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POSITIVE SOLUTIONS FOR NONLINEAR SECOND-ORDER m-POINT BOUNDARY VALUE PROBLEMS WITH SIGN CHANGING NONLINEARITIES

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Abstract. In this paper, we investigate nonlinear second-order m-point boundary value problem

$$\begin{cases} u''(t) + \lambda h(t) f(t,u) = 0, & 0 < t < 1, \\ \beta u(0) - \gamma u'(0) = 0, & u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \end{cases}$$

where the nonlinear term f is allowed to change sign. The existence of an interval of parameters which ensures the problem has at least one positive solution is determined by constructing available operator and combining the method of lower solution with the method of topology degree. Moreover, the associated Green's function for the above problem is also given.

1. Introduction

The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moviseev [5, 6]. Motivated by the study of [5, 6], Gupta [1]studied certain three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary value problems have been studied by several authors. We refer the reader to [2, 9, 7] for some references along this line. Multi-point boundary value problems describe many phenomena in the applied mathematical sciences. For example, the vibrations of a guy

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wire composed of N parts with a uniform cross-section throughout but different densities in different parts can be set up as a multi-point boundary value problem (see [8]). Many problems in the theory of elastic stability can be handle by the method of multi-point boundary value problems (see [11]).

In 1997, Henderson and Wang [4] studied the existence of positive solutions for nonlinear eigenvalue problem

$$\begin{cases} u''(t)+\lambda h(t)f(u)=0, & 0\le t\le 1, \\ u(0)=0, \quad u(1)=0, \end{cases}$$

where $f\in C([0,+\infty),[0,+\infty)), h\in C([0,1],[0,+\infty))$. Authors established the existence of positive solutions theorems under the condition that f is either superlinear or sublinear.

In[9], Ma investigated the following second-order three-point boundary value problem(BVP)

$$\begin{cases} u''(t)+a(t)f(u)=0, & 0\le t\le 1, \\ u(0)=0, \quad u(1)=\alpha u(\eta), \end{cases}$$

where $0<\eta<1, 0<\alpha\eta<1,\ f\in C([0,+\infty),[0,+\infty)),\ a\in C([0,1],[0,+\infty))$. The existence of at least one positive solution is obtained on the conditions that f is either superlinear or sublinear by applying fixed point theorem.

Recently, Ma [10] studied the second-order m-point boundary value problem

$$\begin{cases} u''(t)+a(t)f(u)=0, & 0\le t\le 1, \\ u(0)=0, \quad u(1)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i), \end{cases}$$

where $\alpha_i\ge 0,\ i=1,2,\cdots,m-3,\ \alpha_{m-2}>0,\ 0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1,\ 0<\Sigma_{i=1}^{m-2}\alpha_i\xi_i<1,\ f\in C([0,+\infty),[0,+\infty)),\ a\in C([0,1],[0,+\infty))$. The author obtained the existence of at least one positive solution if f is either superlinear or sublinear by applying a fixed-point theorem in cones.

All the above works were done under the assumption that the nonlinear term is nonnegative due to applying the concavity of solutions in the proofs. In this paper we study the following nonlinear second-order m-point boundary value problem (BVP)

$$u''(t)+\lambda h(t)f(t,u)=0, \qquad 0<t<1, \tag{1.1}$$

$$\beta u(0)-\gamma u'(0)=0, \qquad u(1)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i), \tag{1.2}$$

where the nonlinear term f is allowed to change sign. Firstly we give the associated Green's function for the above problems which makes later discussions more precise. Then by constructing available operator, we combine the method of lower solution with the method of topology degree and show (BVP)(1.1)-(1.2) has at least one positive solution with certain growth conditions imposed on f. In this way we removed the usual restriction on $f \geq 0$.

2. Preliminaries and lemmas

In this section, we present some lemmas that are important to prove our main results.

Lemma 2.1. *Suppose that* $d = \beta(1 - \sum_{i=1}^{m-2} a_i\xi_i) + \gamma(1 - \sum_{i=1}^{m-2} a_i) \neq 0$, $y(t) \in C[0,1]$, *then BVP*

$$u''(t) + y(t) = 0, \qquad 0 < t < 1, \tag{2.1}$$

$$\beta u(0) - \gamma u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(t) \ &= -\textstyle\int_0^t (t-s)y(s)ds + \frac{\beta t+\gamma}{d}\int_0^1 (1-s)y(s)ds \\ &\quad -\textstyle\frac{\beta t+\gamma}{d}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}(\xi_i - s)y(s)ds. \end{aligned} \tag{2.3}$$

Proof. Integrating both sides of (2.1) on $[0,t]$, we have

$$u'(t) = -\int_0^t y(s)ds + u'(0). \tag{2.4}$$

Again integrating (2.4) from 0 to t, we get

$$u(t) = -\int_0^t (t-s)y(s)ds + u'(0)t + u(0). \tag{2.5}$$

In particular,

$$u(1) = -\int_0^1 (1-s)y(s)ds + u'(0) + u(0),$$

and

$$u(\xi_i) = -\int_0^{\xi_i} (\xi_i - s)y(s)ds + u'(0)\xi_i + u(0).$$

By (2.2) we get

$$u'(0) = \frac{\beta}{d}\left[\int_0^1 (1-s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)ds\right].$$

The Lemma 2.1 is proved. □

Lemma 2.2. *Let* $0 < \sum_{i=1}^{m-2} a_i\xi_i < 1,\ d > 0$. *If* $y \in C[0,1]$ *and* $y \geq 0$, *then the unique solution* u *of (BVP)(2.1)-(2.2) satisfies*

$$u(t) \geq 0.$$

Proof. By $u''(t) = -y(t) \leq 0$, we can know that the graph of $u(t)$ is concave down on $(0,1)$. So we only prove $u(0) \geq 0, u(1) \geq 0$.

Firstly, we shall prove $u(0) \geq 0$ by the following two perspectives.
(i) If $0 < \sum_{i=1}^{m-2} a_i \leq 1$, by (2.3) we have

$$\begin{aligned} u(0) &= \frac{\gamma}{d}\left[\int_0^1 (1-s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)ds\right] \\ &\geq \frac{\gamma}{d}\left[\int_0^1 (1-s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_0^1 (1-s)y(s)ds\right] \\ &= \frac{\gamma}{d}\left(1 - \sum_{i=1}^{m-2} a_i\right)\int_0^1 (1-s)y(s)ds \\ &\geq 0. \end{aligned}$$

(ii) If $\sum_{i=1}^{m-2} a_i > 1$, by (2.3) we have

$$\begin{aligned} u(0) &= \frac{\gamma}{d}\left[\int_0^1 (1-s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)ds\right] \\ &\geq \frac{\gamma}{d}\left[\int_0^1 (1-s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_0^1 (\xi_i - s)y(s)ds\right] \\ &= \frac{\gamma}{d}\int_0^1 \left[(1 - \sum_{i=1}^{m-2} a_i\xi_i) + (\sum_{i=1}^{m-2} a_i - 1)s\right] y(s)ds \\ &\geq 0. \end{aligned}$$

On the other hand, by (2.3) we have

$$\begin{aligned} u(1) &= -\int_0^1 (1-s)y(s)ds + \frac{\beta+\gamma}{d}\int_0^1 (1-s)y(s)ds \\ &\quad -\frac{\beta+\gamma}{d}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)ds \\ &\geq \frac{\beta}{d}\Big[\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i(1-s) - (\xi_i - s))y(s)ds \\ &\quad + \sum_{i=1}^{m-2} a_i\xi_i \int_{\xi_i}^1 (1-s)y(s)ds\Big] \\ &\quad + \frac{\gamma}{d}\sum_{i=1}^{m-2} a_i \left[\int_0^1 (1-s)y(s)ds - \int_0^1 (\xi_i - s)y(s)ds\right] \end{aligned}$$

$$= \frac{\beta}{d} \sum_{i=1}^{m-2} a_i \left[\int_0^{\xi_i} (1-\xi_i) s y(s) ds + \xi_i \int_{\xi_i}^1 (1-s) y(s) ds \right]$$
$$+ \frac{\gamma}{d} \sum_{i=1}^{m-2} a_i \left[\int_0^1 (1-\xi_i) y(s) ds \right]$$
$$\geq 0.$$

The proof is completed. □

Lemma 2.3. *Let $\sum_{i=1}^{m-2} a_i\xi_i > 1$, $d \neq 0$. If $y \in C[0,1]$ and $y \geq 0$, then the (BVP)(2.1)-(2.2) has no positive solution.*

Proof. If not, we suppose the (BVP)(2.1)-(2.2) has positive solution u, then $u(\xi_i) > 0, i = 1, 2, \cdots, m-2$ and

$$u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) = \sum_{i=1}^{m-2} a_i \xi_i \frac{u(\xi_i)}{\xi_i} \geq \sum_{i=1}^{m-2} a_i \xi_i \frac{u(\bar{\xi})}{\bar{\xi}} > \frac{u(\bar{\xi})}{\bar{\xi}},$$

where $\bar{\xi} = \min\{\xi_1, \xi_2, \cdots, \xi_{m-2}\}$ satisfies $\frac{u(\bar{\xi})}{\bar{\xi}} = \min\left\{\frac{u(\xi_1)}{\xi_1}, \frac{u(\xi_2)}{\xi_2}, \cdots, \frac{u(\xi_{m-2})}{\xi_{m-2}}\right\}$, which contradicts to the concave of $u(t)$. The proof is completed. □

Lemma 2.4. *Let $a_i \geq 0, i = 1, \cdots, m-2, 0 < \sum_{i=1}^{m-2} a_i\xi_i < 1$, $d > 0$. If $y \in C[0,1]$ and $y \geq 0$, then the unique positive solution $u(t)$ of (BVP)(2.1)-(2.2) satisfies*

$$\inf_{t \in [\xi_{m-2}, 1]} u(t) \geq \sigma ||u||,$$

where $\sigma = \min\left\{\frac{a_{m-2}(1-\xi_{m-2})}{1-a_{m-2}\xi_{m-2}},\ a_{m-2}\xi_{m-2},\ \xi_{m-2}\right\}, ||u|| = \sup_{t\in[0,1]} |u(t)|$.

Proof. Let $u(\bar{t}) = \max_{t\in[0,1]} u(t) = ||u||$, we shall discuss it from the following two perspectives:
Case1: If $0 < \sum_{i=1}^{m-2} a_i < 1$.

Firstly, assume $\bar{t} < \xi_{m-2} < 1$, and so $\min_{t\in[\xi_{m-2},1]} u(t) = u(1)$. By $u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \geq a_{m-2} u(\xi_{m-2})$ we have

$$\begin{aligned} u(\bar{t}) &\leq u(1) + \tfrac{u(1)-u(\xi_{m-2})}{1-\xi_{m-2}}(0-1) \\ &= u(1) - \tfrac{1}{1-\xi_{m-2}} u(1) + \tfrac{1}{1-\xi_{m-2}} u(\xi_{m-2}) \\ &\leq u(1)\left(1 - \tfrac{1}{1-\xi_{m-2}} + \tfrac{1}{a_{m-2}(1-\xi_{m-2})}\right) \\ &= u(1)\tfrac{1-a_{m-2}\xi_{m-2}}{a_{m-2}(1-\xi_{m-2})}. \end{aligned}$$

So

$$\min_{t\in[\xi_{m-2},1]} u(t) \geq \frac{a_{m-2}(1-\xi_{m-2})}{1-a_{m-2}\xi_{m-2}} ||u||. \qquad (2.6)$$

Secondly, assume $\xi_{m-2} < \bar{t} < 1$, then $\min_{t\in[\xi_{m-2},1]} u(t) = u(1)$. Otherwise, we have $\min_{t\in[\xi_{m-2},1]} u(t) = u(\xi_{m-2})$, then $\bar{t} \in [\xi_{m-2}, 1], u(\xi_{m-2}) \geq u(\xi_{m-1}) \geq \cdots \geq u(\xi_2) \geq u(\xi_1)$. By $0 < \sum_{i=1}^{m-2} a_i < 1$ we have

$$u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \leq \sum_{i=1}^{m-2} a_i u(\xi_{m-2}) < u(\xi_{m-2}) \leq u(1)$$

a contradiction.

By concave of $u(t)$ we get $\frac{u(\xi_{m-2})}{\xi_{m-2}} \geq \frac{u(\bar{t})}{\bar{t}} \geq u(\bar{t})$. In fact, since $u(1) \geq a_{m-2}u(\xi_{m-2})$, then $\frac{u(1)}{a_{m-2}\xi_{m-2}} \geq u(\bar{t})$, which implies

$$\min_{t\in[\xi_{m-2},1]} u(t) \geq a_{m-2}\xi_{m-2}||u||. \tag{2.7}$$

Case2: If $\sum_{i=1}^{m-2} a_i > 1$.

Firstly, assume $u(\xi_{m-2}) \leq u(1)$, then $\min_{t\in[\xi_{m-2},1]} u(t) = u(\xi_{m-2})$. By concave of $u(t)$ we have $\bar{t} \in [\xi_{m-2}, 1]$, which implies $\frac{u(\xi_{m-2})}{\xi_{m-2}} \geq \frac{u(\bar{t})}{\bar{t}} \geq u(\bar{t})$, then

$$\min_{t\in[\xi_{m-2},1]} u(t) \geq \xi_{m-2}||u||. \tag{2.8}$$

Secondly, assume $u(\xi_{m-2}) > u(1)$, and so $\min_{t\in[\xi_{m-2},1]} u(t) = u(1)$, and $\bar{t} \in [\xi_1, 1]$. If not, $\bar{t} \in [0, \xi_1)$, then $u(\xi_1) \geq \cdots \geq u(\xi_{m-2}) > u(1)$. So we have

$$u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) > u(1) \sum_{i=1}^{m-2} a_i \geq u(1)$$

a contradiction. By $\sum_{i=1}^{m-2} a_i > 1$ there exists $\bar{\xi} \in \{\xi_1, \xi_2, \cdots, \xi_{m-2}\}$ such that $u(\bar{\xi}) \leq u(1)$, then $u(\xi_1) \leq u(\xi_2) \leq \cdots \leq u(\xi_{m-2}) \leq u(1)$. By concave of $u(t)$ we have $\frac{u(1)}{\xi_1} \geq \frac{u(\xi_1)}{\xi_1} \geq \frac{u(\bar{t})}{\bar{t}} \geq u(\bar{t})$, then

$$\min_{t\in[\xi_{m-2},1]} u(t) \geq \xi_1||u||. \tag{2.9}$$

Therefore, by (2.6)-(2.9) we have

$$\inf_{t\in[\xi_{m-2},1]} u(t) \geq \sigma||u||,$$

where $\sigma = \min\left\{\frac{a_{m-2}(1-\xi_{m-2})}{1-a_{m-2}\xi_{m-2}},\ a_{m-2}\xi_{m-2},\ \xi_{m-2}\right\}$. The proof is completed. □

Lemma 2.5. *Suppose that $d \neq 0$, then the Green's function for the BVP*

$$-u''(t) = 0, \qquad 0 < t < 1,$$

$$\beta u(0) - \gamma u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)$$

is given by

$$G(t,s) = \begin{cases} \dfrac{(\beta s+\gamma)\left[(1-t)-\sum_{j=1}^{m-2} a_j(\xi_j-t)\right]}{d}, \\ 0\le t\le 1,\ s\le \xi_1,\ \ s\le t; \\ \dfrac{(\beta s+\gamma)(1-t)-\sum_{j=i}^{m-2} a_j(\xi_j-t)(\beta s+\gamma)+\sum_{j=1}^{i-1} a_j(\beta\xi_j+\gamma)(t-s)}{d}, \\ \xi_{r-1}\le t\le \xi_r,\ 2\le r\le m-1,\ \xi_{i-1}\le t\le \xi_i,\ 2\le i\le r, s\le t; \\ \dfrac{(\beta t+\gamma)\left[(1-s)-\sum_{j=i}^{m-2} a_j(\xi_j-s)\right]}{d}, \\ \xi_{r-1}\le t\le \xi_r,\ 2\le r\le m-1,\ \xi_{i-1}\le t\le \xi_i,\ 2\le i\le r, t\le s; \\ \dfrac{(\beta t+\gamma)(1-s)}{d}, \\ 0\le t\le 1,\ \xi_{m-2}\le s\le 1,\ t\le s. \end{cases} \tag{2.10}$$

Here for the sake of convenience, we write $\xi_0=0, \xi_{m-1}=1$.

Proof. If $0\le t\le \xi_1$, the unique solution (2.3) given by Lemma 2.1 can be rewritten as

$$\begin{aligned} u(t) &= \int_0^t \tfrac{(\beta s+\gamma)\left[(1-t)-\sum_{j=1}^{m-2} a_j(\xi_j-t)\right]}{d} y(s)ds \\ &\quad + \int_t^{\xi_1} \tfrac{(\beta t+\gamma)\left[(1-s)-\sum_{j=1}^{m-2} a_j(\xi_j-s)\right]}{d} y(s)ds \\ &\quad + \sum_{i=2}^{m-2} \int_{\xi_{i-1}}^{\xi_i} \frac{(\beta t+\gamma)\left[(1-s)-\sum_{j=i}^{m-2} a_j(\xi_j-s)\right]}{d} y(s)ds \\ &\quad + \int_{\xi_{m-2}}^1 \tfrac{(\beta t+\gamma)(1-s)}{d} y(s)ds. \end{aligned}$$

Similarly, if $\xi_{r-1}\le t\le \xi_r, 2\le r\le m-2$, the unique solution (2.3) can be expressed

$$\begin{aligned} u(t) &= \int_0^{\xi_1} \tfrac{(\beta s+\gamma)\left[(1-t)-\sum_{j=1}^{m-2} a_j(\xi_j-t)\right]}{d} y(s)ds \\ &\quad + \sum_{i=2}^{r-1} \int_{\xi_{i-1}}^{\xi_i} \tfrac{(\beta s+\gamma)(1-t)-\sum_{j=i}^{m-2} a_j(\xi_j-t)(\beta s+\gamma)+\sum_{j=1}^{i-1} a_j(\beta\xi_j+\gamma)(t-s)}{d} y(s)ds \\ &\quad + \int_{\xi_{r-1}}^t \tfrac{(\beta s+\gamma)(1-t)-\sum_{j=r}^{m-2} a_j(\xi_j-t)(\beta s+\gamma)+\sum_{j=1}^{i-1} a_j(\beta\xi_j+\gamma)(t-s)}{d} y(s)ds \\ &\quad + \int_t^{\xi_r} \tfrac{(\beta t+\gamma)\left[(1-s)-\sum_{j=r}^{m-2} a_j(\xi_j-s)\right]}{d} y(s)ds \end{aligned}$$

$$+\sum_{i=r+1}^{m-2}\int_{\xi_{i-1}}^{\xi_i}\frac{(\beta t+\gamma)\big[(1-s)-\sum_{j=i}^{m-2}a_j(\xi_j-s)\big]}{d}y(s)ds$$
$$+\int_{\xi_{m-2}}^{1}\frac{(\beta t+\gamma)(1-s)}{d}y(s)ds.$$

If $\xi_{m-2}\le t\le 1$, the unique solution (2.3) can be given in the form

$$\begin{aligned}u(t) &= \int_0^{\xi_1}\frac{(\beta s+\gamma)\big[(1-t)-\sum_{j=1}^{m-2}a_j(\xi_j-t)\big]}{d}y(s)ds\\ &+\sum_{i=2}^{m-2}\int_{\xi_{i-1}}^{\xi_i}\frac{(\beta s+\gamma)(1-t)-\sum_{j=i}^{m-2}a_j(\xi_j-t)(\beta s+\gamma)+\sum_{j=1}^{i-1}a_j(\beta\xi_j+\gamma)(t-s)}{d}y(s)ds\\ &+\int_{\xi_{m-2}}^{t}\frac{(\beta s+\gamma)(1-t)+\sum_{j=1}^{i-1}a_j(\beta\xi_j+\gamma)(t-s)}{d}y(s)ds\\ &+\int_t^1\frac{(\beta t+\gamma)(1-s)}{d}y(s)ds.\end{aligned}$$

Lemma 2.5 is proved. □

By Lemma 2.5, the unique solution of (BVP)(2.1)-(2.2) is $u(t)=\int_0^1 G(t,s)$ $y(s)ds$. Let $\omega(t)=\int_0^1 G(t,s)h(s)ds$. Obviously $\omega(t)$ is the unique solution of (BVP)(2.1)-(2.2) for $y(t)=h(t)$.

Lemma 2.6. *Let $X=C[0,1], K=\{u\in X: u\ge 0\}$. Suppose $T:X\to X$ is completely continuous. Define $\theta: TX\to K$ by*

$$(\theta y)=\max\{y(t),\omega(t)\},\quad \textit{for}\ \ y\in TX,$$

where $\omega\in C^1[0,1], \omega(t)\ge 0$ is given function. Then

$$\theta\circ T: X\to K$$

is also a completely continuous operator.

Proof. The complete continuity of T implies that T is continuous and maps each bounded subset in X to a relatively compact set. Denote θy by $\overline{y}$.

Given a function $h\in C[0,1]$, for each $\varepsilon>0$ there is $\delta>0$ such that

$$||Th-Tg||<\varepsilon,\quad \text{for}\ \ g\in X, ||g-h||<\delta.$$

Since

$$\begin{aligned}|(\theta Th)(t)-(\theta Tg)(t)| &= |\max\{(Th)(t),\omega(t)\}-\max\{(Tg)(t),\omega(t)\}|\\ &\le |(Th)(t)-(Tg)(t)|\\ &<\varepsilon,\end{aligned}$$

we have

$$||(\theta T)h-(\theta T)g||<\varepsilon,\quad \text{for}\ \ g\in X, ||g-h||<\delta,$$

and so θT is continuous.

For any arbitrary bounded set $D \subset X$ and $\forall \varepsilon > 0$, there are $y_i, i = 1, 2, \cdots, m$ such that

$$TD \subset \bigcup_{i=1}^{m} B(y_i, \varepsilon),$$

where $B(y_i, \varepsilon) := \{u \in X : ||u - y_i|| < \varepsilon\}$. Then, for $\forall \overline{y} \in (\theta \circ T)D$, there is a $y \in TD$ such that $\overline{y}(t) = \max\{y(t), \omega(t)\}$. We choose $i \in \{1, 2, \cdots, m\}$ such that $||y - y_i|| < \varepsilon$. The fact

$$\max_{t\in[0,1]} |\overline{y}(t) - \overline{y}_i(t)| \leq \max_{t\in[0,1]} |y(t) - y_i(t)|,$$

which implies $\overline{y} \in B(\overline{y}_i, \varepsilon)$. Hence $(\theta \circ T)D$ has a finite $\varepsilon - net$ and therefore $(\theta \circ T)D$ is relatively compact. $\square$

3. The main results

Let $X = C[0, 1], K = \{u \in X : u \geq 0\}$. Denote by $||.||$ the supremum norm on X.

In the rest of the paper, we make the following assumptions:

(H_1) $\beta,\ \gamma \geq 0, \beta + \gamma > 0, \alpha_i \geq 0, i = 1, 2, \cdots, m-3, \alpha_{m-2} > 0,\ 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, 0 < \sum_{i=1}^{m-2} \alpha_i\xi_i < 1,\ d = \beta(1 - \sum_{i=1}^{m-2} \alpha_i\xi_i) + \gamma(1 - \sum_{i=1}^{m-2} \alpha_i) > 0$;

(H_2) $f : [0, 1] \times [0, +\infty) \to R$ is continuous ;

(H_3) $h(t)$ is a nonnegative measurable function on $[0, 1]$ with $0 < \int_0^1 h(t)dt < \infty$.

Obviously, $G(t, s) \geq 0$. By Hölder's inequality, we have

$$\int_0^1 |G(t, s)h(s)|ds \leq \left(\int_0^1 |G(t, s)|^2 ds\right)^{\frac{1}{2}} \left(\int_0^1 |h(s)|^2 ds\right)^{\frac{1}{2}} < \infty, t \in [0, 1].$$

Let $A = \max_{0\leq t\leq 1} \int_0^1 G(t, s)h(s)ds$.

Theorem 3.1. *Suppose there exist $r > M > 0$ such that*

$$0 < \frac{M}{\min_{0\leq t\leq 1} f(t, M\omega(t))} = a \leq b = \frac{r}{A \max_{0\leq t\leq 1, M\omega(t)\leq u\leq r} f(t, u)}. \tag{3.1}$$

Then, for each $\lambda \in (a, b)$, the (BVP) (1.1)-(1.2) has at least one positive solution $u_1(t)$ such that

$$0 < M\omega(t) \leq u_1(t), \quad and \quad ||u_1|| \leq r.$$

Proof. Let

$$f^*(t,u)=\begin{cases} f(t,u), & u\geq M\omega(t),\\ f(t,M\omega(t)), & u\leq M\omega(t),\end{cases}$$

and define $T:K\to X$ by

$$(Tu)(t)=\lambda\int_0^1 G(t,s)h(s)f^*(s,u(s))dst\in[0,1].$$

Then T is a completely continuous operator on K. For the operator $\theta:X\to K$ defined by

$$(\theta u)(t)=\max\{u(t),0\},$$

by Lemma 2.6 we can know that $\theta\circ T:K\to K$ is also completely continuous.

Take $\Omega=\{u\in K:||u||<r\}$. Given $u\in\partial\Omega$, set $I=\{t\in[0,1]:f^*(t,u(t))\geq 0\}$. Then

$$\begin{aligned}(\theta\circ T)u(t)\ &=\max\left\{\lambda\textstyle\int_0^1 G(t,s)h(s)f^*(s,u(s))ds,0\right\}\\ &\leq\lambda\textstyle\int_I G(t,s)h(s)f^*(s,u(s))ds\\ &\leq b\max_{0\leq t\leq 1,0\leq u\leq r}f^*(t,u)\textstyle\int_I G(t,s)h(s)ds\\ &\leq Ab\max_{0\leq t\leq 1,M\omega(t)\leq u\leq r}f(t,u)\\ &\leq r.\end{aligned}$$

If there is a $u\in\partial\Omega$ such that $(\theta\circ T)u=u$, then $\theta\circ T$ has a fixed point in $\overline{\Omega}$. Suppose for $\forall\ u\in\partial\Omega$ such that $(\theta\circ T)u\neq u$, it follows that

$$deg_K\{I-\theta\circ T,\Omega,0\}=1,$$

where $\deg_K$ stands for the degree on cone K. Then $\theta\circ T$ has a fixed point in Ω. So in both the cases $\theta\circ T$ has a fixed point u_1 in $\overline{\Omega}$.

We claim that

$$(Tu_1)(t)\geq M\omega(t),\quad t\in[0,1]. \tag{3.2}$$

Otherwise, there exists $t_0\in[0,1]$ such that

$$M\omega(t_0)-(Tu_1)(t_0)=\max_{t\in[0,1]}\{M\omega(t)-(Tu_1)(t)\}=L>0. \tag{3.3}$$

Now we prove $t_0\in(0,1)$. Suppose the contrary, if $t_0=0$, then $M\omega'(0)-(Tu_1)'(0)\leq 0$. Since both $M\omega(t)$ and $(Tu_1)(t)$ satisfy the boundary condition (1.2), we have

$$\beta[M\omega(0)-(Tu_1)(0)]-\gamma[M\omega'(0)-(Tu_1)'(0)]=0,$$

which contradicts condition (H_1). If $t_0=1$, we have

$$L=M\omega(1)-(Tu_1)(1)=\sum_{i=1}^{m-2}\alpha_i[M\omega(\xi_i)-(Tu_1)(\xi_i)]\leq\sum_{i=1}^{m-2}\alpha_iL<L$$

a contradiction. So $t_0 \in (0,1)$, and $M\omega'(t_0) - (Tu_1)'(t_0) = 0$.
We prove

$$M\omega(t) > Tu_1(t), \quad t \in [0,1]. \tag{3.4}$$

Otherwise, there exists $t_1 \in [0, t_0) \cup (t_0, 1]$ such that

$$M\omega(t_1) - (Tu_1)(t_1) = 0, \text{ and } M\omega(t) - (Tu_1)(t) > 0, \ t \in (t_1, t_0] \text{ or } t \in [t_0, t_1).$$

Without loss of generality, we suppose $t_1 \in [0, t_0)$. Then for $t \in (t_1, t_0]$,

$$\begin{aligned} M\omega'(t) - (Tu_1)'(t) &= M\omega'(t_0) - (Tu_1)'(t_0) - \textstyle\int_t^{t_0} [M\omega'(s) - (Tu_1)'(s)]' ds \\ &= \textstyle\int_t^{t_0} h(s)[M - \lambda f^*(s, u(s))] ds \\ &= \textstyle\int_t^{t_0} h(s)[M - \lambda f(s, M\omega(t))] ds \\ &\le [M - a \min_{t\in[0,1]} f(t, M\omega(t))] \textstyle\int_t^{t_0} h(s) ds \\ &= 0, \end{aligned}$$

i.e., $M\omega'(t) - (Tu_1)'(t) \le 0$, and then

$$M\omega(t_0) - (Tu_1)(t_0) \le M\omega(t_1) - (Tu_1)(t_1) = 0,$$

which contradicts to (3.3). So (3.4) holds.
However,

$$\begin{aligned} M\omega(t_0) - (Tu_1)(t_0) &= \int_0^1 G(t_0, s)h(s)M ds - \lambda \int_0^1 G(t_0, s)h(s) f^*(s, u_1(s)) ds \\ &= \int_0^1 G(t_0, s)h(s)[M - \lambda f^*(s, u_1(s))] ds \\ &\le [M - a \min_{t\in[0,1]} f(t, M\omega(t))] \int_0^1 G(t_0, s)h(s) ds \\ &= 0. \end{aligned}$$

which contradicts to (3.3). So (3.2) holds. Then $(\theta \circ T)u_1 = Tu_1 = u_1$ and $u_1(t)$ is a solution of (BVP) (1.1)-(1.2). □

Corollary 3.1. *Suppose there exists a constant $M > 0$ such that*

$$a = \frac{M}{\min_{0\le t\le 1} f(t, M\omega(t))} > 0 \tag{3.5}$$

and

$$\lim_{u\to\infty} \frac{\max_{0\le t\le 1} f(t,u)}{u} \le 0. \tag{3.6}$$

Then, for each $\lambda \geq a$, the (BVP) (1.1)-(1.2) has at least one positive solution $u_1(t)$ such that

$$0 < M\omega(t) \leq u_1(t), \qquad ||u_1|| < \infty.$$

Proof. It suffices to show that for $b > a$, there exists $r > 0$ such that

$$b \leq \frac{r}{A \max_{0\leq t\leq 1, M\omega(t)\leq u\leq r} f(t,u)}. \tag{3.7}$$

Fix $b > a > 0$. By condition (3.6), we can know there exists $L > 0$ such that

$$\frac{\max_{0\leq t\leq 1} f(t,u)}{u} < \frac{1}{bA}, \quad \text{for} \quad u \geq L,$$

and there exists $r > L$ such that

$$\frac{\max_{0\leq t\leq 1, M\omega(t)\leq u\leq L} f(t,u)}{r} < \frac{1}{bA}.$$

Hence

$$\begin{aligned}
&\frac{\max_{t\in 0\leq t\leq 1, M\omega(t)\leq u\leq r} f(t,u)}{r} \\
&\leq \max\left\{\frac{\max_{0\leq t\leq 1, M\omega(t)\leq u\leq L} f(t,u)}{r}, \frac{\max_{0\leq t\leq 1, L\leq u\leq r} f(t,u)}{r}\right\} \\
&< \max\left\{\frac{1}{bA}, \max_{L\leq u\leq r}\left[\frac{\max_{0\leq t\leq 1} f(t,u)}{u}\right]\right\} \\
&< \frac{1}{bA},
\end{aligned}$$

and in turn (3.7) holds. Applying Theorem 3.1, we prove this theorem since $b > a$ is arbitrary. □

Theorem 3.2. *Suppose $f(t,0) \geq 0, h(t)f(t,0) \not\equiv 0$, and there exists a constant $r > 0$ such that*

$$b = \frac{r}{A \max_{0\leq t\leq 1, 0\leq u\leq r} f(u)} > 0. \tag{3.8}$$

Then, for each $\lambda \leq b$, the (BVP)(1.1)-(1.2) has at least one positive solution $u_1(t)$ satisfying

$$0 < ||u_1|| < r.$$

Proof. Set

$$f^*(t,u) = \begin{cases} f(t,u), & u \geq 0, \\ f(t,0) - u, & u(t) < 0, \end{cases}$$

The rest proof is similar to Theorem 3.1, we omit it. □

Corollary 3.2. *Suppose condition (3.6) holds and*

$$f(t,0) \geq 0, h(t)f(t,0) \not\equiv 0, t \in (0,1).$$

Then, for each $\lambda \in R$, the (BVP)(1.1)-(1.2) has at least one positive solution $u_1(t)$ satisfying

$$0 < ||u_1|| < r.$$

Proof. Condition (3.8) can be deduced from (3.6) for any $b > 0$. Theorem 3.2 implies this corollary. □

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