \beta\phi_p(u'(\xi)) = 0, \ \gamma\phi_p(u(1)) + \delta\phi_p(u'(\eta)) = 0, \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s, p > 1, \alpha > 0, \beta \geq 0, \gamma > 0, \delta \geq 0, \xi, \eta \in (0,1)$ and $\xi < \eta$. By using monotone iterative technique and fixed point theorem, we establish the existence of two positive solutions for the above problem, one is an iterative positive solution, another is an expansion and compression positive concave solution. In addition, we also give iterative schemes for the first solution, which start off a known simple linear function.

1. Introduction

In this paper, we study the multiplicity of positive solutions for the following quasi-linear equation with p-Laplacian:

$$(\phi_p(u'(t)))' + a(t)f(t,u(t)) = 0, \ 0 < t < 1, \tag{1.1}$$

subject to four point boundary conditions

$$\alpha\phi_p(u(0)) - \beta\phi_p(u'(\xi)) = 0, \ \gamma\phi_p(u(1)) + \delta\phi_p(u'(\eta)) = 0, \tag{1.2}$$

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where $\phi_p(s)$ is p-Laplacian operator, i.e. $\phi_p(s) = |s|^{p-2}s, p > 1,\ \alpha > 0, \beta \geq 0, \gamma > 0, \delta \geq 0, \xi, \eta \in (0,1)$ are given constants, and $\xi < \eta,\ a \in L((0,1), [0,+\infty))$ has countable many singularities on $[0,1]$, $f(t,u)$ is a continuous function mapping $[0,1] \times [0,+\infty)$ to $[0,+\infty)$.

Equations of the above form occur in the study of the n-dimensional p-Laplacian equation, non-Newtonian fluid theory and the turbulent flow of a gas in porous medium, see [1],[2],[11] and [12]. By using the fixed point theorem in cones due to Krasnoselskii, Wang [3], Kong and Wang [4] studied the equation (1.1) subject to one of the following nonlinear boundary conditions:

$$u(0) - g_1(u'(0)) = 0, u(1) + g_2(u'(1)) = 0, \tag{1.3}$$

$$u(0) - g_1(u'(0)) = 0, u'(1) = 0, \tag{1.4}$$

$$u'(0) = 0, u(1) + g_2(u'(1)) = 0. \tag{1.5}$$

By using of the fixed point theorem of three functionals, He and Ge [5] also studied the multiplicity of positive solutions for the equation (1.1) subject to (1.3)-(1.5). However, all the above mentioned references are not allowed to possess countable many singularities on $(0,1)$ for $a(t)$.

Recently, Liu [6] established the existence of positive solutions of the following singular three-point boundary value problems with one-dimensional p-Laplacian:

$$\begin{cases} (\phi_p(u'(t)))' + a(t)f(u(t)) = 0,\ 0 < t < 1, \\ u'(0) = 0,\ u(1) = \beta u(\eta), \end{cases}$$

Under the conditions that $f \in C([0,\infty), [0,\infty)), a \in L([0,1], [0,\infty))$, and $a(t)$ can have countable many singularities on $(0, \frac{1}{2})$. For the other works about the equation with one-dimensional p-Laplacian, we refer the reader to [8]-[10] and their references.

But for equation (1.1), there is few papers dealing with the existence of positive solutions when $a(t)$ possesses countable many singularities on $[0,1]$. In recent papers [7], Su et al. studied the following equation

$$(\phi_p(u'(t)))' + a(t)f(u(t)) = 0, 0 < t < 1, \tag{1.6}$$

subject to the four point boundary condition (1.2). In the corresponding local condition, superlinear and sublinear conditions, Su obtained the existence of the single solution and multiple solutions for the above problems by using the fixed point index theory. It should point out that they didn't give iterative positive solutions and iterative schemes for approximating the solutions.

In this paper, we study the multiplicity of positive solutions for singular nonlinear four point boundary value problem with p-Laplacian when $a(t)$ possesses countable many singularities on $(0,1)$. By using monotone iterative technique and fixed point theorem, we establish the existence of two positive solutions for the BVP (1.1)-(1.2), one is an iterative positive solution, another is an expansion and compression positive concave solution. In addition, we also give iterative schemes for the first solution, which start off a known simple linear function.

2. Preliminaries and Lemmas

In the rest of the paper, we make the following assumptions:

(H_1) $f:[0,1]\times R\to R^+$ is continuous and nondecreasing on u;

(H_2) $a(t)\in L(0,1)$ is nonnegative on $(0,1)$, and $a(t)$ does not vanish identical on any subinterval of $(0,1)$.

Let $E=C[0,1]$ be our Banach space with the maximum norm $\|u\|=\sup_{t\in[0,1]}|u(t)|$.

Definition 2.1. *Let E be a real Banach space. A nonempty closed convex set $K\subset E$ is called a cone of E if it satisfies the following two conditions:*

(1) $x\in K,\lambda^*\geq 0$ *implies* $\lambda^* x\in K$;

(2) $x\in K,-x\in K$ *implies* $x=0$.

Definition 2.2. *A functional $f\in E$ is said to be concave on $[0,1]$ provided $f(tx+(1-t)y)\geq tf(x)+(1-t)f(y)$, for all $x,y\in[0,1]$ and $t\in[0,1]$.*

Definition 2.3. *An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.*

Let

$$K=\{u\in E: u(t)\geq 0, u(t) \text{ is concave function on } [0,1]\},$$

then K is a cone of E.

Lemma 2.4. [11] *Suppose E is a Banach space, $K\in E$ is a cone, let Ω_1,Ω_2 be two bounded open sets of E such that $\theta\in\Omega_1,\overline{\Omega}_1\subset\Omega_2$. Let operator $T: K\cap(\overline{\Omega}_2\backslash\Omega_1)\to K$ be completely continuous. Suppose that one of two conditions hold*

(i) $\|Tx\|\leq\|x\|,\forall x\in K\cap\partial\Omega_1,\quad \|Tx\|\geq\|x\|,\forall x\in K\cap\partial\Omega_2$;

(ii) $\|Tx\|\geq\|x\|,\forall x\in K\cap\partial\Omega_1,\quad \|Tx\|\leq\|x\|,\forall x\in K\cap\partial\Omega_2$,

then T has at least one fixed point in $K\cap(\overline{\Omega}_2\backslash\Omega_1)$.

Lemma 2.5. *Suppose condition (H_2) holds, then*

$$0<\int_\xi^\eta a(s)ds<+\infty.$$

Furthermore, the function

$$A(t)=\int_\xi^t\phi_q\left(\int_s^t a(r)dr\right)ds+\int_t^\eta\phi_q\left(\int_t^s a(r)dr\right)ds,\ t\in[\xi,\eta],$$

is positive continuous function on $[\xi,\eta]$, *therefore* $A(t)$ *has minimum value on* $[\xi,\eta]$, *thus we suppose that this minimum value is* $L>0$, *then*

$$A(t)\ge L,\ \ t\in[\xi,\eta]. \tag{2.1}$$

Proof. At first, it is easy to see that $A(t)$ is continuous on $[\xi,\eta]$. Next, let

$$A_1(t)=\int_\xi^t \phi_q\left(\int_s^t a(r)dr\right)ds,\ A_2(t)=\int_t^\eta \phi_q\left(\int_t^s a(r)dr\right)ds.$$

Then, from the condition (H_2), we have that the function $A_1(t)$ is strictly monotone nondecreasing on $[\xi,\eta]$ and $A_1(\xi)=0$, the function $A_2(t)$ is strictly monotone non-increasing on $[\xi,\eta]$ and $A_1(\eta)=0$, which implies

$$L=\min_{t\in[\xi,\eta]}A(t)>0.$$

The proof is completed. □

Lemma 2.6. *Let* $u\in K$. *Then*

$$u(t)\ge\min\{\xi,1-\eta\}\|u\|,\ \ t\in[\xi,\eta].$$

Proof. Suppose

$$\tau=\inf\left\{\mu\in[0,1]:\sup_{t\in[0,1]}u(t)=u(\mu)\right\}.$$

we shall discuss it from three perspectives.

(i) Let $\tau\in[0,\xi]$. It follows from the concavity of $u(t)$ that each point on chord between $(\tau,u(\tau))$ and $(1,u(1))$ is below the graph of $u(t)$, thus

$$u(t)\ge u(\tau)+\frac{u(1)-u(\tau)}{1-\tau}(t-\tau),t\in[\xi,\eta].$$

Then

$$\begin{aligned}u(t)\ &\ge\min_{t\in[\xi,\eta]}\left[u(\tau)+\tfrac{u(1)-u(\tau)}{1-\tau}(t-\tau)\right]=u(\tau)+\tfrac{u(1)-u(\tau)}{1-\tau}(\eta-\tau)\\&=\tfrac{\eta-\tau}{1-\tau}u(1)+\tfrac{1-\eta}{1-\tau}u(\tau)\ge(1-\eta)u(\tau),\end{aligned}$$

this means $u(t)\ge\min\{\xi,1-\eta\}\|u\|,t\in[\xi,\eta]$.

(ii) Let $\tau\in[\xi,\eta]$. If $t\in[\xi,\tau]$, then we have

$$u(t)\ge u(\tau)+\frac{u(\tau)-u(0)}{\tau}(t-\tau),t\in[\xi,\tau].$$

Then

$$\begin{aligned}u(t)\ &\ge\min_{t\in[\xi,\tau]}\left[u(\tau)+\tfrac{u(\tau)-u(0)}{\tau}(t-\tau)\right]\\&=\tfrac{\xi}{\tau}u(\tau)+\tfrac{\tau-\xi}{\tau}u(0)\ge\xi u(\tau)\ge\min\{\xi,1-\eta\}\|u\|.\end{aligned}$$

If $t \in [\tau, \eta]$, similarly, we have

$$u(t) \geq u(\tau) + \frac{u(1) - u(\tau)}{1 - \tau}(t - \tau), t \in [\tau, \eta].$$

Then

$$\begin{aligned} u(t) &\geq \min_{t \in [\tau, \eta]} \left[u(\tau) + \tfrac{u(1) - u(\tau)}{1 - \tau}(t - \tau) \right] \\ &= \tfrac{1-\eta}{1-\tau} u(\tau) + \tfrac{\eta - \tau}{1 - \tau} u(1) \geq (1 - \eta) u(\tau) \end{aligned}$$

which means $u(t) \geq \min\{\xi, (1 - \eta)\}\|u\|,\ t \in [\xi, \eta]$.

(iii) $\tau \in [\eta, 1]$. Similarly, we have

$$u(t) \geq u(\tau) + \frac{u(\tau) - u(0)}{\tau}(t - \tau), t \in [\xi, \eta],$$

then

$$\begin{aligned} u(t) &\geq \min_{t \in [\xi, \eta]} \left[u(\tau) + \tfrac{u(\tau) - u(0)}{\tau}(t - \tau) \right] \\ &= \tfrac{\xi}{\tau} u(\tau) + \tfrac{\tau - \xi}{\tau} u(0) \geq \xi u(\tau), \end{aligned}$$

this yields $u(t) \geq \xi\|u\|, t \in [\xi, \eta]$. From the above discussion, we obtain $u(t) \geq \min\{\xi, 1 - \eta\}\|u\|, t \in [\xi, \eta]$. The proof is complete. □

For any $u \in K$ satisfying the BVP (1.1), it follows from the boundary condition that we have $u'(\xi) \geq 0, u'(\eta) \leq 0$, then there exists a constant $\sigma \in [\xi, \eta]$ such that $u'(\sigma) = 0$.

Integrate (1.1) from σ to 1, we get

$$\phi_p(u'(t)) = \phi_p(u'(\sigma)) - \int_\sigma^t a(s) f(s, u(s)) ds, \tag{2.2}$$

i.e.,

$$u'(t) = u'(\sigma) - \phi_q \left(\int_\sigma^t a(s) f(s, u(s)) ds \right).$$

thus

$$u(t) = u(\sigma) + u'(\sigma)(t - \sigma) - \int_\sigma^t \phi_q \left(\int_\sigma^s a(r) f(r, u(r)) dr \right) ds. \tag{2.3}$$

Let $t = \eta$ in (2.2), by $u'(\sigma) = 0$, we have

$$\phi_p(u'(\eta)) =) - \int_\sigma^\eta a(s) f(s, u(s)) ds.$$

It follows from the boundary condition $\phi_p(u(1)) = -\frac{\delta}{\gamma}\phi_p(u'(\eta))$ that

$$u(1) = \phi_q \left(\frac{\delta}{\gamma} \int_\sigma^\eta a(s) f(s, u(s)) ds \right). \tag{2.4}$$

Letting $t = 1$ in (2.4), by (2.3), (2.4), we have

$$u(\sigma) = \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta a(s)f(s,u(s))ds\right) + \int_\sigma^\eta \phi_q\left(\int_\sigma^s a(r)f(r,u(r))dr\right)ds. \quad (2.5)$$

So, by (2.5) and (2.3), for $t \in (\sigma, 1)$, we know

$$u(t) = \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta a(s)f(s,u(s))ds\right) + \int_t^1 \phi_q\left(\int_\sigma^s a(r)f(r,u(r))dr\right)ds.$$

Similarly, for $t \in (0, \sigma)$, by integrating the first equation of problems (1.1) on $(0, \sigma)$, we have

$$u(t) = \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(r,u(r))dr\right) + \int_0^t \phi_q\left(\int_s^\sigma a(r)f(r,u(r))dr\right)ds.$$

This implies that

$$u'(t) = \begin{cases} \phi_q\left(\dfrac{\beta}{\alpha}\displaystyle\int_\xi^\sigma a(r)f(r,u(r))dr\right) \\ \quad + \displaystyle\int_0^t \phi_q\left(\int_s^\sigma a(r)f(r,u(r))dr\right)ds, \quad 0 \le t \le \sigma, \\ \phi_q\left(\dfrac{\delta}{\gamma}\displaystyle\int_\sigma^\eta a(r)f(r,u(r))dr\right) \\ \quad + \displaystyle\int_t^1 \phi_q\left(\int_\sigma^s a(r)f(r,u(r))dr\right)ds, \quad \sigma \le t \le 1. \end{cases} \quad (2.6)$$

On the other hand, suppose (2.6) hold, then

$$u'(t) = \begin{cases} \phi_q\left(\displaystyle\int_t^\sigma a(r)f(r,u(r))dr\right) \ge 0, \quad 0 \le t \le \sigma, \\ -\phi_q\left(-\displaystyle\int_\sigma^t a(r)f(r,u(r))dr\right) \le 0, \quad \sigma \le t \le 1. \end{cases} \quad (2.7)$$

It is easy from (2.7) to be obtained $(\phi_p(u'(t)))' + a(t)f(t,u(t)) = 0, 0 < t < 1$. Moreover, letting $t = 0$ and $t = 1$ in (2.6) and (2.7), we can see the boundary value conditions of (1.1) hold.

Now, we define an operator $T : K \to E$ by

$$(Tu)(t) = \begin{cases} \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(r,u(r))dr\right) \\ +\int_0^t \phi_q\left(\int_s^\sigma a(r)f(r,u(r))dr\right)ds, 0 \le t \le \sigma, \\ \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta a(r)f(r,u(r))dr\right) \\ +\int_t^1 \phi_q\left(\int_\sigma^s a(r)f(r,u(r))dr\right)ds, \sigma \le t \le 1. \end{cases} \tag{2.8}$$

Then each fixed point of T is a positive solution of the BVP (1.1)-(1.2).

Lemma 2.7. *$T : K \to K$ is continuous, compact and nondecreasing.*

Proof. At first, since

$$(Tu)'(t) = \begin{cases} \phi_q\left(\int_t^\sigma a(r)f(r,u(r))dr\right) \ge 0, 0 \le t \le \sigma, \\ -\phi_q\left(-\int_\sigma^t a(r)f(r,u(r))dr\right) \le 0, \sigma \le t \le 1. \end{cases} \tag{2.9}$$

is continuous and nonincreasing in $[0,1]$ and $(Tu)'(\sigma) = 0$, which implies that $(Tu)''(t) \le 0$. At the same time, for any $u \in K$, we have

$$(\phi_q(Tu)'(t))' = -a(t)f(u(t)), t \in (0,1)$$

and

$$(Tu)(\sigma) = \|Tu\|. \tag{2.10}$$

This implies that $T(K) \subset K$, and that each fixed point of T is a positive solution of BVP (1.1).

Next, suppose $D \subset K$ is a bounded set, then for any $u \in D$, there exists a constant $M > 0$ such that $\|u\| \le M$. Thus for any $u \in D$, we have

$$\begin{aligned} &\|Tu\| = (Tu)(\sigma) \\ &= \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(r,u(r))dr\right) + \int_0^t \phi_q\left(\int_s^\sigma a(r)f(r,u(r))dr\right)ds \\ &\le \left[\phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)dr\right) + \int_0^t \phi_q\left(\int_s^\sigma a(r)dr\right)ds\right]\phi_q\left(\sup_{[0,1]\times[0,M]} f(t,x)\right) \end{aligned}$$

which implies $T(D)$ is bounded.

On the other hand, according to the Arezela-Ascoli theorem and Lebesgue dominated convergence theorem, we easily see $T : K \to K$ is completely continuous. In the end, noticing the monotonicity of f on u and the definition of T, we also have the operator T is nondecreasing. □

3. Main results

Theorem 3.1. *Suppose conditions (H_1) and (H_2) hold. If there exist two positive constants $n < m$ such that*

$$\sup_{t\in[0,1]} f(t,m) \leq (mM)^{p-1}, \inf_{t\in[\xi,\eta]} f(t, \min\{\xi, 1-\eta\}n) \geq (nN)^{p-1}, \tag{3.1}$$

where

$$M = \left(\phi_q\left(\frac{\beta}{\alpha}\int_\xi^\eta a(r)dr\right) + \int_0^\eta \phi_q\left(\int_s^\eta a(r)dr\right)ds\right)^{-1}, N = \frac{2}{L},$$

then BVP (1.1)-(1.2) has at least two positive and concave solutions u^*, v^* such that $n \leq \|u^*\| \leq m, n \leq \|v^*\| \leq m$. Moreover one positive and concave solution u^* can be obtained through iterative sequence $\{u_n\}$,

$$u_n(t) = (Tu_{n-1})(t) = \begin{cases} \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(r,u_{n-1}(r))dr\right) \\ + \int_0^t \phi_q\left(\int_s^\sigma a(r)f(r,u_{n-1}(r))dr\right)ds, 0 \leq t \leq \sigma, \\ \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta a(r)f(r,u_{n-1}(r))dr\right) \\ + \int_t^1 \phi_q\left(\int_\sigma^s a(r)f(r,u_{n-1}(r))dr\right)ds, \sigma \leq t \leq 1. \end{cases}$$

and

$$\lim_{n\to+\infty} T^n u_0 = u^*,$$

where the initial value $u_0 = m$.

Proof. Let $K[n,m] = \{u \in K : n \leq \|u\| \leq m\}$, we firstly prove $TKn, m \subset K[n,m]$.

In fact, for any $u \in K[n,m]$, we have

$$0 \leq u(t) \leq \max_{t\in[0,1]} u(t) = \|u\| \leq m. \tag{3.2}$$

On the other hand, for any $u \in K[n,m]$, it follows from Lemma 2.6 that

$$\min_{t\in[\xi,\eta]} u(t) \geq \min\{\xi, 1-\eta\}\|u\| \geq \min\{\xi, 1-\eta\}n. \tag{3.3}$$

So by the assumptions (H_1) and (3.1)-(3.2), we have

$$0 \le f(t,u(t)) \le f(t,m) \le \sup_{t\in[0,1]} f(t,m) \le (mM)^{p-1}, \tag{3.4}$$

and from (3.3), for any $t \in [\xi,\eta]$, we also have

$$\begin{aligned} f(t,u(t)) &\ge f(t,\min\{\xi,1-\eta\}n) \\ &\ge \inf_{t\in[\xi,\eta]} f(t,\min\{\xi,1-\eta\}n) \\ &\ge (nN)^{-1}, t\in[\xi,\eta]. \end{aligned} \tag{3.5}$$

It is easy to know by Lemma 2.7 that $Tu \in K$, and by (3.4) and Lemma 2.7, we have

$$\begin{aligned} \|Tu\| &= (Tu)(\sigma) \\ &= \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(r,u(r))dr\right) + \int_0^\sigma \phi_q\left(\int_s^\sigma a(r)f(r,u(r))dr\right)ds, \\ &\le \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\eta a(r)f(r,u(r))dr\right) + \int_0^\eta \phi_q\left(\int_s^\eta a(r)f(r,u(r))dr\right)ds, \\ &\le mM\left(\phi_q\left(\frac{\beta}{\alpha}\int_\xi^\eta a(r)dr\right) + \int_0^\eta \phi_q\left(\int_s^\eta a(r)dr\right)ds\right) \\ &= m. \end{aligned}$$

On the other hand, noting that $\sigma \in [\xi,\eta]$, by Lemma 2.5-2.7 and (3.5), for any $u \in K[n,m]$, we have

$$\begin{aligned} 2\|Tu\| &= 2(Tu)(\sigma) \\ &\ge \int_0^\sigma \phi_q\left(\int_s^\sigma a(r)f(r,u(r))dr\right)ds + \int_\sigma^1 \phi_q\left(\int_\sigma^s a(r)f(r,u(r))dr\right)ds, \\ &\ge \int_\xi^\sigma \phi_q\left(\int_s^\sigma a(r)f(r,u(r))dr\right)ds + \int_\sigma^\eta \phi_q\left(\int_\sigma^s a(r)f(r,u(r))dr\right)ds, \\ &\ge nN\left[\int_\xi^\sigma \phi_q\left(\int_s^\sigma a(r)dr\right)ds + \int_\sigma^\eta \phi_q\left(\int_\sigma^s a(r)dr\right)ds\right] \\ &= NnA(\sigma) \ge nML = 2n. \end{aligned}$$

Thus we have $\|Tu\| \ge n$. From the above expressions, we obtain $n \le \|Tu\| \le m$ for $u \in K[n,m]$, which implies $TK[n,m] \subset K[n,m]$.

Let $u_0(t) = m, t \in [0,1]$. Then $u_0(t) \in K[n,m]$. Let $u_1 = Tu_0$. Then $u_1 \in K[n,m]$. Thus we denote

$$u_{n+1} = Tu_n = T^{n+1}u_0, n = 1,2,\cdots.$$

It follows from $TK[n,m] \subset K[n,m]$ that

$$u_n \in K[n,m], n = 0,1,2,\cdots.$$

Since T is compact by Lemma 2.7, we can assert that u_n is a sequentially compact set.

Now, since $u_1 \in K[n,m]$, we have

$$0 \leq u_1(t) \leq \|u_1\| \leq m = u_0(t).$$

It follows from Lemma 2.7 that $T : K \to K$ is nondecreasing, so

$$u_2 = Tu_1 \leq Tu_0 = u_1.$$

By the induction, we have

$$u_{n+1} \leq u_n, n = 0, 1, 2, \cdots .$$

Consequently, there exists $u^* \in K[n,m]$ such that $u_n \to u^*$. Letting $n \to +\infty$, from the continuity of T and $Tu_n = u_{n-1}$, we obtain $Tu^* = u^*$, which implies that u^* is a nonnegative solution of boundary value problem (1.1)-(1.2). Since $\|u^*\| \geq n > 0$, we conclude $u^*(t) > 0, t \in (0,1)$, thus u^* is a positive concave solution of boundary value problem (1.1)-(1.2).

Next we define two open subsets Ω_1 and Ω_2 of E :

$$\Omega_1 = \{u \in K : \|u\| < n\}, \Omega_2 = \{u \in K : \|u\| < m\}.$$

For any $u \in \partial\Omega_1$, since $\sigma \in [\xi, \eta]$, by Lemma 2.5-2.7 and (3.5), we have

$$\begin{aligned} 2\|Tu\| &= 2(Tu)(\sigma) \\ &\geq \int_0^\sigma \phi_q\left(\int_s^\sigma a(r)f(r,u(r))dr\right)ds + \int_\sigma^1 \phi_q\left(\int_\sigma^s a(r)f(r,u(r))dr\right)ds, \\ &\geq \int_\xi^\sigma \phi_q\left(\int_s^\sigma a(r)f(r,u(r))dr\right)ds + \int_\sigma^\eta \phi_q\left(\int_\sigma^s a(r)f(r,u(r))dr\right)ds, \\ &\geq nN\left[\int_\xi^\sigma \phi_q\left(\int_s^\sigma a(r)dr\right)ds + \int_\sigma^\eta \phi_q\left(\int_\sigma^s a(r)dr\right)ds\right] \\ &= NnA(\sigma) \geq nML = 2n. \end{aligned}$$

Thus we have $\|Tu\| \geq \|u\|, \forall u \in \partial\Omega_1$.

On the other hand, for any $u \in \partial\Omega_2$, by (3.4), we have

$$\begin{aligned} \|Tu\| &= (Tu)(\sigma) \\ &= \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(r,u(r))dr\right) + \int_0^\sigma \phi_q\left(\int_s^\sigma a(r)f(r,u(r))dr\right)ds, \\ &\leq \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\eta a(r)f(r,u(r))dr\right) + \int_0^\eta \phi_q\left(\int_s^\eta a(r)f(r,u(r))dr\right)ds, \\ &\leq mM\left(\phi_q\left(\frac{\beta}{\alpha}\int_\xi^\eta a(r)dr\right) + \int_0^\eta \phi_q\left(\int_s^\eta a(r)dr\right)ds\right) \\ &= m = \|u\|, \end{aligned}$$

which implies that

$$\|Tu\| \leq \|u\|, \forall u \in \partial\Omega_2.$$

It follows from Lemma 2.4 that T has a fixed point $v^* \in (\Omega_2)\backslash\Omega_1$. Obviously, v^* is a concave solution of problem (1.1)-(1.2) and $n < \|v\| < m$. Thus the BVP (1.1)-(1.2) has two positive concave solutions.

Remark 3.2. In Theorem 3.1, although we obtain two positive concave solutions of the BVP (1.1)-(1.2), it is a pity that we dont know whether the two solutions may coincide, if they coincide, then the BVP has only one solution in $K[n, m]$.

Remark 3.3. In Theorem 3.1, we establish iterative sequence of one of solution, which start off with known simple linear function. It is more important that the figure of iterative sequence of solution is explicit, which is helpful to compute for us.

Corollary 3.4. *Assume that (H_1) and (H_2) hold. Further*

$$\begin{aligned} &\underline{\lim}_{x\to+\infty} \sup_{0\leq t\leq 1} \tfrac{f(t,x)}{x^{p-1}} \leq M^{p-1}, \\ &(particularly, \underline{\lim}_{x\to+\infty} \sup_{0\leq t\leq 1} \frac{f(t,x)}{x^{p-1}} = 0), \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\overline{\lim}_{x\to 0} \inf_{\xi\leq t\leq \eta} \frac{f(t,x)}{x^{p-1}} \geq \frac{N^{p-1}}{\min\{\xi, 1-\eta\}}, \\ &(particularly, \overline{\lim}_{x\to 0} \inf_{\xi\leq t\leq \eta} \frac{f(t,x)}{x^{p-1}} = +\infty), \end{aligned} \tag{3.7}$$

where

$$M = \left(\phi_q\left(\frac{\beta}{\alpha}\int_\xi^\eta a(r)dr\right) + \int_0^\eta \phi_q\left(\int_s^\eta a(r)dr\right)ds\right)^{-1}, N = \frac{2}{L}.$$

Then there exist two constants $m > n$ such that the BVP (1.1)-(1.2) has two positive concave solutions u^*, v^* such that $n < \|u^*\| \leq m, n < \|v^*\| \leq m$.

Moreover one concave positive solution u^* can be obtained through iterative sequence $\{u_n\}$,

$$u_n(t) = (Tu_{n-1})(t) = \begin{cases} \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(r,u_{n-1}(r))dr\right) \\ +\int_0^t \phi_q\left(\int_s^\sigma a(r)f(r,u_{n-1}(r))dr\right)ds, 0 \le t \le \sigma, \\ \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta a(r)f(r,u_{n-1}(r))dr\right) \\ +\int_t^1 \phi_q\left(\int_\sigma^s a(r)f(r,u_{n-1}(r))dr\right)ds, \sigma \le t \le 1. \end{cases}$$

and

$$\lim_{n\to+\infty} T^n u_0 = u^*,$$

where the initial value $u_0 = m$.

Proof. It follows from (3.6) and (3.7) that (3.1) in Theorem 3.1 holds, copy the proof of Theorem 3.1, we know the conclusion of corollary 3.5 is true. □

Corollary 3.5. *Suppose conditions (H_1) and (H_2) hold. If there exists $2k$ constants $0 < n_1 < m_1 < n_2 < m_2 < \cdots < n_k < m_k$ such that*

$$\begin{aligned} &\sup_{t\in[0,1]} f(t,m_i) \le (m_iM)^{p-1}, \\ &\inf_{t\in[\xi,\eta]} f(t,\min\{\xi,1-\eta\}n_i) \ge (n_iN)^{p-1}, \\ &i = 1,2,\cdots,k, \end{aligned} \tag{3.8}$$

where

$$M = \left(\phi_q\left(\frac{\beta}{\alpha}\int_\xi^\eta a(r)dr\right) + \int_0^\eta \phi_q\left(\int_s^\eta a(r)dr\right)ds\right)^{-1}, N = \frac{2}{L},$$

then BVP (1.1)-(1.2) has at least $2k$ positive and concave solutions u_i^*, v_i^* such that $n < \|u_i^*\| \le m, n < \|v_i^*\| \le m$.

Moreover k concave positive solution $u_i^*(i = 1, 2, \cdots, k)$ *can be obtained through iterative sequence* $\{u_{i_n}\}$, *which is defined by*

$$u_{i_j}(t) = (Tu_{i_j-1})(t) = \begin{cases} \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma a(r)f(r, u_{i_j-1}(r))dr\right) \\ +\int_0^t \phi_q\left(\int_s^\sigma a(r)f(r, u_{i_j-1}(r))dr\right) ds, 0 \le t \le \sigma, \\ \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta a(r)f(r, u_{i_j-1}(r))dr\right) \\ +\int_t^1 \phi_q\left(\int_\sigma^s a(r)f(r, u_{i_j-1}(r))dr\right) ds, \sigma \le t \le 1. \end{cases}$$

and

$$\lim_{j\to+\infty} T^j u_{i_0} = u_i^*, i = 1, 2, \cdots, k,$$

where the initial value $u_{i_0} = m, i = 1, 2, \cdots, k$.

Remark 3.6. In Corollary 3.5, although there exist $2k$ positive, concave solutions for the BVP (1.1)-(1.2), but these solutions may coincide in $K[n_i, m_i]$, in this case the BVP has only k solutions.

Remark 3.7. In our main results, since we only require $a \in L(0, 1)$, which implies that $a(t)$ can be singular in some countable subset $t \in D$ of $(0, 1)$, for example, $a(t)$ can be singular in all of rational number points of $(0, 1)$. Thus we allow $a(t)$ to possess countable many singularities on $(0, 1)$.

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