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($GDDVIP$;λ) AND THE VARIABLE STEP ITERATIVE METHOD FOR T-η-INVEX FUNCTION OF ORDER λ IN HILBERT SPACES

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Abstract. The aim of this paper is to introduce the concept of generalized differential dominated variational inequality problem of order $\lambda > 0$ ($GDDVIP; \lambda$) and T-η-invex function of order λ and study them by using a function of proportionality. The problems such as Minimization Problem with variational inequality condition ($MPVIC$), and generalized differential inequality problem ($GDIP$) are studied in the presence of T-η-invex function of order λ by using the function of proportionality. The existence of the T-η-invex function of order λ is studied in Hilbert space by using the variable step iterative method. The iterative process considered in the paper admits the presence of variable iteration parameters, which can be useful in numerical implementation to find T-η-invex function of order λ. Finally the existence theorem of T-η-invex function of order λ is studied with a concrete example.

1. Introduction

In recent decades, the study of variational inequalities introduced by Stampacchia [19] has become a part of development in the theory of optimization theory because optimization problems can often be reduced to the solution of variational inequalities. It is important to remark that these subjects pertant

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to more than just optimization problems and there in lies much of their attractiveness. Several authors have proved many fascinating results on variational inequality problem. We list some of them, which are used frequently in this paper. The existence of the solution to the problem is studied by many authors such as, D. Kinderlehrer and G. Stampacchia [15], M. Chipot [6], J.L. Lions and G. Stampacchia [16], R.W. Cottle, F. Giannessi and J.L. Lions [8], F.E. Browder [5], Ky Fan [12], U. Mosco [18], A. Behera and G.K. Panda [1, 2], G. Isac[14], to name only a few.

Let X be a reflexive real Banach space with its dual X^*. Let K be a nonempty subset of X. Let $T : K \to X^*$ be a nonlinear mapping. The pair $\langle f, x\rangle$ denotes the value of $f \in X^*$ at $x \in K$. The variational inequality problem is to: Find $x_0 \in K$ such that

(VIP) $$\langle T(x_0), x - x_0\rangle \geq 0 \quad \forall x \in K.$$

The notion of invexity was introduced by M.A. Hanson [13] in 1981 as a generalization of the concept of convexity. The concept of invexity of a function brought a new edge to generalize the variational inequality problem which is a general case of optimization problem, complementarity problem and fixed point problem. Many authors have studied different types of convex and invex functions in vector spaces with different assumptions. For the concept of invexity, we refer to [4, 17]. For the generalization of the differentiable invex function [13], Behera and Das [3] introduced the T-η-invex as an operator invex function in ordered topological vector spaces and differentiable manifolds to study various types of generalized vector variational inequality problems in ordered topological vector spaces, H-differentiable manifolds, n-manifolds and $\mathbb{S}^n$. For our study, we refer to [9, 11].

In [11], the *generalized differential dominated variational inequality problems*$(GDDVIP)$ is defined as follows. Let $F : K \to \mathbb{R}$ be a differentiable map where ∇F is the derivative of F. Let $\eta : K \times K \to X$ be a vector valued mapping.

Find $x_0 \in K$ such that

(GDDVIP) $$\langle (\nabla F - T)(x_0), \eta(x, x_0)\rangle \geq 0, \quad \forall x \in K.$$

In this paper, the generalized differential dominated variational inequality problems $(GDDVIP)$ is extended as the *generalized dominated differential variational inequality problems* of order λ $(GDDVIP; \lambda)$, $\lambda > 0$ given by

$(GDDVIP;\lambda)$ for any $\lambda > 0$, find $x_0 \in K$ such that

$$(\mathbf{GDDVIP};\lambda) \qquad \langle (\nabla F - \lambda T)(x_0), \eta(x, x_0)\rangle \geq 0 \quad \forall x \in K.$$

We recall some known definitions and results for our need.

Definition 1.1. ([13]) The mapping $F : M \to Y \subset \mathbb{R}^n$ is

(a) η-invex on M if

$$F(x) - F(u) \geq \langle \nabla F(u), \eta(x, u)\rangle \quad \text{for all } x, u \in M,$$

(b) η-pseudoinvex on M if

$$\langle \nabla F(u), \eta(x, u)\rangle \geq 0 \quad \Rightarrow F(x) - F(u) \geq 0 \quad \text{for all } x, u \in M.$$

(b) η-quasiinvex on M if

$$F(x) - F(u) \leq 0 \quad \Rightarrow \langle \nabla F(u), \eta(x, u)\rangle \leq 0 \quad \text{for all } x, u \in M.$$

Definition 1.2. ([13], η-invex set) Let K be any subset of the vector space X. Let $\eta : K \times K \to X$ be continuous vector valued mapping. Then K is said to be η-invex if for all $x, u \in K$ and for all $t \in (0, 1)$, we have

$$u + t\eta(x, u) \in K.$$

Definition 1.3. ([10], weakly η-invex set) The set K is said to be weakly η-invex set if for all $x, u \in K$, there exists a $t \in (0,\ 1)$ such that

$$z + t\eta(x, u) \in K \quad \textit{where } z \in \{u, x\}.$$

The classical problems are defined as follows. Let X be a reflexive real Banach space with its dual X^*. Let K be a nonempty subset of X. Let $T : K \to X^*$ be a nonlinear mapping. Let $F : K \to \mathbb{R}^n$ be a differentiable map where $\nabla F(u)$ is the differential of $u \in K$.

The *generalized variational inequality problem* is of: finding $x_0 \in K$ such that

$$(\mathbf{GVIP}) \qquad \langle T(x_0), \eta(x, x_0)\rangle \geq 0 \quad \forall x \in K.$$

The *generalized differential inequality problem* is of: finding $x_0 \in K$ such that

$$(\mathbf{GDIP}) \qquad \langle \nabla F(x_0), \eta(x, x_0)\rangle \geq 0 \quad \forall x \in K.$$

The *minimization problem* is of: finding $x_0 \in K$ such that

(**MP**) $F(x) \geq F(x_0) \quad \forall x \in K.$

The concept of *Minimization Problem* given with *Variational Inequality Constraints* is defined to: find $u \in M$ such that

(**MPVIC**) $F(x) - F(u) \geq 0$

subject to

$$\langle T(u), \eta(x,u)\rangle \geq 0 \quad \forall\ x \in M.$$

Let there exist a multiplier vector $\lambda > 0$ such that

$$F(x) - F(u) - \lambda\langle T(u), \eta(x,u)\rangle \geq 0$$

for all $x \in M$ and fixed $u \in M$; hence $(MPVIC)$ problem is to find $u \in M$ such that F is T-η-invex of order $\lambda > 0$ at point $u \in M$. In other words, if F is T-η-invex of order $\lambda > 0$ at point $u \in M$ and u satisfies the constraints of $(MPVIC)$, then u is a minimal solution of $(MPVIC)$.

In this paper we deal with the above problems and prove their existence theorems under certain conditions in the presence of T-η-invex function.

2. T-η-invex of order λ and Function of Proportionality

For our need, we recall the concept of T-η-invexity of any function F and η-monotoneness of T. Let X be a topological vector space, $M \subset X$, (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $intP \neq \varnothing$, $L(X,Y)$ be the set of continuous linear functionals from X to Y, $\eta : M \times M \to X$ be a vector-valued function, and $T : M \to L(X,Y)$ be an operator.

Definition 2.1. ([3]) The mapping $F : M \to Y$ is

(a) T-η-invex on M if

$$F(x) - F(u) - \langle T(u), \eta(x,u)\rangle \geq_P 0 \quad \text{for all } x, u \in M,$$

(b) T-η-invex at point $u \in M$ if

$$F(x) - F(u) - \langle T(u), \eta(x,u)\rangle \geq_P 0 \quad \text{for all } x \in M.$$

Definition 2.2. ([3]) The mapping $T : M \to L(X,Y)$ is η-monotone if

$$\langle T(u), \eta(x,u)\rangle + \langle T(x), \eta(u,x)\rangle \leq_P 0 \quad \text{for all } x \in M.$$

The definition of T-η-equiinvex and T-η-invex of order λ are defined as follows.

Definition 2.3. The mapping $F : M \to Y$ is

(a) T-η-equiinvex on M if

$$F(x) - F(u) - \langle T(u), \eta(x,u)\rangle =_P 0 \quad \text{for all } x, u \in M,$$

(b) T-η-invex of order λ on M if there exist some $\lambda > 0$ such that

$$F(x) - F(u) - \lambda\langle T(u), \eta(x,u)\rangle \geq_P 0 \quad \text{for all } x, u \in M,$$

(c) T-η-invex of order λ at point $u \in M$ if there exist some $\lambda > 0$ such that

$$F(x) - F(u) - \lambda\langle T(u), \eta(x,u)\rangle \geq_P 0 \quad \text{for all } x \in M.$$

Here the multiplier λ is called the bound of proportionality of the fraction $F(x) - F(u)$ by $\langle T(u), \eta(x,u)\rangle$ for all $x, u \in M$.

Remark 2.4. If F is T-η-equiinvex on M, then F is T-η-invex on M but not conversely.

If X is a reflexive real Banach space, $Y = \mathbb{R}^n$ and $P = \mathbb{R}^n_+$, the set of nonnegative numbers in $\mathbb{R}^n$, then in the Definition 2.3, the symbols $\geq_P$, $\leq_P$ will be replaced by $\geq$ and $\leq$ respectively.

Let X be a reflexive real Banach space and $M \subset X$. Let $Y = \mathbb{R}^n$ and $P = \mathbb{R}^n_+$. Consider a function of proportionality $Q : Y \times Y \to \mathbb{R}$ as a fractional function in Y defined by the rule

$$Q(y_1, y_2) = \frac{y_1}{y_2} \quad \text{for all } y_1,\ y_2 \in Y.$$

Thus

$$Q(y_1, ty_2) = \frac{y_1}{ty_2} = Q\left(\frac{y_1}{t}, y_2\right) \quad \text{for all } y_1,\ y_2 \in Y.$$

Obviously Q is a homogeneous function of degree 0 as for any $t \in \mathbb{R}$,

$$Q(ty_1, ty_2) = Q(y_1, y_2).$$

Proposition 2.5. *Let X be a reflexive real Banach space and $M \subset X$. Let $T : M \to L(X, \mathbb{R}^n) \equiv \mathbb{R}^n$ be a map and $\eta : M \times M \to X$ be a vector valued map. If the mapping $F : M \to \mathbb{R}^n$ is T-η-equiinvex on M, then F is T-η-invex of order $\lambda \in (0,1)$ on M.*

Proof. The mapping $F : M \to \mathbb{R}^n$ is T-η-equiinvex on M, i.e.,

$$F(x) - F(u) - \langle T(u), \eta(x,u)\rangle = 0$$

for all $x, u \in M$, then

$$Q\left(F(x) - F(u), \langle T(u), \eta(x,u)\rangle\right) = 1.$$

Thus

$$Q\left(F(x) - F(u), \langle T(u), \eta(x,u)\rangle\right) \geq \lambda$$

for all $\lambda \in (0,1)$, i.e.,

$$F(x) - F(u) - \lambda\langle T(u), \eta(x,u)\rangle \geq 0$$

for all $x, u \in M$ and $\lambda \in (0,1)$. Thus F is T-η-invex on M of order $\lambda \in (0,1)$. This completes the proof. □

Remark 2.6. If $\lambda = 1$, then F is the T-η-invex function where bound of proportionality is $\lambda = 1$ [3].

We consider the problem as follows.

Problem 2.7. *Let for each $u \in M$, the set $K(u)$ be defined by*

$$K(u) = \{x \in M : F(x) - F(u) \geq 0\},$$

and the set $C(u)$ be defined by

$$C(u) = \{x \in M : \langle T(u), \eta(x,u)\rangle \geq 0\}.$$

Consider a problem as follows: find $x_0 \in M$ such that

$$Q\left(F(x) - F(x_0), \langle T(x_0), \eta(x, x_0)\rangle\right) \geq \lambda$$

for all $x \in K(x_0) \bigcap C(x_0)$ and $\lambda > 0$.

The following theorem shows the existence of T-η-invex function of order λ through the function of proportionality Q on $Y \times Y$ where Q satisfies the property:

$$y \geq y^* \in Y \Rightarrow Q(y,z) \geq Q(y^*,z) \text{ for all } z \in Y. \tag{2.1}$$

Theorem 2.8. *Let X be a reflexive real Banach space and $M \subset X$. Let X^* be the dual of X. Let $\eta : M \times M \to X$ be a vector valued function. Let $T : M \to X^*$ be a nonlinear map. Let $F : M \to \mathbb{R}^n$ be a η-invex function where $\nabla F(u)$ is the differential of F at $u \in K$. Let M be a η-invex cone. Let $\langle T(u), \eta(x,u)\rangle \geq 0$ for all $x, u \in M$. Then for all $x, u \in M$ and $\lambda > 0$, the following statements are equivalent:*

(a) $Q\left(F(x) - F(u), \langle T(u), \eta(x,u)\rangle\right) \geq \lambda$,
(b) $Q\left(\langle \nabla F(u), \eta(x,u)\rangle, \langle T(u), \eta(x,u)\rangle\right) \geq \lambda$.

Proof. Let for some $\lambda > 0$,

$$Q(F(x) - F(u), \langle T(u), \eta(x,u)\rangle) \geq \lambda \tag{2.2}$$

for all $x, u \in M$. Since the set M is a η-invex cone, $u + t\eta(x,u) \in M$ and

$$\eta(u + t\eta(x,u), u) = t\eta(x,u)$$

for all $x, u \in M$, and $t \in (0,1)$. Replacing x by $u + t\eta(x,u)$ in (2.2) and using the property of η-invex cone, we get

$$Q(F(u + t\eta(x,u)) - F(u), \langle T(u), \eta(u + t\eta(x,u), u)\rangle) \geq \lambda,$$

i.e.,

$$Q(F(u + t\eta(x,u)) - F(u), \langle T(u), t\eta(x,u)\rangle) \geq \lambda.$$

Thus

$$Q(F(u + t\eta(x,u)) - F(u), t\langle T(u), \eta(x,u)\rangle) \geq \lambda,$$

i.e.,

$$Q\left(\frac{F(u + t\eta(x,u)) - F(u)}{t}, \langle T(u), \eta(x,u)\rangle\right) \geq \lambda.$$

Taking limit as $t \to 0$ in the above inequality, we get

$$Q\left(\langle \nabla F(u), \eta(x,u)\rangle, \langle T(u), \eta(x,u)\rangle\right) \geq \lambda$$

for all $x, u \in M$.

Conversely, let

$$Q\left(\langle \nabla F(u), \eta(x,u)\rangle, \langle T(u), \eta(x,u)\rangle\right) \geq \lambda$$

for all $x, u \in M$. Given F is η-invex on M, i.e.,

$$F(x) - F(u) \geq \langle \nabla F(u), \eta(x,u)\rangle$$

for all $x, u \in M$. Thus by (2.1), we get

$$\begin{aligned} Q\left(F(x) - F(u), \langle T(u), \eta(x,u)\rangle\right) &\geq Q\left(\langle \nabla F(u), \eta(x,u)\rangle, \langle T(u), \eta(x,u)\rangle\right) \\ &\geq \lambda \end{aligned}$$

for all $x, u \in M$. This completes the proof of the theorem. □

For any $\lambda > 0$, the problem of finding $u \in M$ such that

$$Q\left(\langle \nabla F(u), \eta(x,u)\rangle, \langle T(u), \eta(x,u)\rangle\right) \geq \lambda \quad \text{for all } x \in M,$$

can be written as: for any $\lambda > 0$, find $u \in M$ such that

$$\langle (\nabla F - \lambda T)(u), \eta(x,u)\rangle \geq 0 \quad \text{for all } x \in M \tag{2.3}$$

which is the generalized differential dominated variational inequality problem of order $\lambda > 0$ and the problem stated in (2.3) coincides with the generalized differential dominated variational inequality problem if $\lambda = 1$ [11].

Theorem 2.9. *Let X be a reflexive real Banach space with its dual X^* and $M \subset X$. Let $T : M \to X^*$ be a nonlinear map. Let $\eta : M \times M \to X$ be a vector valued function. Let M be a η-invex cone. Let $F : M \to \mathbb{R}^n$ be a η-pseudoinvex function on M. Let the following conditions hold:*

(*a*) $\langle T(x), \eta(x,x)\rangle = 0$ *for all* $x \in M$,

(*b*) *for each $x \in M$, there exists a $\lambda > 0$ such that the set*

$$\Gamma(u) = \{u \in M : Q(\langle \nabla F(u), \eta(x,u)\rangle, \langle T(u), \eta(x,u)\rangle) \geq \lambda\}$$

is compact,

(c) *for each fixed* $x \in M$ *and for all* $t \in (0,1)$, *the map* $\langle T(x), \eta(-,x)\rangle : M \to \mathbb{R}^n$ *satisfies the condition*

$$t\langle T(x), \eta(v,x)\rangle + (1-t)\langle T(x), \eta(u,x)\rangle$$
$$-\langle T(x), \eta(v + t\eta(u,v), x))\rangle \geq 0 \quad \text{for all } u, v \in M,$$

(d) $\bigcap\{\Gamma(u) : u \in M\}$ *is contractible.*

Then there exists $x_0 \in \Gamma(x_0)$ *such the* x_0 *solves* $(GVIP)$, $(GDIP)$ *and* (MP), *i.e.,*

(A) $\langle T(x_0), \eta(x, x_0)\rangle \geq 0$ *for all* $x \in M$,
(B) $\langle \nabla F(x_0), \eta(x, x_0)\rangle \geq 0$ *for all* $x \in M$,
(C) $F(x) \geq F(x_0)$ *for all* $x \in M$.

Proof. By (b), for each $x \in M$, there exists a $\lambda > 0$ such that the set

$$\Gamma(u) = \{u \in M : Q(\langle \nabla F(u), \eta(x,u)\rangle, \langle T(u), \eta(x,u)\rangle) \geq \lambda\}$$

is compact, i.e., the set (by (2.3))

$$\Gamma(u) = \{u \in M : \langle(\nabla F - \lambda T)(u), \eta(x,u)\rangle \geq 0\}$$

is compact for each $x \in M$. Let

$$N = \{(x,u) : x \in M \text{ and } u \in \Gamma(u)\} \subset M \times M.$$

To prove $\Gamma(u)$ is nonempty, we show N is nonempty. By Theorem 2.8 and (b),

$$\begin{aligned}\Gamma(u) &= \{u \in M : Q(F(x) - F(u), \langle T(u), \eta(x,u)\rangle) \geq \lambda\} \\ &= \{u \in M : F(x) - F(u) - \lambda\langle T(u), \eta(x,u)\rangle \geq 0\}\end{aligned}$$

is compact for each $x \in M$. By condition (a), we have

$$\langle T(x), \eta(x,x)\rangle = 0$$

for all $x \in M$. Therefore, at $x = u$, $\langle T(u), \eta(u,u)\rangle = 0$, that is,

$$F(u) - F(u) - \langle T(u), \eta(u,u)\rangle = 0.$$

Thus, by condition (b), N is nonempty.

Since $\bigcap\{\Gamma(u) : u \in M\}$ is contractible, the set $\bigcap\{\Gamma(u) : u \in M\}$ is homotopically equivalent to a point, say x_0, then $x_0 \in \Gamma(x_0)$. Thus we have

$$Q(F(x) - F(x_0), \langle T(x_0), \eta(x,x_0)\rangle) \geq \lambda$$

for all $x \in M$, that is,

$$\langle \nabla F(x_0), \eta(x,x_0)\rangle - \lambda\langle T(x_0), \eta(x,x_0)\rangle \geq 0 \quad \text{for all } x \in M.$$

Now we show, x_0 solves $(GVIP)$, that is, to show

$$\langle T(x_0), \eta(x,x_0)\rangle \geq 0 \quad \text{for all } x \in M.$$

We have M is a η-invex cone, which means M is a η-invex set and for all $x, u \in M$ and for all $t \in (0, 1)$, we have

$$\eta(u + t\eta(x, u), u) = t\eta(x, u) \in M.$$

By (a), we get

$$\langle T(x), \eta(x, x)\rangle = 0 \quad \text{for all } x \in M. \tag{2.4}$$

Since M is a η-invex set, so for all $x, u \in M$ and for all $t \in (0,\ 1)$, we have $u + t\eta(x, u) \in M$. Therefore, for $x_0 \in \Gamma(x_0) \subset M$ and for all $x \in M$, let $x_t = x_0 + t\eta(x, x_0)$. Replacing x by x_t in (2.4) and using (c), we have

$$\begin{aligned} 0 &= \langle T(x_t), \eta(x_t, x_t)\rangle \\ &= \langle T(x_t), \eta(x_0 + t\eta(x, x_0), x_t)\rangle \\ &\leq t\langle T(x_t), \eta(x_0, x_t)\rangle + (1 - t)\langle T(x_t), \eta(x, x_t)\rangle \end{aligned}$$

for all $x \in M$ (by (c)). Taking limit as $t \to 0^+$ in the above inequality and using (2.4), we get

$$\langle T(x_0), \eta(x, x_0)\rangle \geq 0 \quad \text{for all } x \in M,$$

that is, x_0 solves $(GVIP)$. Finally for any $\lambda > 0$, we have

$$\langle(\nabla F - \lambda T)(x_0), \eta(x, x_0)\rangle \geq 0,$$

i.e.,

$$\langle\nabla F(x_0), \eta(x, x_0)\rangle \geq \lambda\langle T(x_0), \eta(x, x_0)\rangle \geq 0$$

for all $x \in M$. Thus $x_0 \in \Gamma(x_0)$ solves $(GDIP)$. Since F is η-pseudoinvex on M, we get

$$F(x) - F(x_0) \geq 0 \quad \text{for all } x \in M,$$

i.e.,

$$F(x) \geq F(x_0)$$

for all $x \in M$. Thus $x_0 \in \Gamma(x_0)$ solves $(MPVIC)$. This completes the proof of the theorem. □

3. Iterative method for T-η-invex function of order λ

In [11], Das has introduced a variable step iterative method to find the solution for the T-η-invex function. In this section, the result is extended. For general purpose, we have obtained the numerical solution for the T-η-invex function of order λ using the variable step iterative method with some additional conditions.

Let V be a Hilbert space with the inner product $\langle\ ,\ \rangle$ satisfies the Euclidean norm $\|\ .\ \|$ by the rule $\|v\| = \sqrt{\langle v, v\rangle}$ and M be a nonempty subset of V. Let V^* be the dual of V. Let $\eta : M \times M \to V$ be a vector valued function. Let

$T : M \to V^*$ be a nonlinear map. Let $F : M \to \mathbb{R}$ is a differentiable function where $\nabla F(u)$ is the differential of F at $u \in M$.

Definition 3.1. ([3]) Let $F : M \to \mathbb{R}$ be a function. Then,

(a) F is T-η-invex on M if

$$F(x) - F(u) - \langle T(u), \eta(x, u)\rangle \geq 0 \quad \forall x, u \in M,$$

(b) F is T-η-invex at point $u \in M$ if

$$F(x) - F(u) - \langle T(u), \eta(x, u)\rangle \geq 0 \quad \forall x \in M.$$

For our aims, we define the generalized Lipschitz continuous function as follows.

Definition 3.2. The mapping $F : V \to Y$ is η-Lipschitz continuous of rank $L > 0$ if there exists a vector function $\eta : V \times V \to X$ such that

$$\|F(x) - F(u)\|_Y \leq L\|\eta(x, u)\|_V \quad \forall\ x, u \in V.$$

Remark 3.3. If $\eta(x, u) = x - u$, then the definition of η-Lipschitz continuity coincides with the definition of Lipschitz continuity [7].

To study the iterative process of T-η-invex function of order λ, it can be considered as a generalized vector F-λ-variational inequality problem (in short $(GVVIP_F; \lambda)$ given by:

$(GVVIP_F; \lambda)$: for any given $\lambda > 0$, find $u \in M$ such that

$$F(v) - F(u) - \lambda\langle T(u), \eta(v, u)\rangle \geq 0 \quad \forall v \in M. \tag{3.1}$$

To analyze the convergence of the iterative process, we need the following assertion.

Lemma 3.4. ([20], pp. 93) *Let $\{a_k\}_{k=0}^{\infty}$ be a numerical sequence such that $a_{k+1} \leq a_k + \delta_k$ where $\delta_k \geq 0$, $k = 0, 1, 2, \cdots$, and $\sum\limits_{k=0}^{\infty} \delta_k < \infty$. Then there exists a limit $\lim\limits_{k\to\infty} a_k < \infty$. In addition, the sequence $\{a_k\}_{k=0}^{\infty}$ is bounded below then the limit is finite.*

Suppose that the following properties are satisfied for the problem given in (3.1):

(P$_1$) T satisfies the following properties:

(a) for all $u, v \in M$, and $t \in (0, 1)$,

$$\langle T(tu), \eta(u, u)\rangle - \langle T(tv), \eta(v, v)\rangle = \langle T(v + t\eta(v, u)), \eta(v, u)\rangle,$$

(b) for all $u, v \in M$,

$$\|T(v + t\eta(v, u))\| \leq \|T(u + t\eta(v, u))\|,$$

(c) for any $z \in M - \{u, v\}$, there exists a $\lambda > 0$ such that

$$\lambda\langle T(u), \eta(v, u)\rangle \quad \leq \lambda\langle T(u), \eta(v, z)\rangle + \lambda\langle T(u), \eta(u, z)\rangle$$

for all $u, v \in M$.

($\mathbf{P}_2$) η is a absolutely bounded function satisfying the condition

$$\frac{2}{\gamma}c(\epsilon) \leq \|\eta(v, u)\| < 1 \quad \forall u, v \in M, \tag{3.2}$$

where for each $\epsilon > 0$, $c(\epsilon)$ is a continuous function.

($\mathbf{P}_3$) T is Lipschitz continuous with constant $L > 0$, i.e., satisfies the condition

$$\|T(x) - T(u)\| \leq L\|x - u\| \quad \forall x, u \in M. \tag{3.3}$$

Furthermore, we assume that $F : V \to \mathbb{R}$ is a convex(not necessarily differentiable) functional such that for each $\epsilon > 0$, there exists a functional F_ϵ satisfying the conditions:

$$|F_\epsilon(x) - F(x)| \leq c(\epsilon) \quad \forall x \in V, \tag{3.4}$$

$$|F_\epsilon(x) - F_\epsilon(u)| \leq \gamma\|\eta(x, u)\| \quad \forall x, u \in V, \tag{3.5}$$

where $\gamma > 0$(constant). Using (3.2), (3.4) and (3.5), we get

$$\begin{aligned} |F(x) - F(u)| \quad &= |F(x) - F_\epsilon(x) + F_\epsilon(x) - F_\epsilon(u) + F_\epsilon(u) - F(u)| \\ &\leq |F(x) - F_\epsilon(x)| + |F_\epsilon(x) - F_\epsilon(u)| + |F_\epsilon(u) - F(u)| \\ &\leq c(\epsilon) + \gamma\|\eta(x, u)\| + c(\epsilon) \\ &= 2c(\epsilon) + \gamma\|\eta(x, u)\| \\ &\leq 2\gamma\|\eta(x, u)\| \quad (\text{since } \frac{2}{\gamma}c(\epsilon) \leq \|\eta(v, u)\|), \end{aligned}$$

that is, F is η-Lipschitz continuous with constant 2γ.

Under the assumptions imposed on M, T and F, there exists a solution $u \in M$ of the problem ($GVVIP_F$; λ) given in (3.1).

Introduce a functional $\Gamma : V \to \mathbb{R}$ by the relation

$$\Gamma(u) = F(u) - A_\eta(u),$$

where

$$A_\eta(u) = \int_0^1 \langle T(tu), \eta(u, u)\rangle \, dt \quad \text{for all } u \in M.$$

By property $P_1(a)$, we have

$$\begin{aligned} A_\eta(u) - A_\eta(v) &= \int_0^1 \langle T(tu), \eta(u,u)\rangle dt - \int_0^1 \langle T(tv), \eta(v,v)\rangle \, dt \\ &= \int_0^1 \{\langle T(tu), \eta(u,u)\rangle - \langle T(tv), \eta(v,v)\rangle\} \, dt \\ &= \int_0^1 \langle T(v + t\eta(v,u)), \eta(v,u)\rangle \, dt. \end{aligned} \tag{3.6}$$

Hence,

$$\begin{aligned} \Gamma(u) - \Gamma(v) &= F(u) - A_\eta(u) - F(v) + A_\eta(v), \\ &= F(u) - F(v) - (A_\eta(u) - A_\eta(v)), \\ &= F(u) - F(v) - \int_0^1 \langle T(v + t\eta(v,u)), \eta(v,u)\rangle \, dt. \end{aligned} \tag{3.7}$$

To solve the problem $(GVVIP_F;\ \lambda)$, we consider the following iterative process.

Let u_0 be an arbitrary element of M. For $n = 0, 1, 2, \cdots$, we define $u_{n+1} \in M$ as the solution of the variational inequality problem

$$\langle \eta(u_n, u_{n+1}), \eta(v, u_{n+1})\rangle + \rho_n \left(F_{\epsilon_n}(u_{n+1}) - F_{\epsilon_n}(v)\right) \le 0, \quad \forall v \in M, \tag{3.8}$$

where the sequence $\{\rho_n\}_{n=0}^{\infty}$ of the iteration parameters satisfies the conditions

$$0 \le \rho_* \le \rho_n \le \rho^* \le \frac{2}{L}. \tag{3.9}$$

For the sequence $\{\epsilon_n\}_{n=1}^{\infty}$, we assume that

$$\sum_{n=1}^{\infty} c(\epsilon_n) = \sigma < \infty. \tag{3.10}$$

Theorem 3.5. *Let $M \subset V$ be a weakly η-invex set of V. Let the condition given in* (3.9) *be satisfied. Then, the iterative sequence $\{u_n\}_{n=0}^{\infty}$ given by* (3.8) *is bounded in V, and all its weak limit points are solutions of the problem $(GVVIP_F;\ \lambda)$.*

Proof. Let

$$S(u_0) = \{u \in M : \Gamma(u) \le \Gamma(u_0) + 2\sigma\} \subset M.$$

Then, $S(u_0)$ is nonempty because by definition of $S(u_0)$, we have $u_0 \in S(u_0)$ and it is bounded. Next, we show that the iterative sequence defined by

$$\{u_n\}_{n=0}^{\infty} \subset S(u_0)$$

is bounded, i.e, to show if $u_n \in S(u_0)$ then

$$u_{n+1} \in S(u_0).$$

Assume that

$$\|T(u_n)\| \geq \frac{1}{\rho_n} \text{ for all } n.$$

Taking $v = u_n$ in (3.8), we obtain

$$\begin{aligned}
&\langle \eta(u_n, u_{n+1}), \eta(u_n, u_{n+1}) \rangle + \rho_n \left(F_{\epsilon_n}(u_{n+1}) - F_{\epsilon_n}(u_n)\right) \leq 0 \\
&\Rightarrow \|\eta(u_n, u_{n+1})\|^2 + \rho_n \left(F_{\epsilon_n}(u_{n+1}) - F_{\epsilon_n}(u_n)\right) \leq 0 \\
&\Rightarrow -\|\eta(u_n, u_{n+1})\|^2 - \rho_n \left(F_{\epsilon_n}(u_{n+1}) - F_{\epsilon_n}(u_n)\right) \geq 0 \\
&\Rightarrow -F_{\epsilon_n}(u_{n+1}) + F_{\epsilon_n}(u_n) - \frac{1}{\rho_n}\|\eta(u_n, u_{n+1})\|^2 \geq 0 \\
&\Rightarrow F_{\epsilon_n}(u_n) \geq F_{\epsilon_n}(u_{n+1}) + \frac{1}{\rho_n}\|\eta(u_n, u_{n+1})\|^2 \\
&\Rightarrow F_{\epsilon_n}(u_n) \geq F_{\epsilon_n}(u_{n+1}) \\
&\Rightarrow -F_{\epsilon_n}(u_n) \leq -F_{\epsilon_n}(u_{n+1}). \qquad (3.11)
\end{aligned}$$

Again, since M is a weakly η-invex set, for $x, u \in M$, there exists a $t \in (0,1)$ such that $x + t\eta(x,u) \in M$. By property P_3 given in (3.3), we get

$$\|T(x + t\eta(x,u)) - T(x)\| \leq Lt\|\eta(x,u)\| \qquad (3.12)$$

for all $x, u \in M$ and for each $t \in (0,1)$. Replacing x by u_n and u by u_{n+1}, in (3.12), we get

$$\|T(u_n + t\eta(u_n, u_{n+1})) - T(u_n)\| \leq Lt\|\eta(u_n, u_{n+1})\|.$$

Hence using the above equation, we have

$$\begin{aligned}
&\|\langle T(u_n + t\eta(u_n, u_{n+1})) - T(u_n), \eta(u_n, u_{n+1})\rangle\| \\
&\leq \|T(u_n + t\eta(u_n, u_{n+1})) - T(u_n)\| \; \|\eta(u_n, u_{n+1})\| \\
&= Lt\|\eta(u_n, u_{n+1})\|^2, \qquad (3.13)
\end{aligned}$$

for each $t \in (0,1)$. From (3.2), we have $\|\eta(u_n, u_{n+1})\| < 1$, i.e.,

$$-\|\eta(u_n, u_{n+1})\| < -\|\eta(u_n, u_{n+1})\|^2. \qquad (3.14)$$

Therefore substituting $v = u_n$ and $u = u_{n+1}$ in (3.7) and using (3.6) and (3.11), it follows that

$\Gamma(u_{n+1}) - \Gamma(u_n)$

$= F(u_{n+1}) - A_\eta(u_{n+1}) - F(u_n) + A_\eta(u_n)$

$$
\begin{aligned}
&= F(u_{n+1}) - F(u_n) - (A_\eta(u_{n+1}) - A_\eta(u_n)) \\
&= F(u_{n+1}) - F(u_n) - \int_0^1 \langle T(u_n + t\eta(u_n, u_{n+1})), \eta(u_n, u_{n+1})\rangle dt \\
&= [F(u_{n+1}) - F_{\epsilon_n}(u_n)] + [F_{\epsilon_n}(u_n) - F(u_n)] \\
&\qquad - \int_0^1 \langle T(u_n + t\eta(u_n, u_{n+1})), \eta(u_n, u_{n+1})\rangle dt \\
&\leq [F(u_{n+1}) - F_{\epsilon_n}(u_{n+1})] + [F_{\epsilon_n}(u_n) - F(u_n)] \\
&\qquad - \int_0^1 \langle T(u_n + t\eta(u_n, u_{n+1})), \eta(u_n, u_{n+1})\rangle dt \\
&= [F(u_{n+1}) - F_{\epsilon_n}(u_{n+1})] + [F_{\epsilon_n}(u_n) - F(u_n)] \\
&\qquad - \int_0^1 \langle T(u_n + t\eta(u_n, u_{n+1})) - T(u_n), \eta(u_n, u_{n+1})\rangle dt \\
&\qquad - \int_0^1 \langle T(u_n), \eta(u_n, u_{n+1})\rangle dt \\
&\leq |F(u_{n+1}) - F_{\epsilon_n}(u_{n+1})| + |F_{\epsilon_n}(u_n) - F(u_n)| \\
&\qquad + \int_0^1 \|T(u_n + t\eta(u_n, u_{n+1})) - T(u_n)\| \, \|\eta(u_n, u_{n+1})\| dt \\
&\qquad - \int_0^1 \|T(u_n)\| \, \|\eta(u_n, u_{n+1})\| \, dt \\
&\leq 2c(\epsilon_n) + \int_0^1 Lt\|\eta(u_n, u_{n+1})\|^2 \, dt - \int_0^1 \tfrac{1}{\rho_n}\|\eta(u_n, u_{n+1})\| \, dt \\
&\leq 2c(\epsilon_n) + L\|\eta(u_n, u_{n+1})\|^2 \int_0^1 t \, dt - \tfrac{1}{\rho_n}\|\eta(u_n, u_{n+1})\| \int_0^1 dt \qquad \text{(by (3.13))} \\
&= 2c(\epsilon_n) + \tfrac{L}{2}\|\eta(u_n, u_{n+1})\|^2 - \tfrac{1}{\rho_n}\|\eta(u_n, u_{n+1})\| \\
&\leq 2c(\epsilon_n) + \tfrac{L}{2}\|\eta(u_n, u_{n+1})\|^2 - \tfrac{1}{\rho_n}\|\eta(u_n, u_{n+1})\|^2 \qquad \text{(by (3.14))} \\
&= 2c(\epsilon_n) + \left(\tfrac{L}{2} - \tfrac{1}{\rho_n}\right)\|\eta(u_n, u_{n+1})\|^2 \\
&= 2c(\epsilon_n) - \mu\|\eta(u_n, u_{n+1})\|^2,
\end{aligned}
$$

where $\mu = \frac{1}{\rho_n} - \frac{L}{2} \geq 0$. Thus, we obtained the relation

$$\Gamma(u_{n+1}) + \mu\|\eta(u_n, u_{n+1})\|^2 \leq \Gamma(u_n) + 2c(\epsilon_n) \tag{3.15}$$

is valid for all $n = 0, 1, 2, \cdots$. Putting $n = 0, 1, 2, \cdots, N$ in (3.15), we get

$$\Gamma(u_{N+1}) + \mu\sum_{n=0}^{N} \|\eta(u_n, u_{n+1})\|^2 \leq \Gamma(u_0) + 2\sum_{n=0}^{N} c(\epsilon_n) \leq \Gamma(u_0) + 2\sigma,$$

$$\Rightarrow \Gamma(u_{N+1}) + \mu\sum_{n=0}^{N} \|\eta(u_n, u_{n+1})\|^2 \leq \Gamma(u_0) + 2\sigma, \tag{3.16}$$

$$\Rightarrow \Gamma(u_{N+1}) \leq \Gamma(u_0) + 2\sigma - \mu\sum_{n=0}^{N} \|\eta(u_n, u_{n+1})\|^2 \leq \Gamma(u_0) + 2\sigma.$$

Thus, $u_{N+1} \in S(u_0) = \{u \in M : \Gamma(u) \leq \Gamma(u_0) + 2\sigma\}$. Since N is arbitrary, so replacing N by n, we get $u_{n+1} \in S(u_0)$ and hence,

$$\{u_n\}_{n=0}^{\infty} \subset S(u_0).$$

Now, by (3.10), the assumptions of Lemma 3.4 are valid the sequence

$$\{\Gamma(u_n)\}_{n=1}^{\infty}.$$

Next to show, the sequence $\{\Gamma(u_n)\}_{n=1}^{\infty}$ is bounded above and has a finite limit. Since $\|\eta(u, v)\| < 1$, so we have

$$\|\eta(u_n, u_{n+1})\| < 1 \ \Rightarrow \lim_{n\to\infty} \|\eta(u_n, u_{n+1})\| = 0.$$

Hence, the series

$$\sum_{n=0}^{\infty} \|\eta(u_n, u_{n+1})\|^2$$

is convergent. Now, taking limit $N \to \infty$ in (3.16), we get

$$\lim_{N\to\infty} \Gamma(u_{N+1}) + \mu\sum_{n=0}^{\infty} \|\eta(u_n, u_{n+1})\|^2 \leq \Gamma(u_0) + 2\sigma,$$

i.e.,

$$\lim_{n\to\infty} \Gamma(u_n) \leq \Gamma(u_0) + 2\sigma - \mu\sum_{n=0}^{\infty} \|\eta(u_n, u_{n+1})\|^2.$$

Hence, the sequence $\{\Gamma(u_n)\}_{n=1}^{\infty}$ is bounded above. Again, by the property $P_1(c)$, for any $z \in M - \{u, v\}$, there exists a $\lambda > 0$ such that

$$\lambda\langle T(u), \eta(v, u)\rangle \leq \lambda\langle T(u), \eta(v, z)\rangle + \lambda\langle T(u), \eta(u, z)\rangle$$

for all $u, v \in M$. Taking $u = u_n$, $z = u_{n+1}$ in the above inequality, we get

$$
\begin{aligned}
&\lambda\langle T(u_n), \eta(v, u_n)\rangle \\
&\le \lambda\langle T(u_n), \eta(v, u_{n+1})\rangle + \lambda\langle T(u_n), \eta(u_n, u_{n+1})\rangle \\
&\le \lambda\langle T(u_n), \eta(u_n, u_{n+1})\rangle + \lambda\langle T(u_n), \eta(v, u_{n+1})\rangle + F_{\epsilon_n}(v) - F_{\epsilon_n}(u_{n+1}) \\
&\qquad -\frac{1}{\rho_n}\langle \eta(u_n, u_{n+1}), \eta(v, u_{n+1})\rangle \quad \forall\ v \in M \qquad \text{(from (3.8))} \\
&= \lambda\langle T(u_n), \eta(u_n, u_{n+1})\rangle + \lambda\langle T(u_n), \eta(v, u_{n+1})\rangle - \tfrac{1}{\rho_n}\langle \eta(u_n, u_{n+1}), \eta(v, u_{n+1})\rangle \\
&\qquad + [F_{\epsilon_n}(v) - F_{\epsilon_n}(u_n)] + [F_{\epsilon_n}(u_n) - F_{\epsilon_n}(u_{n+1})] \\
&\le \lambda\|T(u_n)\|\ \|\eta(u_n, u_{n+1})\| + \lambda\|T(u_n)\|\ \|\eta(v, u_{n+1})\| \\
&\qquad + \frac{1}{\rho_n}\|\eta(u_n, u_{n+1})\|\ \|\eta(v, u_{n+1})\| + [F_{\epsilon_n}(v) - F_{\epsilon_n}(u_n)] \\
&\qquad + [F_{\epsilon_n}(u_n) - F_{\epsilon_n}(u_{n+1})] \quad \forall\ v \in M \\
&\le (\lambda\|T(u_n)\| + \tfrac{1}{\rho_n}\|\eta(v, u_{n+1})\|)\ \|\eta(u_n, u_{n+1})\| \\
&\qquad + \lambda\|T(u_n)\|\ \|\eta(v, u_{n+1})\| + F(v) - F(u_n) + 2c(\epsilon_n)
\end{aligned}
$$

for all $v \in M$, that is,

$$
\begin{aligned}
\lambda\langle T(u_n), \eta(v, u_n)\rangle \le C_v\|\eta(u_n, u_{n+1})\| + S_v\|T(u_n)\|\ \|\eta(v, u_{n+1})\| \\
+ [F(v) - F(u_n)] + 2c(\epsilon_n)
\end{aligned} \tag{3.17}
$$

for all $v \in M$, where

$$C_v = \lambda\|T(u_n)\| + \frac{1}{\rho_n}\|\eta(v, u_{n+1})\|$$

and

$$S_v = \lambda\|\eta(v, u_{n+1})\|$$

are the nonnegative constants limits to 0 as $n \to \infty$ depending on $v \in M$. Since the iterative sequence is bounded, it has a subsequence which is of finite limit. We claim that, there exists a finite subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of M which converges weakly to u^* in M, i.e.,

$$u_{n_k} \underset{\rightarrow}{w} u^* \quad \text{as} \quad k \to \infty$$

in M and satisfying the inequality

$$\lim_{k\to\infty} \sup\ \langle T(u_{n_k}), \eta(u^*, u_{n_k})\rangle \le 0.$$

Taking $v = u^*$ in (3.17), we have

$$
\begin{aligned}
&\lambda \lim_{k\to\infty} \sup \ \langle T(u_{n_k}), \eta(u^*, u_{n_k})\rangle \\
&\leq \lim_{k\to\infty} \sup \ C_{u^*}\|\eta(u_{n_k}, u_{n_k+1})\| + \lim_{k\to\infty} \sup S_{u^*}\|T(u_{n_k})\| \\
&\qquad + \lim_{k\to\infty} \sup \ [F(u^*) - F(u_{n_k})] + 2 \lim_{k\to\infty} \sup \ c(\epsilon_{n_k}) \\
&\leq F(u^*) - \lim_{k\to\infty} \inf F(u_{n_k}) \\
&\leq F(u^*) - F(u^*) = 0.
\end{aligned}
$$

Next, we show that u^* solves the problem $(GVVIP_F;\lambda)$. From (3.17), we have

$$
\begin{aligned}
&C_v\|\eta(u_n, u_{n+1})\| + S_v\|T(u_n)\| + 2c(\epsilon_n) \\
&\geq \lambda\langle T(u_n), \eta(v, u_n)\rangle - F(v) + F(u_n)
\end{aligned}
$$

giving

$$
\begin{aligned}
&C_v\|\eta(u_{n_k}, u_{n_k+1})\| + S_v\|T(u_{n_k})\| + 2c(\epsilon_{n_k}) \\
&\geq \lambda\langle T(u_{n_k}), \eta(v, u_{n_k})\rangle - F(v) + F(u_{n_k}).
\end{aligned}
$$

Taking limit as $k \to \infty$, we get

$$
\begin{aligned}
&\lim_{k\to\infty} \inf \ [\ C_v\|\eta(u_{n_k}, u_{n_k+1})\| + S_v\|T(u_{n_k})\| + 2c(\epsilon_{n_k})] \\
&\geq \lambda \lim_{k\to\infty} \inf \ [\langle T(u_{n_k}), \eta(v, u_{n_k})\rangle - F(v) + F(u_{n_k})],
\end{aligned}
$$

i.e.,

$$
0 \geq \lambda \lim_{k\to\infty} \inf \ \langle T(u_{n_k}), \eta(v, u_{n_k})\rangle + \lim_{k\to\infty} \inf \ [\ -F(v) + F(u_{n_k})\ ].
$$

Thus

$$
0 \geq \lambda\langle T(u^*), \eta(v, u^*)\rangle - F(v) + F(u^*),
$$

i.e.,

$$
F(v) - F(u^*) - \lambda\langle T(u^*), \eta(v, u^*)\rangle \geq 0
$$

for all $v \in M$. Thus, $u^* \in M$ solves the problem $(GVVIP_F;\lambda)$, that is, F is T-η-invex of order λ at $u^* \in M$. This completes the proof of the theorem. □

4. The Existence Theorem and Example

In this section, we prove an existence theorem of T-η-invex function of order λ followed by a concrete example.

Theorem 4.1. *Let X be a real Banach space and $K \subset X$. Let $\eta : K\times K \to X$ be a vector valued function. Let K be a η-invex set. Let the mapping $F : K \to \mathbb{R}^n$ be η-invex on K, and $T : K \to L(X, \mathbb{R}^n) \equiv \mathbb{R}^n$ be a map. Assume that for $x, u \in K$, there exists at least one $\lambda \in (0, 1)$ such that* *(a)* $\eta(u+\lambda\eta(x,u),u)=\lambda\eta(x,u)$,

(b) *F satisfies*

$$F(u+\lambda\eta(x,u))\leq F(x),$$

(b) *F and T satisfies*

$$\langle\nabla F(u),\eta(x,u)\rangle\geq\langle T(u),\eta(x,u)\rangle.$$

Then F is T-η-invex of order λ on K.

Proof. The mapping $F:K\to\mathbb{R}^n$ is η-invex on K, i.e.,

$$F(x)-F(u)-\langle\nabla F(u),\eta(x,u)\rangle\geq 0$$

for all $x,u\in K$. Since the set K is η-invex, we have

$$u+t\eta(x,u)\in K$$

for all $x,u\in K$, and $t\in(0,1)$. By (c), we have

$$\langle\nabla F(u),\eta(x,u)\rangle\geq\langle T(u),\eta(x,u)\rangle,$$

i.e.,

$$-\langle T(u),\eta(x,u)\rangle\geq-\langle\nabla F(u),\eta(x,u)\rangle$$

for all $x,u\in K$. Therefore,

$$\begin{aligned}F(x)-F(u)-\langle T(u),\eta(x,u)\rangle &\geq F(x)-F(u)-\langle\nabla F(u),\eta(x,u)\rangle\\ &\geq 0\end{aligned}$$

for all $x,u\in K$. Replacing x by $u+t\eta(x,u)$ in the above inequality, we get

$$F(u+t\eta(x,u))-F(u)-\langle T(u),\eta(u+t\eta(x,u),u)\rangle\geq 0$$

for all $x,u\in K$, and $t\in(0,1)$. At $t=\lambda$, we have

$$F(u+\lambda\eta(x,u))-F(u)-\langle T(u),\eta(u+\lambda\eta(x,u),u)\rangle\geq 0$$

for all $x,u\in K$, i.e.,

$$\langle T(u),\eta(u+\lambda\eta(x,u),u)\rangle\leq F(u+\lambda\eta(x,u))-F(u)$$

for all $x,u\in K$. By (a), we get

$$\langle T(u),\lambda\eta(x,u)\rangle\leq F(u+\lambda\eta(x,u))-F(u)$$

for all $x,u\in K$, i.e.,

$$\lambda\langle T(u),\eta(x,u)\rangle\leq F(u+\lambda\eta(x,u))-F(u)$$

for all $x,u\in K$. Again by (b), we have

$$F(u+\lambda\eta(x,u))\leq F(x)$$

for all $x,u\in K$, i.e.,

$$F(u+\lambda\eta(x,u))-F(u)\leq F(x)-F(u)$$

for all $x, u \in K$. Hence

$$\begin{aligned}\lambda\langle T(u), \eta(x,u)\rangle &\leq F(u+\lambda\eta(x,u)) - F(u)\\ &\leq F(x) - F(u)\end{aligned}$$

for all $x, u \in K$, i.e.,

$$F(x) - F(u) - \lambda\langle T(u), \eta(x,u)\rangle \geq 0$$

for all $x, u \in K$. Thus F is T-η-invex of order λ on K. This completes the proof of the theorem. □

Example 4.2. Let $X = \mathbb{R}$. Then the dual of X is $X^* \equiv X$. Let $K \subset \mathbb{R}$. Let $F : K \to \mathbb{R}$, $\eta : K \times K \to X$, and $T : K \to X^*$ be the functions defined by

$$F(x) = e^x, \quad \eta(x,u) = |x-u|$$

and

$$T(x) = x$$

for all $x, u \in K$. Let

$$\langle f, z\rangle = f \cdot z$$

for all $f \in X^*$ and $z \in X$. Here K is a η-invex cone. Let F be η-invex on K, i.e.,

$$\begin{aligned}F(x) - F(u) - \langle \nabla F(u), \eta(x,u)\rangle &= e^x - e^u - e^u|x-u|\\ &\geq 0\end{aligned}$$

for all $x, u \in K$, implying $x \geq u$. Let for each $u \in K$, K_u be the set defined by

$$K_u = \{x \in K : x \geq u\} \neq \varnothing.$$

For each $x \in K_u$, $u \in K$, we have $e^u \geq u$, so

$$\begin{aligned}\langle \nabla F(u), \eta(x,u)\rangle &= e^u|x-u|\\ &\geq u|x-u| = \langle T(u), \eta(x,u)\rangle\end{aligned}$$

for each $x \in K_u$ and for all $u \in K$. Thus

$$\begin{aligned}0 &\leq F(x) - F(u) - \langle \nabla F(u), \eta(x,u)\rangle\\ &= e^x - e^u - e^u(x-u)\\ &\leq e^x - e^u - u(x-u)\\ &= F(x) - F(u) - \langle T(u), \eta(x,u)\rangle\end{aligned}$$

for each $u \in K$ and $x \in K_u$ which means that F is T-η-invex at each $x \in K_u$ for all $u \in K$. Now we have

$$u \leq u + \lambda\eta(x,u) \leq x$$

for at least one $\lambda \in [0, 1]$. Thus

$$\begin{aligned} F(u + \lambda\eta(x, u) - F(x) &= e^{u+\lambda\eta(x,u)} - e^{x} \\ &\leq 0 \end{aligned}$$

for at least one $\lambda \in (0, 1)$. Thus all the conditions of Theorem 4.1 are satisfied. So by Theorem 4.1, F is T-η-invex of order λ at $x \in K_u$ for all $u \in K$.

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