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THE N-ORDER ITERATIVE SCHEME FOR A SYSTEM OF NONLINEAR WAVE EQUATIONS ASSOCIATED WITH THE HELICAL FLOWS OF MAXWELL FLUID

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Abstract. In this paper, we study a system of nonlinear wave equations associated with the helical flows of Maxwell fluid. By constructing a N-order iterative scheme, we prove the local existence and uniqueness of a weak solution. Furthermore, we show that the sequence associated with N-order iterative scheme converges to the unique weak solution at a rate of N-order.

1. Introduction

In this paper, we consider the following initial-boundary value problem for the system of nonlinear wave equations

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$$\begin{cases} u_{tt} - a_1\left(u_{xx} + \frac{1}{x}u_x - \frac{1}{x^2}u\right) = f(x,t,u,v), \\ \qquad\qquad\qquad\qquad\qquad\qquad x \in \Omega = (1,R),\ 0 < t < T, \\ v_{tt} - a_2\left(v_{xx} + \frac{1}{x}v_x\right) = g(x,t,u,v),\ x \in \Omega,\ 0 < t < T, \\ u_x(1,t) - b_1 u(1,t) = v_x(1,t) = u(R,t) = v(R,t) = 0, \\ (u(x,0), v(x,0)) = (\tilde{u}_0(x), \tilde{v}_0(x)), \\ (u_t(x,0), v_t(x,0)) = (\tilde{u}_1(x), \tilde{v}_1(x)), \end{cases} \tag{1.1}$$

where $a_1 > 0$, $a_2 > 0$, $b_1 > 0$, $R > 1$ are given constants and $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{v}_0$, $\tilde{v}_1$, f, g are given functions.

Problem (1.1) here is studied in literature for Maxwell fluid between two infinite coaxial circular cylinders. It is well known that there is a great interest of theoretical and applied scientists relating to the fluid flows in the neighborhood of translating or oscillating bodies, in which, Maxwell fluid has received special attention, see for [3]-[6], [13], [19], [21]-[24] and the references therein. In [5], Jamil and Fetecau studied the following problem:

$$\begin{cases} \lambda u_{tt} + u_t = \nu\left(u_{xx} + \frac{1}{x}u_x - \frac{1}{x^2}u\right),\ 1 < x < R,\ t > 0, \\ \lambda V_{tt} + V_t = \nu\left(V_{xx} + \frac{1}{x}V_x\right),\ 1 < x < R,\ t > 0, \\ u_x(1,t) - u(1,t) = \frac{F}{\mu}t,\ \ V_x(1,t) = \frac{G}{\mu}t,\ \ t > 0, \\ u(R,t) = V(R,t) = 0,\ t > 0, \\ u(x,0) = u_t(x,0) = 0,\ 1 < x < R, \\ V(x,0) = V_t(x,0) = 0,\ 1 < x < R, \end{cases} \tag{1.2}$$

where λ, μ, ν, F, G are the given constants, this is a mathematical model describing the helical flows of Maxwell fluid in the annular region between two infinite coaxial circular cylinders of radii 1 and $R > 1$. The authors have obtained an exact solution for the problem (1.2) by means of finite Hankel transforms and presented under series form in terms of Bessel functions $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$, $J_2(x)$ and $Y_2(x)$, satisfying all imposed initial and boundary conditions. Extending the results of Jamil and Fetecau [5], in [24], Truong et al. have established the global existence, uniqueness, regularity and decay of solutions of Problem (1.1), where

$$\begin{aligned} f = &\ \ f(x,t,u,v,u_x,v_x,u_t,v_t) = -\lambda_1 u_t - f_1(u,v) + F_1(x,t), \\ g = &\ \ g(x,t,u,v,u_x,v_x,u_t,v_t) = -\lambda_2 v_t - f_2(u,v) + F_2(x,t), \end{aligned} \tag{1.3}$$

and $f_1(u,v)$, $f_2(u,v)$ have been assumed that $(f_1,\ f_2) = \left(\frac{\partial \mathcal{F}}{\partial u}, \frac{\partial \mathcal{F}}{\partial v}\right)$ with $\mathcal{F}(u,v) \le C_1\left(1+u^2+v^2\right)$, for all $u,v \in \mathbb{R}$, $C_1>0$. This paper is inspired by the results of [24], we continue to extend the results of [5] to obtain a weak solution (u,v) of Problem (1.1) in the sense as in Remark 2.2 below. The main tools used here are the Galerkin method associated with a priori estimates, the weak convergence and the compactness techniques. Furthermore, in case of $f \in C^N([0,1]\times\mathbb{R}_+\times\mathbb{R}^2)$, under suitable assumptions we construct a $N-$order iterative scheme to have a convergent sequence at a rate of order N to a local weak solution of Problem (1.1). This scheme is established based on a high-order method for solving the operator equation $F(x)=0$, it also has been applied in some works, for example see [11], [15]-[18], [25] and the references therein.

2. Preliminaries

The notation we use in this paper is standard and can be found in [1] or Lions's book [8], with $\Omega=(1,R)$, $Q_T=\Omega\times(0,T)$, $T>0$ and $\|\cdot\|$ is the norm in L^2.

On $H^1 \equiv H^1(\Omega)$, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2+\|v_x\|^2\right)^{1/2}. \tag{2.1}$$

We put

$$V_R=\{v\in H^1 : v(R)=0\}. \tag{2.2}$$

V_R is a closed subspace of H^1 and on V_R two norms $\|v\|_{H^1}$ and $\|v_x\|$ are equivalent norms.

Note that L^2, H^1 are also the Hilbert spaces with respect to the corresponding scalar products:

$$\langle u,v\rangle = \int_1^R xu(x)v(x)dx,\ \langle u,v\rangle + \langle u_x,v_x\rangle. \tag{2.3}$$

The norms in L^2 and H^1 induced by the corresponding scalar products (2.3) are denoted by $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. H^1 is continuously and densely embedded in L^2. Identifying L^2 with $(L^2)'$ (the dual of L^2), we have $H^1 \hookrightarrow L^2 \hookrightarrow (H^1)'$; on the other hand, the notation $\langle\cdot,\cdot\rangle$ is used for the pairing between H^1 and $(H^1)'$.

We then have the following lemmas, the proofs of which can be found in the paper [24].

Lemma 2.1. *We have the following inequalities*

$$\begin{array}{ll} \text{(i)} & \|v\| \leq \|v\|_0 \leq \sqrt{R}\,\|v\|\,, \text{ for all } v \in L^2, \\ \text{(ii)} & \|v\|_{H^1} \leq \|v\|_1 \leq \sqrt{R}\,\|v\|_{H^1}\,, \text{ for all } v \in H^1. \end{array} \tag{2.4}$$

Lemma 2.2. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \leq \alpha_0 \|v\|_{H^1} \text{ for all } v \in H^1, \tag{2.5}$$

where $\alpha_0 = \frac{1}{\sqrt{2(R-1)}}\sqrt{1+\sqrt{1+16(R-1)^2}}$.

Lemma 2.3. *The imbedding $V_R \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\begin{array}{ll} \text{(i)} & \|v\|_{C^0(\overline{\Omega})} \leq \sqrt{R-1}\,\|v_x\| \leq \sqrt{R-1}\,\|v_x\|_0 \text{ for all } v \in V_R, \\ \text{(ii)} & \|v\|_0 \leq \sqrt{\frac{R+1}{2}}(R-1)\,\|v_x\|_0 \text{ for all } v \in V_R, \\ \text{(iii)} & \int_1^R x\,|v(x)|^\gamma\,dx \leq \frac{R^2-1}{2}\left(\sqrt{R-1}\right)^\gamma \|v_x\|_0^\gamma \text{ for all } v \in V_R,\ \forall \gamma > 0. \end{array} \tag{2.6}$$

Put

$$\begin{cases} a(u,w) = a_1\left[\langle u_x, w_x\rangle + b_1 u(1)w(1) + \langle \frac{1}{x^2}u, w\rangle\right], \\ b(v,\phi) = a_2\langle v_x, \phi_x\rangle, \text{ for all } u,\ v,\ w,\ \phi \in V_R, \end{cases} \tag{2.7}$$

and

$$\begin{cases} \|v\|_a = \sqrt{a(v,v)} = \sqrt{a_1}\left[\|v_x\|_0^2 + b_1 v^2(1) + \left\|\frac{1}{x}v\right\|_0^2\right]^{1/2}, \\ \|v\|_b = \sqrt{b(v,v)} = \sqrt{a_2}\,\|v_x\|_0\,,\ v \in V_R, \end{cases} \tag{2.8}$$

with $a_1 > 0$, $a_2 > 0$, $b_1 > 0$ are given constants. Then, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are the symmetric bilinear forms on $V_R \times V_R$.

We also have the following lemmas.

Lemma 2.4. *For* $a_1^* = \left[1 + \left(b_1 + \dfrac{R^2-1}{2}\right)(R-1)\right]^{1/2}$ *and* $\bar{a}_1^* = \left[1 + \dfrac{R+1}{2}(R-1)^2\right]^{1/2}$, *we have*

$$\begin{cases} \text{(i)} & \sqrt{a_1}\,\|v_x\|_0 \leq \|v\|_a \leq a_1^*\,\|v_x\|_0\,, \text{ for all } v \in V_R, \\ \text{(ii)} & \|v_x\|_0 \leq \|v\|_1 \leq \bar{a}_1^*\,\|v_x\|_0\,, \text{ for all } v \in V_R. \end{cases} \tag{2.9}$$

Remark 2.5. On L^2, two norms $v \longmapsto \|v\|$ and $v \longmapsto \|v\|_0$ are equivalent. So are two norms $v \longmapsto \|v\|_{H^1}$ and $v \longmapsto \|v\|_1$ on H^1, and five norms $v \longmapsto \|v\|_{H^1}$, $v \longmapsto \|v\|_1$, $v \longmapsto \|v_x\|$, $v \longmapsto \|v_x\|_0$ and $v \longmapsto \|v\|_a$ on V_R.

Lemma 2.6. *There exists the Hilbert orthonormal base $\{w_j\}$ of L^2 consisting of the eigenfunctions w_j corresponding to the eigenvalue $\bar{\lambda}_j$ such that*

$$\begin{cases} 0 < \bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \cdots \leq \bar{\lambda}_j \leq \bar{\lambda}_{j+1} \leq \cdots, \ \lim_{j\to+\infty} \bar{\lambda}_j = +\infty, \\ a(w_j, w) = \bar{\lambda}_j \langle w_j, w\rangle \ \textit{for all} \ w \in V_R, \ j = 1, 2, \cdots. \end{cases} \tag{2.10}$$

Furthermore, the sequence $\{w_j/\sqrt{\bar{\lambda}_j}\}$ is also the Hilbert orthonormal base of V_R with respect to the scalar product $a(\cdot,\cdot)$.

On the other hand, we also have w_j satisfying the following boundary value problem

$$\begin{cases} L_1 w_j \equiv -a_1(w_{jxx} + \frac{1}{x} w_{jx} - \frac{1}{x^2} w_j) = \bar{\lambda}_j w_j, \ \textit{in} \ (1, R), \\ w_{jx}(1) - b_1 w_j(1) = w_j(R) = 0, \ w_j \in C^\infty([1, R]). \end{cases} \tag{2.11}$$

The proof of Lemma 2.6 can be found in [[20], p.87, Theorem 7.7], with $H = L^2$ and $V = V_R$, and $a(\cdot,\cdot)$ is defined as in (2.7).

Similarly, we also obtain the following lemma.

Lemma 2.7. *There exists the Hilbert orthonormal base $\{\phi_j\}$ of L^2 consisting of the eigenfunctions ϕ_j corresponding to the eigenvalue $\bar{\mu}_j$ such that*

$$\begin{cases} 0 < \bar{\mu}_1 \leq \bar{\mu}_2 \leq \cdots \leq \bar{\mu}_j \leq \bar{\mu}_{j+1} \leq \cdots, \ \lim_{j\to+\infty} \bar{\mu}_j = +\infty, \\ b(\phi_j, \phi) = \bar{\mu}_j \langle \phi_j, \phi\rangle \ \textit{for all} \ \phi \in V_R, \ j = 1, 2, \cdots. \end{cases} \tag{2.12}$$

Furthermore, the sequence $\{\phi_j/\sqrt{\bar{\mu}_j}\}$ is also the Hilbert orthonormal base of V_R with respect to the scalar product $b(\cdot,\cdot)$.

On the other hand, we also have ϕ_j satisfying the following boundary value problem

$$\begin{cases} L_2 \phi_j \equiv -a_2(\phi_{jxx} + \frac{1}{x} \phi_{jx}) = \bar{\mu}_j \phi_j, \ \textit{in} \ (1, R), \\ \phi_{jx}(1) = \phi_j(R) = 0, \ \phi_j \in C^\infty([1, R]). \end{cases} \tag{2.13}$$

Remark 2.8. The weak formulation of the initial-boundary value problem (1.1) can be given in the following manner: Find $(u, v) \in \bar{W}_T = \{(u, v) \in L^\infty\big(0, T; (H^2 \cap V_R)^2\big) : (u', v') \in L^\infty\Big(0, T; (V_R)^2\Big), (u'', v'') \in L^\infty\Big(0, T; \big(L^2\big)^2\Big)\}$, such that (u, v) satisfies the following variational equation

$$\begin{cases} \langle u''(t), w\rangle + a(u(t), w) = \langle f[u, v](t), w\rangle, \\ \langle v''(t), \phi\rangle + b(v(t), \phi) = \langle g[u, v](t), \phi\rangle, \end{cases} \tag{2.14}$$

for all $(w, \phi) \in V_R \times V_R$, a.e., $t \in (0, T)$, together with the initial conditions

$$\big(u(0), u'(0)\big) = (\tilde{u}_0, \tilde{u}_1), \ \ \big(v(0), v'(0)\big) = (\tilde{v}_0, \tilde{v}_1), \tag{2.15}$$

where we use the notations $f[u, v](x, t) = f(x, t, u, v)$, $g[u, v](x, t) = g(x, t, u, v)$.

Remark 2.9. We note that (see [8])

$$\bar{W}_T = \Big\{(u,v) \in L^\infty\left(0,T;(H^2\cap V_R)^2\right) \cap C([0,T];V_R\times V_R) \cap C^1([0,T];L^2\times L^2) : (u',v') \in L^\infty\left(0,T;(V_R)^2\right) \cap C([0,T];L^2\times L^2), (u'',v'') \in L^\infty\left(0,T;\left(L^2\right)^2\right)\Big\}.$$

3. The N-order iterative schemes

In this section, we consider Problem (1.1) with a_1, a_2, b_1 are positive constants and give the following assumptions:

(A_1) $(\tilde{u}_0,\tilde{u}_1)$, $(\tilde{v}_0,\tilde{v}_1) \in (V_R\cap H^2)\times V_R$, $\tilde{u}_{0x}(1) - b_1\tilde{u}_0(1) = \tilde{v}_{0x}(1) = 0$;
(A_2) f, $g \in C^N([0,1]\times\mathbb{R}_+\times\mathbb{R}^2)$, $f(R,t,0,0) = g(R,t,0,0) = 0$, $\forall t \geq 0$.

Consider $T^* > 0$ fixed, let $T \in (0,T^*]$, we define

$$W_T = \Big\{(u,v) \in L^\infty\left(0,T;(H^2\cap V_R)^2\right) : (u',v') \in L^\infty(0,T;V_R\times V_R), (u'',v'') \in L^2(0,T;L^2\times L^2)\Big\}, \tag{3.1}$$

then W_T is the Banach space with norm

$$\|(u,v)\|_{W_T} = \max\Big\{\|(u,v)\|_{L^\infty(0,T;(H^2\cap V_R)^2)}, \left\|(u',v')\right\|_{L^\infty(0,T;(V_R)^2)}, \left\|(u'',v'')\right\|_{L^2(0,T;(L^2)^2)}\Big\}. \tag{3.2}$$

For $M > 0$, we put

$$\begin{aligned} W(M,T) &= \Big\{v \in W_T : \|v\|_{W_T} \leq M\Big\}, \\ W_1(M,T) &= \left\{(u,v) \in W(M,T) : (u'',v'') \in L^\infty(0,T;(L^2)^2)\right\}. \end{aligned} \tag{3.3}$$

Now, we construct the recurrent sequence $\{(u_m,v_m)\}$ defined by $(u_0,v_0) = (0,0)$, and suppose that

$$(u_{m-1},v_{m-1}) \in W_1(M,T), \tag{3.4}$$

and associate with Problem (2.13), (2.15) the following problem:

Find $(u_m,v_m) \in W_1(M,T)$ $(m \geq 1)$ which satisfies the following linear variational problem:

$$\begin{cases} \langle u_m''(t),w\rangle + a(u_m(t),w) = \langle F_m(t),w\rangle, \\ \langle v_m''(t),\phi\rangle + b(v_m(t),\phi) = \langle G_m(t),\phi\rangle, \ \ \forall (w,\phi) \in V_R\times V_R, \\ (u_m(0),u_m'(0)) = (\tilde{u}_0,\tilde{u}_1), \ (v_m(0),v_m'(0)) = (\tilde{v}_0,\tilde{v}_1), \end{cases} \tag{3.5}$$

where

$$\begin{cases} F_m(x,t) = \sum\limits_{|\alpha|\le N-1} \dfrac{1}{\alpha!} D^\alpha f[u_{m-1}, v_{m-1}](x,t)(u_m - u_{m-1})^{\alpha_1}(v_m - v_{m-1})^{\alpha_2}, \\ G_m(x,t) = \sum\limits_{|\alpha|\le N-1} \dfrac{1}{\alpha!} D^\alpha g[u_{m-1}, v_{m-1}](x,t)(u_m - u_{m-1})^{\alpha_1}(v_m - v_{m-1})^{\alpha_2}, \\ D^\alpha f = D_1^{\alpha_1} D_2^{\alpha_2} f = \dfrac{\partial^{\alpha_1+\alpha_2} f}{\partial u^{\alpha_1} \partial v^{\alpha_2}}, \\ \alpha! = \alpha_1!\alpha_2!,\ |\alpha| = \alpha_1 + \alpha_2,\ \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2. \end{cases} \tag{3.6}$$

Then, we have the following theorem.

Theorem 3.1. *Let* (A_1)-(A_2) *hold. Then there exist positive constants* M, $T > 0$ *such that, for* $(u_0, v_0) = (\tilde{u}_0, \tilde{v}_0)$, *there exists a recurrent sequence* $\{(u_m, v_m)\} \subset W_1(M,T)$ *defined by* (3.5), (3.6).

Proof. The proof consists of several steps.

Step 1. The Faedo-Galerkin approximation (introduced by Lions [8]).

Put

$$u_m^{(k)}(t) = \sum\nolimits_{j=1}^k c_{mj}^{(k)}(t) w_j,\ v_m^{(k)}(t) = \sum\nolimits_{j=1}^k d_{mj}^{(k)}(t) \phi_j, \tag{3.7}$$

where the coefficients $c_{mj}^{(k)}(t)$, $d_{mj}^{(k)}(t)$ satisfy the system of nonlinear differential equations:

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j \rangle + a(u_m^{(k)}(t), w_j) = \langle F_m^{(k)}(t), w_j \rangle, \\ \langle \ddot{v}_m^{(k)}(t), \phi_j \rangle + b(v_m^{(k)}(t), \phi_j) = \langle G_m^{(k)}(t), \phi_j \rangle,\ 1 \le j \le k, \\ (u_m^{(k)}(0), \dot{u}_m^{(k)}(0)) = (\tilde{u}_{0k}, \tilde{u}_{1k}),\ (v_m^{(k)}(0), \dot{v}_m^{(k)}(0)) = (\tilde{v}_{0k}, \tilde{v}_{1k}), \end{cases} \tag{3.8}$$

where

$$\begin{cases} (\bar{u}_{0k}, \bar{u}_{1k}) = \sum\limits_{j=1}^k (\alpha_j^{(k)}, \beta_j^{(k)}) w_j \to (\tilde{u}_0, \tilde{u}_1) \text{ strongly in } \left(H^2 \cap V_R\right) \times V_R, \\ (\bar{v}_{0k}, \bar{v}_{1k}) = \sum\limits_{j=1}^k (\tilde{\alpha}_j^{(k)}, \tilde{\beta}_j^{(k)}) \phi_j \to (\tilde{v}_0, \tilde{v}_1) \text{ strongly in } \left(H^2 \cap V_R\right) \times V_R, \end{cases} \tag{3.9}$$

and

$$\begin{cases} F_m^{(k)}(x,t) = \sum\limits_{|\alpha|\le N-1} \dfrac{1}{\alpha!} D^\alpha f[u_{m-1}, v_{m-1}](u_m^{(k)} - u_{m-1})^{\alpha_1}(v_m^{(k)} - v_{m-1})^{\alpha_2}, \\ G_m^{(k)}(x,t) = \sum\limits_{|\alpha|\le N-1} \dfrac{1}{\alpha!} D^\alpha g[u_{m-1}, v_{m-1}](u_m^{(k)} - u_{m-1})^{\alpha_1}(v_m^{(k)} - v_{m-1})^{\alpha_2}. \end{cases} \tag{3.10}$$

Let us suppose that (u_{m-1}, v_{m-1}) satisfies (3.4). Then it is clear that the system (3.8) has a solution $\left(u_m^{(k)}, v_m^{(k)}\right)$ on an interval $0 \le t \le T_m^{(k)} \le T$. The following estimates allow one to take constant $T_m^{(k)} = T$ for all m and k.

Step 2. A priori estimates.

First, we put

$$\begin{cases} \|f\|_{C^0(A_M)} = \sup_{(x,t,u,v)\in A_M} |f(x,t,u,v)|, \\ K_N(M,f) = \|D^\alpha f\|_{C^N(A_M)} = \sum_{|\alpha|\le N} \|D^\alpha f\|_{C^0(A_M)}, \\ A_M = [0,1]\times[0,T^*]\times[-\sqrt{R-1}M, \sqrt{R-1}M]^2, \\ f = f(x,t,u,v),\ D_1 f = \dfrac{\partial f}{\partial x},\ D_2 f = \dfrac{\partial f}{\partial t},\ D_3 f = \dfrac{\partial f}{\partial u},\ D_4 f = \dfrac{\partial f}{\partial v} \end{cases} \tag{3.11}$$

and

$$\begin{aligned} S_m^{(k)}(t) &= \left\|\dot u_m^{(k)}(t)\right\|_0^2 + \left\|\dot v_m^{(k)}(t)\right\|_0^2 + \left\|\dot u_m^{(k)}(t)\right\|_a^2 + \left\|\dot v_m^{(k)}(t)\right\|_b^2 \\ &\quad + \left\|u_m^{(k)}(t)\right\|_a^2 + \left\|v_m^{(k)}(t)\right\|_b^2 + \left\|L_1 u_m^{(k)}(t)\right\|_0^2 + \left\|L_2 v_m^{(k)}(t)\right\|_0^2 \\ &\quad + \int_0^t \left\|\ddot u_m^{(k)}(s)\right\|_0^2 ds + \int_0^t \left\|\ddot v_m^{(k)}(s)\right\|_0^2 ds. \end{aligned} \tag{3.12}$$

Then, it follows from (3.8), (3.12) that

$$\begin{aligned} S_m^{(k)}(t) &= S_m^{(k)}(0) + 2\int_0^t \left[\langle F_m^{(k)}(s), \dot u_m^{(k)}(s)\rangle + \langle G_m^{(k)}(s), \dot v_m^{(k)}(s)\rangle\right] ds \\ &\quad + 2\int_0^t \left[a(F_m^{(k)}(s), \dot u_m^{(k)}(s)) + b(G_m^{(k)}(s), \dot v_m^{(k)}(s))\right] ds \\ &\quad + \int_0^t \left\|\ddot u_m^{(k)}(s)\right\|_0^2 ds + \int_0^t \left\|\ddot v_m^{(k)}(s)\right\|_0^2 ds \\ &= S_m^{(k)}(0) + \sum_{j=1}^4 I_j. \end{aligned} \tag{3.13}$$

We shall estimate the terms of (3.13) as follows.

We can easily check that for

$$\begin{aligned} \|v\|_{H^2\cap V_R} &= \sqrt{\|v_x\|_0^2 + \|v_{xx}\|_0^2}, \\ L_1 v &\equiv -a_1\left(v_{xx} + \frac{1}{x}v_x - \frac{1}{x^2}v\right), \\ L_2 v &\equiv -a_2\left(v_{xx} + \frac{1}{x}v_x\right), \end{aligned} \tag{3.14}$$

there exist two constants $\gamma_1,\ \gamma_2 > 0$ such that

$$\begin{array}{ll} (1) & \|L_1 v\|_0^2 + \|v\|_a^2 \geq \gamma_1 \|v\|_{H^2\cap V_R}^2, \ \forall v \in H^2 \cap V_R, \\ (2) & \|L_2 v\|_0^2 + \|v\|_b^2 \geq \gamma_2 \|v\|_{H^2\cap V_R}^2, \ \forall v \in H^2 \cap V_R. \end{array} \tag{3.15}$$

We shall estimate the terms $S_m^{(k)}(t)$, $S_m^{(k)}(0)$, I_j of (3.13) as follows.

(i) Estimate of $S_m^{(k)}(t)$.

By above inequalities, we deduce from (3.12) that

$$\begin{aligned} S_m^{(k)}(t) \geq & \left\|\dot{u}_m^{(k)}(t)\right\|_0^2 + \left\|\dot{v}_m^{(k)}(t)\right\|_0^2 + \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 + \left\|\dot{v}_m^{(k)}(t)\right\|_b^2 \\ & + \gamma_1 \left\|u_m^{(k)}(t)\right\|_{H^2\cap V_R}^2 + \gamma_2 \left\|v_m^{(k)}(t)\right\|_{H^2\cap V_R}^2 \\ & + \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 ds + \int_0^t \left\|\ddot{v}_m^{(k)}(s)\right\|_0^2 ds \\ \geq & \ \gamma_* \bar{S}_m^{(k)}(t), \end{aligned} \tag{3.16}$$

where $\gamma_* = \min\{1, \gamma_1, \gamma_2\}$ and

$$\begin{aligned} \bar{S}_m^{(k)}(t) = & \left\|\dot{u}_m^{(k)}(t)\right\|_0^2 + \left\|\dot{v}_m^{(k)}(t)\right\|_0^2 + \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 + \left\|\dot{v}_m^{(k)}(t)\right\|_b^2 \\ & + \left\|u_m^{(k)}(t)\right\|_{H^2\cap V_R}^2 + \left\|v_m^{(k)}(t)\right\|_{H^2\cap V_R}^2 \\ & + \int_0^t \left(\left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 + \left\|\ddot{v}_m^{(k)}(s)\right\|_0^2\right) ds. \end{aligned} \tag{3.17}$$

(ii) In order to estimate the terms $I_1, \cdots, I_4$, we prove that the followings:

$$\begin{array}{ll} (a) & \left|F_m^{(k)}(x,t)\right| \leq f_M^{(0)} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right], \\ (b) & \left|G_m^{(k)}(x,t)\right| \leq g_M^{(0)} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right], \\ (c) & \left\|F_{mx}^{(k)}(t)\right\|_0 \leq f_M^{(1)} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right], \\ (d) & \left\|G_{mx}^{(k)}(t)\right\|_0 \leq g_M^{(1)} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right], \end{array} \tag{3.18}$$

where

$$\begin{aligned} f_M^{(0)} &= K_N(M,f)\left(1+M^{N-1}\right)\sum_{k=0}^{N-1}\frac{R_*^k}{k!}, \\ g_M^{(0)} &= K_N(M,g)\left(1+M^{N-1}\right)\sum_{k=0}^{N-1}\frac{R_*^k}{k!}, \\ f_M^{(1)} &= K_N(M,f)\left(1+d_M^*\right)\left(1+M^{N-1}\right)R_{**}, \\ g_M^{(1)} &= K_N(M,f)\left(1+d_M^*\right)\left(1+M^{N-1}\right)R_{**}, \\ R_* &= 4\sqrt{R-1},\ R_{**} = 4+\frac{d_M^* R_*^{N-1}}{(N-1)!}+\left(4+d_M^*\right)\sum_{k=1}^{N-2}\frac{R_*^k}{k!}. \end{aligned} \tag{3.19}$$

(a) Estimate of $\left|F_m^{(k)}(x,t)\right|$. By using the inequalities

$$\begin{aligned} |u_{m-1}(x,t)| &\leq \|u_{m-1}(t)\|_{C^0(\overline{\Omega})} \leq \sqrt{R-1}\,\|\nabla u_{m-1}(t)\|_0 \leq \sqrt{R-1}M, \\ |v_{m-1}(x,t)| &\leq \sqrt{R-1}M, \\ \left|u_m^{(k)}(x,t)\right| &\leq \left\|u_m^{(k)}(t)\right\|_{C^0(\overline{\Omega})} \leq \sqrt{R-1}\left\|u_{mx}^{(k)}(t)\right\|_0 \\ &\leq \sqrt{R-1}\left\|u_m^{(k)}(t)\right\|_{H^2\cap V_R} \leq \sqrt{R-1}\sqrt{\bar{S}_m^{(k)}(t)}, \\ \left|v_m^{(k)}(x,t)\right| &\leq \sqrt{R-1}\sqrt{\bar{S}_m^{(k)}(t)}, \\ (a+b)^p &\leq 2^{p-1}(a^p+b^p),\ \forall a,b\geq 0,\ \forall p\geq 1, \\ a &\leq 1+a^p,\ \forall a\geq 0,\ \forall p\geq 1, \end{aligned}$$

it follows from $(3.10)_1$ that

$$\begin{aligned} &\left|F_m^{(k)}(x,t)\right| \\ &\leq |f[u_{m-1},v_{m-1}](x,t)| \\ &\quad + \sum_{1\leq|\alpha|\leq N-1}\frac{1}{\alpha!}\left|D^\alpha f[u_{m-1},v_{m-1}]\right|\left(\left|u_m^{(k)}\right|+|u_{m-1}|\right)^{\alpha_1}\left(\left|v_m^{(k)}\right|+|v_{m-1}|\right)^{\alpha_2} \\ &\leq K_N(M,f)+K_N(M,f)\sum_{1\leq|\alpha|\leq N-1}\frac{1}{\alpha!}\left(\sqrt{R-1}\right)^{|\alpha|}\left(M+\sqrt{\bar{S}_m^{(k)}(t)}\right)^{|\alpha|} \end{aligned}$$

$$
\begin{aligned}
&\le K_N(M,f) \\
&\quad + K_N(M,f) \sum_{1\le|\alpha|\le N-1} \frac{1}{\alpha!}\left(\sqrt{R-1}\right)^{|\alpha|} 2^{|\alpha|-1}\left[M^{|\alpha|}+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{|\alpha|}\right] \\
&\le K_N(M,f) \\
&\quad + K_N(M,f) \sum_{1\le|\alpha|\le N-1} \frac{1}{\alpha!}\left(\sqrt{R-1}\right)^{|\alpha|} 2^{|\alpha|-1}\left[2+M^{N-1}+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\
&\le K_N(M,f) \\
&\quad + K_N(M,f) \sum_{1\le|\alpha|\le N-1} \frac{1}{\alpha!}\left(2\sqrt{R-1}\right)^{|\alpha|}\left(1+M^{N-1}\right)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right].
\end{aligned}
$$

It is known that $\sum_{|\alpha|=k} \frac{1}{\alpha!} = \frac{2^k}{k!}$, hence

$$
\sum_{1\le|\alpha|\le N-1} \frac{1}{\alpha!}\left(2\sqrt{R-1}\right)^{|\alpha|} = \sum_{k=1}^{N-1}\sum_{|\alpha|=k} \frac{1}{\alpha!}\left(2\sqrt{R-1}\right)^{k} = \sum_{k=1}^{N-1} \frac{R_*^k}{k!},
$$

we deduce that

$$
\begin{aligned}
&\left|F_m^{(k)}(x,t)\right| \\
&\le |f[u_{m-1},v_{m-1}](x,t)| \\
&\quad + \sum_{1\le|\alpha|\le N-1} \frac{1}{\alpha!}\left|D^\alpha f[u_{m-1},v_{m-1}]\right|\left(\left|u_m^{(k)}\right|+|u_{m-1}|\right)^{\alpha_1}\left(\left|v_m^{(k)}\right|+|v_{m-1}|\right)^{\alpha_2} \\
&\le K_N(M,f) + K_N(M,f)\sum_{k=1}^{N-1} \frac{R_*^k}{k!}\left(1+M^{N-1}\right)\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\
&\le f_M^{(0)}\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right].
\end{aligned}
\tag{3.20}
$$

(b) Estimate of $\left|G_m^{(k)}(x,t)\right|$. Similar to $F_m^{(k)}(x,t)$, we also have a estimate $G_m^{(k)}(x,t)$ as in (3.18)(b).

(c) Estimate of $\left\| F_{mx}^{(k)}(t) \right\|_0$. We have

$$
\begin{aligned}
& F_{mx}^{(k)}(x,t) \\
&= \frac{\partial}{\partial x} f[u_{m-1}, v_{m-1}] \\
&\quad + \sum_{1 \le |\alpha| \le N-1} \frac{1}{\alpha!} \left(\frac{\partial}{\partial x} D^{\alpha} f[u_{m-1}, v_{m-1}] \right) (u_m^{(k)} - u_{m-1})^{\alpha_1} (v_m^{(k)} - v_{m-1})^{\alpha_2} \\
&\quad + \sum_{1 \le |\alpha| \le N-1} \frac{\alpha_1}{\alpha!} D^{\alpha} f[u_{m-1}, v_{m-1}] (u_m^{(k)} - u_{m-1})^{\alpha_1 - 1} (u_{mx}^{(k)} - \nabla u_{m-1}) \\
&\quad \times (v_m^{(k)} - v_{m-1})^{\alpha_2} \\
&\quad + \sum_{1 \le |\alpha| \le N-1} \frac{\alpha_2}{\alpha!} D^{\alpha} f[u_{m-1}, v_{m-1}] (u_m^{(k)} - u_{m-1})^{\alpha_1} (v_m^{(k)} - v_{m-1})^{\alpha_2 - 1} \\
&\quad \times (v_{mx}^{(k)} - \nabla v_{m-1}) \\
&= \frac{\partial}{\partial x} f[u_{m-1}, v_{m-1}] + J_1^* + J_2^* + J_3^*.
\end{aligned} \tag{3.21}
$$

We shall estimate the terms $\frac{\partial}{\partial x} f[u_{m-1}, v_{m-1}]$, J_1^*, J_2^*, J_3^* on the right-hand side of (3.21) as follows.

(c)-1 Estimate of $\frac{\partial}{\partial x} f[u_{m-1}, v_{m-1}]$. We have

$$
\begin{aligned}
& \left\| \frac{\partial}{\partial x} f[u_{m-1}, v_{m-1}] \right\|_0 \\
&= \| D_1 f[u_{m-1}, v_{m-1}] + D_3 f[u_{m-1}, v_{m-1}] \nabla u_{m-1} + D_4 f[u_{m-1}, v_{m-1}] \nabla_{m-1} \|_0 \\
&\le K_N(M,f) \left[\sqrt{\frac{R^2-1}{2}} + \| \nabla u_{m-1} \|_0 + \| \nabla_{m-1} \|_0 \right] \\
&\le K_N(M,f) \left[\sqrt{\frac{R^2-1}{2}} + 2M \right] = K_N(M,f) d_M^*,
\end{aligned} \tag{3.22}
$$

where $d_M^* = \sqrt{\frac{R^2-1}{2}} + 2M$.

(c)-2 Estimate of J_1^*. Similarly

$$
\|J_1^*\|_0 \le \sum_{1 \le |\alpha| \le N-1} \frac{1}{\alpha!} \left\| \frac{\partial}{\partial x} D^{\alpha} f[u_{m-1}, v_{m-1}] (u_m^{(k)} - u_{m-1})^{\alpha_1} (v_m^{(k)} - v_{m-1})^{\alpha_2} \right\|_0 \tag{3.23}
$$

$$
\begin{aligned}
&\leq \sum_{1\leq|\alpha|\leq N-1} \frac{1}{\alpha!} \left\| \frac{\partial}{\partial x} D^{\alpha} f[u_{m-1}, v_{m-1}] \left(\left| u_m^{(k)} \right| + |u_{m-1}| \right)^{\alpha_1} \left(\left| v_m^{(k)} \right| + |v_{m-1}| \right)^{\alpha_2} \right\|_0 \\
&\leq \sum_{1\leq|\alpha|\leq N-1} \frac{1}{\alpha!} \left\| \frac{\partial}{\partial x} D^{\alpha} f[u_{m-1}, v_{m-1}] \right\|_0 \left(\sqrt{R-1} \right)^{|\alpha|} \left(M + \sqrt{\bar{S}_m^{(k)}(t)} \right)^{|\alpha|} \\
&\leq K_N(M,f) d_M^* \sum_{1\leq|\alpha|\leq N-1} \frac{1}{\alpha!} \left(\sqrt{R-1} \right)^{|\alpha|} 2^{|\alpha|-1} \left[M^{|\alpha|} + \left(\sqrt{\bar{S}_m^{(k)}(t)} \right)^{|\alpha|} \right] \\
&\leq K_N(M,f) d_M^* \sum_{1\leq|\alpha|\leq N-1} \frac{1}{\alpha!} \left(\sqrt{R-1} \right)^{|\alpha|} 2^{|\alpha|-1} \left[2 + M^{N-1} + \left(\sqrt{\bar{S}_m^{(k)}(t)} \right)^{N-1} \right] \\
&\leq K_N(M,f) d_M^* \sum_{1\leq|\alpha|\leq N-1} \frac{1}{\alpha!} \left(2\sqrt{R-1} \right)^{|\alpha|} \left(1 + M^{N-1} \right) \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)} \right)^{N-1} \right] \\
&= K_N(M,f) \left(1 + M^{N-1} \right) d_M^* \sum_{k=1}^{N-1} \frac{R_*^k}{k!} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)} \right)^{N-1} \right].
\end{aligned}
$$

(c)-3 Estimate of $J_2^* + J_3^*$. We have

$$
\begin{aligned}
&\|J_2^*\|_0 \\
&\leq \sum_{1\leq|\alpha|\leq N-1} \frac{\alpha_1}{\alpha!} \Big\| D^{\alpha} f[u_{m-1}, v_{m-1}] (u_m^{(k)} - u_{m-1})^{\alpha_1 - 1} \\
&\qquad \times (u_{mx}^{(k)} - \nabla u_{m-1})(v_m^{(k)} - v_{m-1})^{\alpha_2} \Big\|_0 \\
&\leq \sum_{1\leq|\alpha|\leq N-1} \frac{\alpha_1}{\alpha!} \Big\| D^{\alpha} f[u_{m-1}, v_{m-1}] \left(\left| u_m^{(k)} \right| + |u_{m-1}| \right)^{\alpha_1 - 1} \\
&\qquad \times \left(\left| u_{mx}^{(k)} \right| + |\nabla u_{m-1}| \right) \left(\left| v_m^{(k)} \right| + |v_{m-1}| \right)^{\alpha_2} \Big\|_0 \qquad (3.24) \\
&\leq K_N(M,f) \sum_{1\leq|\alpha|\leq N-1} \frac{\alpha_1}{\alpha!} \left(\sqrt{R-1} \right)^{|\alpha|-1} \left(M + \sqrt{\bar{S}_m^{(k)}(t)} \right)^{|\alpha|-1} \\
&\qquad \times \left\| \left| u_{mx}^{(k)} \right| + |\nabla u_{m-1}| \right\|_0 \\
&\leq K_N(M,f) \sum_{1\leq|\alpha|\leq N-1} \frac{\alpha_1}{\alpha!} \left(\sqrt{R-1} \right)^{|\alpha|-1} \left(M + \sqrt{\bar{S}_m^{(k)}(t)} \right)^{|\alpha|}
\end{aligned}
$$

$$\leq K_N(M,f) \sum_{1\leq|\alpha|\leq N-1} \frac{\alpha_1}{\alpha!} \left(2\sqrt{R-1}\right)^{|\alpha|-1} \left[M^{|\alpha|} + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{|\alpha|}\right]$$
$$\leq 2K_N(M,f)\left(1+M^{N-1}\right) \sum_{1\leq|\alpha|\leq N-1} \frac{\alpha_1}{\alpha!} \left(2\sqrt{R-1}\right)^{|\alpha|-1}$$
$$\times \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right].$$

Similarly

$$\|J_3^*\|_0 \leq 2K_N(M,f)\left(1+M^{N-1}\right) \sum_{1\leq|\alpha|\leq N-1} \frac{\alpha_2}{\alpha!} \left(2\sqrt{R-1}\right)^{|\alpha|-1} \times \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right]. \tag{3.25}$$

Hence, we deduce from (3.24) and (3.25) that

$$\begin{aligned}
&\|J_2^* + J_3^*\|_0 \\
&\leq \|J_2^*\|_0 + \|J_3^*\|_0 \\
&\leq 2K_N(M,f)\left(1+M^{N-1}\right) \sum_{1\leq|\alpha|\leq N-1} \frac{|\alpha|}{\alpha!} \left(2\sqrt{R-1}\right)^{|\alpha|-1} \\
&\quad \times \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\
&= 4K_N(M,f)\left(1+M^{N-1}\right) \sum_{k=1}^{N-1} \frac{1}{(k-1)!} R_*^{k-1} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\
&= 4K_N(M,f)\left(1+M^{N-1}\right) \left[1 + \sum_{k=1}^{N-2} \frac{1}{k!} R_*^k\right] \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right].
\end{aligned} \tag{3.26}$$

Combining (3.21), (3.22), (3.23) and (3.26), we obtain

$$\begin{aligned}
&\left\|F_{mx}^{(k)}(t)\right\|_0 \\
&\leq \left\|\frac{\partial}{\partial x} f[u_{m-1}, v_{m-1}]\right\|_0 + \|J_1^*\|_0 + \|J_2^* + J_3^*\|_0
\end{aligned} \tag{3.27}$$

$$
\begin{aligned}
&\leq K_N(M,f)d_M^* + K_N(M,f)\left(1+M^{N-1}\right)d_M^* \\
&\quad\times\left[\sum_{k=1}^{N-2}\frac{R_*^k}{k!}+\frac{R_*^{N-1}}{(N-1)!}\right]\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\
&\quad + K_N(M,f)\left(1+M^{N-1}\right)\left[4+4\sum_{k=1}^{N-2}\frac{1}{k!}R_*^k\right]\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\
&\leq K_N(M,f)d_M^* + K_N(M,f)\left(1+M^{N-1}\right) \\
&\quad\times\left[4+d_M^*\frac{R_*^{N-1}}{(N-1)!}+(4+d_M^*)\sum_{k=1}^{N-2}\frac{R_*^k}{k!}\right]\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\
&\leq K_N(M,f)\left(1+d_M^*\right)\left(1+M^{N-1}\right) \\
&\quad\times\left[4+\frac{d_M^*R_*^{N-1}}{(N-1)!}+(4+d_M^*)\sum_{k=1}^{N-2}\frac{R_*^k}{k!}\right]\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\
&= f_M^{(1)}\left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right].
\end{aligned}
$$

(d) Estimate of $\left\|G_{mx}^{(k)}(t)\right\|_0$. Similar to $\left\|F_{mx}^{(k)}(t)\right\|_0$, we also have a estimate $\left\|G_{mx}^{(k)}(t)\right\|_0$ as in (3.18)(d).

Next, we estimate the I_i $(i=1,2,3,4)$.

Estimate of $I_1 = 2\int_0^t\left[\langle F_m^{(k)}(s),\dot{u}_m^{(k)}(s)\rangle+\langle G_m^{(k)}(s),\dot{v}_m^{(k)}(s)\rangle\right]ds$.

By the Cauchy inequality, we deduce from (3.18) (a), (b) that

$$
\begin{aligned}
I_1 &= 2\int_0^t\left[\langle F_m^{(k)}(s),\dot{u}_m^{(k)}(s)\rangle+\langle G_m^{(k)}(s),\dot{v}_m^{(k)}(s)\rangle\right]ds \\
&\leq 2\int_0^t\left[\left\|F_m^{(k)}(s)\right\|_0\left\|\dot{u}_m^{(k)}(s)\right\|_0+\left\|G_m^{(k)}(s)\right\|_0\left\|\dot{v}_m^{(k)}(s)\right\|_0\right]ds \\
&\leq 2\sqrt{\frac{R^2-1}{2}}\left(f_M^{(0)}+g_M^{(0)}\right)\int_0^t\left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right]\sqrt{\bar{S}_m^{(k)}(s)}ds \qquad (3.28)\\
&= 2\sqrt{\frac{R^2-1}{2}}\left(f_M^{(0)}+g_M^{(0)}\right)\int_0^t\left[\sqrt{\bar{S}_m^{(k)}(s)}+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N}\right]ds \\
&= 4\sqrt{\frac{R^2-1}{2}}\left(f_M^{(0)}+g_M^{(0)}\right)\int_0^t\left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N}\right]ds.
\end{aligned}
$$

Estimate of $I_2 = 2\int_0^t\left[a(F_m^{(k)}(s),\dot{u}_m^{(k)}(s))+b(G_m^{(k)}(s),\dot{v}_m^{(k)}(s))\right]ds$.

Similar to I_1, we have

$$\begin{aligned}
I_2 &= 2\int_0^t \left[a(F_m^{(k)}(s), \dot{u}_m^{(k)}(s)) + b(G_m^{(k)}(s), \dot{v}_m^{(k)}(s)) \right] ds \\
&\leq 2\int_0^t \left[\left\| F_m^{(k)}(s) \right\|_a \left\| \dot{u}_m^{(k)}(s) \right\|_a + \left\| G_m^{(k)}(s) \right\|_b \left\| \dot{v}_m^{(k)}(s) \right\|_b \right] ds \\
&= 2\int_0^t \left[a_1^* \left\| F_{mx}^{(k)}(s) \right\|_0 + \sqrt{a_2} \left\| G_{mx}^{(k)}(s) \right\|_0 \right] \sqrt{\bar{S}_m^{(k)}(s)} ds \\
&\leq 2\left(a_1^* f_M^{(1)} + \sqrt{a_2} g_M^{(1)} \right) \int_0^t \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(s)} \right)^{N-1} \right] \sqrt{\bar{S}_m^{(k)}(s)} ds \\
&= 2\left(a_1^* f_M^{(1)} + \sqrt{a_2} g_M^{(1)} \right) \int_0^t \left[\sqrt{\bar{S}_m^{(k)}(s)} + \left(\sqrt{\bar{S}_m^{(k)}(s)} \right)^{N} \right] ds \\
&\leq 4\left(a_1^* f_M^{(1)} + \sqrt{a_2} g_M^{(1)} \right) \int_0^t \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(s)} \right)^{N} \right] ds.
\end{aligned} \tag{3.29}$$

Estimate of $I_3 = \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|_0^2 ds$. Eq. $(3.8)_1$ is rewritten as follows

$$\langle \ddot{u}_m^{(k)}(t), w_j \rangle + \langle L_1 u_m^{(k)}(t), w_j \rangle = \langle F_m(t), w_j \rangle, \ 1 \leq j \leq k. \tag{3.30}$$

Then, it follows after replacing w_j with $\ddot{u}_m^{(k)}(t)$ and integrating that

$$\begin{aligned}
I_3 &= \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|_0^2 ds \\
&\leq 2\int_0^t \left\| L_1 u_m^{(k)}(s) \right\|_0^2 ds + 2\int_0^t \left\| F_m(s) \right\|_0^2 ds \\
&\leq 2\int_0^t \bar{S}_m^{(k)}(s) ds + T(R^2 - 1) K_N^2(M, f).
\end{aligned} \tag{3.31}$$

Estimate of $I_4 = \int_0^t \left\| \ddot{v}_m^{(k)}(s) \right\|_0^2 ds$. Similarly, we get

$$I_4 = \int_0^t \left\| \ddot{v}_m^{(k)}(s) \right\|_0^2 ds \leq 2\int_0^t \bar{S}_m^{(k)}(s) ds + T(R^2 - 1) K_N^2(M, g). \tag{3.32}$$

On the other hand, we have

$$\begin{aligned}
S_m^{(k)}(0) &= \left\| \tilde{u}_{1k} \right\|_0^2 + \left\| \tilde{v}_{1k} \right\|_0^2 + \left\| \tilde{u}_{1k} \right\|_a^2 + \left\| \tilde{v}_{1k} \right\|_b^2 \\
&\quad + \left\| \tilde{u}_{0k} \right\|_a^2 + \left\| \tilde{v}_{0k} \right\|_b^2 + \left\| L_1 \tilde{u}_{0k} \right\|_0^2 + \left\| L_2 \tilde{v}_{0k} \right\|_0^2 .
\end{aligned} \tag{3.33}$$

By means of the convergences in (3.9), we deduce the existence of a constant $M > 0$ independent of k and m such that

$$S_m^{(k)}(0) \leq \frac{\gamma_*}{2} M^2, \text{ for all } k \text{ and } m \in \mathbb{N}. \tag{3.34}$$

Combining (3.13), (3.16), (3.17), (3.28), (3.29), (3.31), (3.32) and (3.34), the result is

$$\bar{S}_m^{(k)}(t) \leq \frac{1}{2} M^2 + T\bar{D}_3(M) + \bar{D}_2(M) \int_0^t \left(\sqrt{\bar{S}_m^{(k)}(s)} \right)^N ds, \tag{3.35}$$

where

$$\begin{cases} \bar{D}_1(M) = \dfrac{1}{\gamma_*}(R^2 - 1) \left[K_N^2(M, f) + K_N^2(M, g) \right], \\ \bar{D}_2(M) = \dfrac{4}{\gamma_*} \left[1 + \sqrt{\dfrac{R^2 - 1}{2}} \left(f_M^{(0)} + g_M^{(0)} \right) + \left(a_1^* f_M^{(1)} + \sqrt{a_2} g_M^{(1)} \right) \right], \\ \bar{D}_3(M) = \bar{D}_1(M) + \bar{D}_2(M). \end{cases} \tag{3.36}$$

Then, by solving a nonlinear Volterra integral equation (based on the methods in [7]), there exists a constant $T > 0$ depending on T_* (*independent of* m) such that

$$\bar{S}_m^{(k)}(t) \leq M^2, \ \forall m \in \mathbb{N}, \ \forall t \in [0, T], \tag{3.37}$$

where C_T is a constant depending only on T.

Step 3. (Limiting process). From (3.37), we deduce the existence of a subsequence of $\{(u_m^{(k)}, v_m^{(k)})\}$, denoted by the same symbol such that

$$\begin{cases} (u_m^{(k)}, v_m^{(k)}) \to (u_m, v_m) & \text{in} \quad L^\infty(0, T; \left(H^2 \cap V_R\right)^2) \text{ weak*}, \\ (\dot{u}_m^{(k)}, \dot{v}_m^{(k)}) \to (u'_m, v'_m) & \text{in} \quad L^\infty(0, T; V_R \times V_R) \text{ weak*}, \\ (\ddot{u}_m^{(k)}, \ddot{v}_m^{(k)}) \to (u''_m, v''_m) & \text{in} \quad L^2(Q_T) \times L^2(Q_T) \text{ weak}, \\ (u_m, v_m) \in W(M, T). \end{cases} \tag{3.38}$$

By the compactness of Lemma ([8], p. 57) and the compact imbedding $H^1(0, T_*) \hookrightarrow C^0([0, T_*])$, we can deduce from $(3.38)_{1,2}$ the existence of a subsequence still denoted by $\{u_m\}$ such that

$$(u_m^{(k)}, v_m^{(k)}) \to (u_m, v_m) \text{ strongly in } L^2(0, T; V_R \times V_R) \text{ and a.e. in } Q_T. \tag{3.39}$$

On the other hand

$$\begin{aligned}
&\left|F_m^{(k)}(x,t) - F_m(x,t)\right| \\
&\leq \sum_{|\alpha|\leq N-1} \frac{1}{\alpha!} \left|D^\alpha f[u_{m-1}, v_{m-1}]\right| \\
&\quad \times \left|\left[(u_m^{(k)} - u_{m-1})^{\alpha_1}(v_m^{(k)} - v_{m-1})^{\alpha_2} - (u_m - u_{m-1})^{\alpha_1}(v_m - v_{m-1})^{\alpha_2}\right]\right| \\
&\leq K_N(M,f) \sum_{|\alpha|\leq N-1} \frac{1}{\alpha!} \\
&\quad \times \left|\left[(u_m^{(k)} - u_{m-1})^{\alpha_1}(v_m^{(k)} - v_{m-1})^{\alpha_2} - (u_m - u_{m-1})^{\alpha_1}(v_m - v_{m-1})^{\alpha_2}\right]\right| \\
&= K_N(M,f) \sum_{|\alpha|\leq N-1} \frac{1}{\alpha!} \left|\Psi_m^{(k)}(x,t)\right|,
\end{aligned} \tag{3.40}$$

where

$$\begin{aligned}
\Psi_m^{(k)}(x,t) &= (u_m^{(k)} - u_{m-1})^{\alpha_1}(v_m^{(k)} - v_{m-1})^{\alpha_2} - (u_m - u_{m-1})^{\alpha_1}(v_m - v_{m-1})^{\alpha_2} \\
&= \left[(u_m^{(k)} - u_{m-1})^{\alpha_1} - (u_m - u_{m-1})^{\alpha_1}\right](v_m^{(k)} - v_{m-1})^{\alpha_2} \\
&\quad + (u_m - u_{m-1})^{\alpha_1}\left[(v_m^{(k)} - v_{m-1})^{\alpha_2} - (v_m - v_{m-1})^{\alpha_2}\right].
\end{aligned} \tag{3.41}$$

By using the inequalities

$$\begin{aligned}
|u_{m-1}| &\leq \sqrt{R-1}M, \\
|u_m - u_{m-1}| &\leq 2\sqrt{R-1}M, \\
\left|u_m^{(k)}\right| &\leq \sqrt{R-1}\left\|u_{mx}^{(k)}(t)\right\|_0 \leq \sqrt{R-1}\sqrt{\bar{S}_m^{(k)}(t)} \leq \sqrt{R-1}M, \\
\left|u_m^{(k)} - u_{m-1}\right| &\leq \left|u_m^{(k)}\right| + |u_{m-1}| \\
&\leq \sqrt{R-1}\left(\left\|u_{mx}^{(k)}(t)\right\|_0 + \|\nabla u_{m-1}\|_0\right) \leq 2\sqrt{R-1}M, \\
|x^\alpha - y^\alpha| &\leq \alpha M_1^{\alpha-1}|x-y|,
\end{aligned}$$

for all $x, y \in [-M_1, M_1]$, $M_1 > 0$, $\alpha \in \mathbb{N}$, we obtain

$$\begin{aligned}
\left|(u_m^{(k)} - u_{m-1})^{\alpha_1} - (u_m - u_{m-1})^{\alpha_1}\right| &\leq \alpha_1 M_1^{\alpha_1-1}\left|u_m^{(k)} - u_m\right| \\
&\leq \alpha_1 M_1^{\alpha_1-1}\sqrt{R-1}\left\|u_{mx}^{(k)} - u_{mx}\right\|_0 \\
&\leq \frac{\alpha_1}{2} M_1^{\alpha_1}\left\|u_{mx}^{(k)} - u_{mx}\right\|_0,
\end{aligned}$$

where $M_1 = 2\sqrt{R-1}M$, hence

$$\left\|(u_m^{(k)} - u_{m-1})^{\alpha_1} - (u_m - u_{m-1})^{\alpha_1}\right\|_0 \le \frac{\alpha_1}{2} M_1^{\alpha_1} \sqrt{\frac{R^2-1}{2}} \left\|u_{mx}^{(k)} - u_{mx}\right\|_0 .$$

This implies

$$\begin{aligned} &\left\|(u_m^{(k)} - u_{m-1})^{\alpha_1} - (u_m - u_{m-1})^{\alpha_1}\right\|_{L^2(0,T;L^2)} \\ &\le \frac{\alpha_1}{2} M_1^{\alpha_1} \sqrt{\frac{R^2-1}{2}} \left\|u_m^{(k)} - u_m\right\|_{L^2(0,T;V_R)} . \end{aligned}$$

Similarly, it is clear to see that

$$\begin{aligned} &\left\|(v_m^{(k)} - v_{m-1})^{\alpha_2} - (v_m - v_{m-1})^{\alpha_2}\right\|_{L^2(0,T;L^2)} \\ &\le \frac{\alpha_2}{2} M_1^{\alpha_2} \sqrt{\frac{R^2-1}{2}} \left\|v_m^{(k)} - v_m\right\|_{L^2(0,T;V_R)} . \end{aligned}$$

By the inequalities $\left|v_m^{(k)} - v_{m-1}\right|^{\alpha_2} \le M_1^{\alpha_2}$, $|u_m - u_{m-1}|^{\alpha_1} \le M_1^{\alpha_1}$, it follows that

$$\begin{aligned} \left\|\Psi_m^{(k)}\right\|_{L^2(0,T;L^2)} &\le \frac{|\alpha|}{2} M_1^{|\alpha|} \sqrt{\frac{R^2-1}{2}} \\ &\quad \times \left[\left\|u_m^{(k)} - u_m\right\|_{L^2(0,T;V_R)} + \left\|v_m^{(k)} - v_m\right\|_{L^2(0,T;V_R)}\right] \to 0. \end{aligned}$$

It follows that

$$F_m^{(k)} \to F_m \quad \text{strongly in} \quad L^2(0,T;V_R \times V_R). \tag{3.42}$$

Similarly, by (3.39), we deduce from $(3.6)_2$ and $(3.10)_2$ that

$$G_m^{(k)} \to G_m \quad \text{strongly in} \quad L^2(0,T;V_R \times V_R). \tag{3.43}$$

Passing to limit in (3.8), we have (u_m, v_m) satisfying (3.5), (3.6) in $L^2(0,T)$.

On the other hand, it follows from (3.5)-(3.8) and $(3.38)_4$ that

$$u_m'' = -L_1 u_m + F_m \in L^\infty(0,T;L^2)$$

and

$$v_m'' = -L_2 v_m + G_m \in L^\infty(0,T;L^2).$$

Hence $(u_m, v_m) \in W_1(M,T)$ and the proof of Theorem 3.1 is complete. □

Next, we state and prove the main theorem in this section, in which

$$W_1(T) = C([0,T];V_R \times V_R) \cap C^1([0,T];L^2 \times L^2), \tag{3.44}$$

it is well known that $W_1(T)$ is a Banach space with respect to the norm (see Lions [8]):

$$\|(u,v)\|_{W_1(T)} = \|(u,v)\|_{C([0,T];V_R\times V_R)} + \|(u,v)\|_{C^1([0,T];L^2\times L^2)}\,. \tag{3.45}$$

Theorem 3.2. *Let* (A_1)-(A_2) *hold. Then, there exist positive constants* M, $T>0$ *such that*

(i) *the problem* (1.1) *has a unique weak solution* $(u,v)\in W_1(M,T)$.

(ii) *the recurrent sequence* $\{(u_m,v_m)\}$ *defined by* (3.5)-(3.6) *converges to the weak solution* (u,v) *of Problem* (1.1) *strongly in the space* $W_1(T)$.

Furthermore, we have the estimate

$$\|(u_m,v_m)-(u,v)\|_{W_1(T)} \le C\,(k_T)^{N^m}\,,\ \forall m\in\mathbb{N}, \tag{3.46}$$

where $k_T\in(0,1)$ *and* C *are chosen such that* k_T, C *depend only on* T, f, g, $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{v}_0$, $\tilde{v}_1$.

Proof. (a) Existence of the solution.

We shall prove that $\{(u_m,v_m)\}$ is a Cauchy sequence in $W_1(T)$. Let $\bar{u}_m = u_{m+1}-u_m$, $\bar{v}_m = v_{m+1}-v_m$. Then $(\bar{u}_m,\bar{v}_m)$ satisfies the variational problem:

$$\begin{cases} \langle \bar{u}_m''(t),w\rangle + a(\bar{u}_m(t),w) = \langle F_{m+1}(t)-F_m(t),w\rangle\,, \\ \langle \bar{v}_m''(t),\phi\rangle + b(\bar{v}_m(t),\phi) = \langle G_{m+1}(t)-G_m(t),\phi\rangle\,,\ \forall (w,\phi)\in V_R\times V_R, \\ (\bar{u}_m(0),\bar{v}_m(0)) = (\bar{u}_m'(0),\bar{v}_m'(0)) = (0,0). \end{cases} \tag{3.47}$$

Taking $(w,\phi)=(\bar{u}_m'(t),\bar{v}_m'(t))$ in (3.47), after integrating in t, we get

$$\begin{aligned} Z_m(t) &= 2\int_0^t \left\langle F_{m+1}(s)-F_m(s),\bar{u}_m'(s)\right\rangle ds \\ &\quad + 2\int_0^t \left\langle G_{m+1}(s)-G_m(s),\bar{v}_m'(s)\right\rangle ds \\ &\equiv J_1+J_2, \end{aligned} \tag{3.48}$$

where

$$Z_m(t) = \left\|\bar{u}_m'(t)\right\|_0^2 + \left\|\bar{v}_m'(t)\right\|_0^2 + \|\bar{u}_m(t)\|_a^2 + \|\bar{v}_m(t)\|_b^2\,. \tag{3.49}$$

And all terms of (3.48) are estimated as follows.

(1) The term $Z_m(t)$. We have

$$\begin{aligned} Z_m(t) &= \left\|\bar{u}_m'(t)\right\|_0^2 + \left\|\bar{v}_m'(t)\right\|_0^2 + \|\bar{u}_m(t)\|_a^2 + \|\bar{v}_m(t)\|_b^2 \\ &\ge \left\|\bar{u}_m'(t)\right\|_0^2 + \left\|\bar{v}_m'(t)\right\|_0^2 + a_1\|\bar{u}_{mx}(t)\|_0^2 + a_2\|\bar{v}_{mx}(t)\|_0^2 \\ &\ge a_*\bar{Z}_m(t), \end{aligned} \tag{3.50}$$

where

$$\begin{aligned}\bar{Z}_m(t) &= \left\|\bar{u}_m'(t)\right\|_0^2 + \left\|\bar{v}_m'(t)\right\|_0^2 + \|\bar{u}_{mx}(t)\|_0^2 + \|\bar{v}_{mx}(t)\|_0^2, \\ a_* &= \min\{1, a_1, a_2\}.\end{aligned} \tag{3.51}$$

(2) First integral $J_1 = 2\int_0^t \langle F_{m+1}(s) - F_m(s), \bar{u}_m'(s)\rangle\, ds$. We have

$$\begin{aligned}&F_{m+1}(t) - F_m(t) \\ &= f[u_m, v_m](x,t) - f[u_{m-1}, v_{m-1}](x,t) \\ &\quad + \sum_{1\le|\alpha|\le N-1} \frac{1}{\alpha!} D^\alpha f[u_m, v_m](x,t)\, (\bar{u}_m)^{\alpha_1} (\bar{v}_m)^{\alpha_2} \\ &\quad - \sum_{1\le|\alpha|\le N-1} \frac{1}{\alpha!} D^\alpha f[u_{m-1}, v_{m-1}](x,t)\, (\bar{u}_{m-1})^{\alpha_1} (\bar{v}_{m-1})^{\alpha_2}.\end{aligned} \tag{3.52}$$

By using Taylor's expansion of the function $f[u_m, v_m] = f[u_{m-1}+\bar{u}_{m-1}, v_{m-1}+\bar{v}_{m-1}]$ around the point $[u_{m-1}, v_{m-1}] = (x, t, u_{m-1}, v_{m-1})$ up to order N, we obtain

$$\begin{aligned}&f[u_m, v_m] - f[u_{m-1}, v_{m-1}] \\ &= \sum_{1\le|\alpha|\le N-1} \frac{1}{\alpha!} D^\alpha f[u_{m-1}, v_{m-1}]\, (\bar{u}_{m-1})^{\alpha_1} (\bar{v}_{m-1})^{\alpha_2} + R_m[f],\end{aligned} \tag{3.53}$$

where for $0 < \theta < 1$,

$$R_m[f] = \sum_{|\alpha|=N} \frac{1}{\alpha!} D^\alpha f[u_{m-1} + \theta\bar{u}_{m-1}, v_{m-1} + \theta\bar{v}_{m-1}]\, (\bar{u}_{m-1})^{\alpha_1} (\bar{v}_{m-1})^{\alpha_2}. \tag{3.54}$$

Then, $F_{m+1}(t) - F_m(t)$ is rewritten as follows:

$$F_{m+1}(t) - F_m(t) = \sum_{1\le|\alpha|\le N-1} \frac{1}{\alpha!} D^\alpha f[u_m, v_m](x,t)\, (\bar{u}_m)^{\alpha_1} (\bar{v}_m)^{\alpha_2} + R_m[f]. \tag{3.55}$$

Thus

$$|F_{m+1}(x,t) - F_m(x,t)| \le K_N(M, f) \sum_{1\le|\alpha|\le N-1} \frac{1}{\alpha!} \left|(\bar{u}_m)^{\alpha_1} (\bar{v}_m)^{\alpha_2}\right| + |R_m[f](x,t)|. \tag{3.56}$$

(3) Estimate of $\sum_{1\le|\alpha|\le N-1}\frac{1}{\alpha!}\left|(\bar{u}_m)^{\alpha_1}(\bar{v}_m)^{\alpha_2}\right|$. Note that

$$\begin{aligned}
\left|(\bar{u}_m)^{\alpha_1}(\bar{v}_m)^{\alpha_2}\right| &\le \left(\sqrt{R-1}\right)^{|\alpha|}\|\bar{u}_{mx}(t)\|_0^{\alpha_1-1}\|\bar{v}_{mx}(t)\|_0^{\alpha_2}\|\bar{u}_{mx}(t)\|_0 \\
&\le \left(\sqrt{R-1}\right)^{|\alpha|}M^{\alpha_1-1}M^{\alpha_2}\|\bar{u}_{mx}(t)\|_0 \\
&\le \left(\sqrt{R-1}\right)^{|\alpha|}M^{|\alpha|-1}\sqrt{\bar{Z}_m(t)}.
\end{aligned}\tag{3.57}$$

Therefore, by (3.57), we obtain

$$\begin{aligned}
\sum_{1\le|\alpha|\le N-1}\frac{1}{\alpha!}\left|(\bar{u}_m)^{\alpha_1}(\bar{v}_m)^{\alpha_2}\right| &\le \sum_{1\le|\alpha|\le N-1}\frac{1}{\alpha!}\left(\sqrt{R-1}\right)^{|\alpha|}M^{|\alpha|-1}\sqrt{\bar{Z}_m(t)} \\
&= \sum_{k=1}^{N-1}\sum_{|\alpha|=k}\frac{1}{\alpha!}\left(\sqrt{R-1}\right)^{k}M^{k-1}\sqrt{\bar{Z}_m(t)} \\
&= \frac{1}{M}\sum_{k=1}^{N-1}\frac{\left(2M\sqrt{R-1}\right)^k}{k!}\sqrt{\bar{Z}_m(t)}.
\end{aligned}\tag{3.58}$$

(4) Estimate of $R_m[f](x,t)$. We have

$$\begin{aligned}
|R_m[f](x,t)| &\le K_N(M,f)\sum_{|\alpha|=N}\frac{1}{\alpha!}\left|(\bar{u}_{m-1})^{\alpha_1}(\bar{v}_{m-1})^{\alpha_2}\right| \\
&\le K_N(M,f)\sum_{|\alpha|=N}\frac{1}{\alpha!}\left(\sqrt{R-1}\right)^{|\alpha|}\|\nabla\bar{u}_{m-1}(t)\|_0^{\alpha_1}\|\nabla\bar{v}_{m-1}(t)\|_0^{\alpha_2} \\
&\le K_N(M,f)\sum_{|\alpha|=N}\frac{1}{\alpha!}\left(\sqrt{R-1}\right)^{|\alpha|}\|(\bar{u}_{m-1},\bar{v}_{m-1})\|_{W_1(T)}^{|\alpha|} \\
&= K_N(M,f)\frac{\left(2\sqrt{R-1}\right)^N}{N!}\|(\bar{u}_{m-1},\bar{v}_{m-1})\|_{W_1(T)}^N.
\end{aligned}\tag{3.59}$$

It follows from (3.56), (3.58) and (3.59) that

$$\begin{aligned}
|F_{m+1}(x,t)-F_m(x,t)| \;\le\; & \frac{K_N(M,f)}{M}\sum_{k=1}^{N-1}\frac{\left(2M\sqrt{R-1}\right)^k}{k!}\sqrt{\bar{Z}_m(t)} \\
& +K_N(M,f)\frac{\left(2\sqrt{R-1}\right)^N}{N!}\|(\bar{u}_{m-1},\bar{v}_{m-1})\|_{W_1(T)}^N.
\end{aligned}$$

Hence

$$\|F_{m+1}(t)-F_m(t)\|_0 \leq E_1(M,f)\sqrt{\bar{Z}_m(t)} + E_2(M,f)\left\|(\bar{u}_{m-1},\bar{v}_{m-1})\right\|_{W_1(T)}^N, \tag{3.60}$$

where

$$\begin{aligned} E_1(M,f) &= \frac{K_N(M,f)}{M}\sqrt{\frac{R^2-1}{2}}\sum_{k=1}^{N-1}\frac{\left(2M\sqrt{R-1}\right)^k}{k!}, \\ E_2(M,f) &= K_N(M,f)\sqrt{\frac{R^2-1}{2}}\frac{\left(2\sqrt{R-1}\right)^N}{N!}. \end{aligned} \tag{3.61}$$

Now, we can estimate the integral J_1 as follows:

$$\begin{aligned} J_1 &= 2\int_0^t \left\langle F_{m+1}(s)-F_m(s), \bar{u}_m'(s)\right\rangle ds \\ &\leq 2\int_0^t \|F_{m+1}(s)-F_m(s)\|_0 \left\|\bar{u}_m'(s)\right\|_0 ds \\ &\leq 2\int_0^t \left(E_1(M,f)\sqrt{\bar{Z}_m(s)} + E_2(M,f)\left\|(\bar{u}_{m-1},\bar{v}_{m-1})\right\|_{W_1(T)}^N\right)\sqrt{\bar{Z}_m(s)}ds \\ &= 2E_1(M,f)\int_0^t \bar{Z}_m(s)ds + 2E_2(M,f)\left\|(\bar{u}_{m-1},\bar{v}_{m-1})\right\|_{W_1(T)}^N \int_0^t\sqrt{\bar{Z}_m(s)}ds \\ &\leq TE_2^2(M,f)\left\|(\bar{u}_{m-1},\bar{v}_{m-1})\right\|_{W_1(T)}^{2N} + (1+2E_1(M,f))\int_0^t \bar{Z}_m(s)ds. \end{aligned} \tag{3.62}$$

Next integral J_2. Similarly

$$\begin{aligned} J_2 &= 2\int_0^t \left\langle G_{m+1}(s)-G_m(s), \bar{v}_m'(s)\right\rangle ds \\ &\leq TE_2^2(M,g)\left\|(\bar{u}_{m-1},\bar{v}_{m-1})\right\|_{W_1(T)}^{2N} + (1+2E_1(M,g))\int_0^t \bar{Z}_m(s)ds, \end{aligned} \tag{3.63}$$

where

$$\begin{aligned} E_1(M,g) &= \frac{K_N(M,f)}{M}\sqrt{\frac{R^2-1}{2}}\sum_{k=1}^{N-1}\frac{\left(2M\sqrt{R-1}\right)^k}{k!}, \\ E_2(M,g) &= K_N(M,f)\sqrt{\frac{R^2-1}{2}}\frac{\left(2\sqrt{R-1}\right)^N}{N!}. \end{aligned} \tag{3.64}$$

Combining (3.48), (3.50), (3.51), (3.62) and (3.63), we obtain

$$\bar{Z}_m(t) \leq \tilde{E}_2(M,f,g)T\left\|(\bar{u}_{m-1},\bar{v}_{m-1})\right\|_{W_1(T)}^{2N} + 2\tilde{E}_1(M,f,g)\int_0^t \bar{Z}_m(s)ds, \tag{3.65}$$

where

$$\begin{aligned}\tilde{E}_1(M,f,g) &= \frac{1}{a_*}\left(1+E_1(M,f)+E_1(M,g)\right),\\ \tilde{E}_2(M,f,g) &= \frac{E_2^2(M,f)+E_2^2(M,g)}{a_*}.\end{aligned} \tag{3.66}$$

By Gronwall's lemma, we deduce from (3.65) that

$$\|(\bar{u}_m,\bar{v}_m)\|_{W_1(T)} \le \mu_T \|(\bar{u}_{m-1},\bar{v}_{m-1})\|_{W_1(T)}^N, \tag{3.67}$$

where $\mu_T = 4\sqrt{\tilde{E}_2(M,f,g)}\sqrt{T}\exp(T\tilde{E}_1(M,f,g))$ with $\beta_T = M\mu_T^{\frac{1}{N-1}} < 1$, which implies that

$$\|(u_m,v_m)-(u_{m+p},v_{m+p})\|_{W_1(T)} \le (1-\beta_T)^{-1}(\mu_T)^{\frac{-1}{N-1}}(\beta_T)^{N^m}, \ \forall m,p\in\mathbb{N}. \tag{3.68}$$

It follows that $\{(u_m,v_m)\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $(u,v)\in W_1(T)$ such that

$$(u_m,v_m)\to(u,v) \text{ strongly in } W_1(T). \tag{3.69}$$

Note that $(u_m,v_m)\in W_1(M,T)$, then there exists a subsequence $\{(u_{m_j},v_{m_j})\}$ of $\{(u_m,v_m)\}$ such that

$$\begin{cases} (u_{m_j},v_{m_j})\to(u,v) & \text{in } L^\infty(0,T;(H^2\cap V_R)^2) \text{ weak*},\\ (u'_{m_j},v'_{m_j})\to(u',v') & \text{in } L^\infty(0,T;V_R\times V_R) \text{ weak*},\\ (u''_{m_j},v''_{m_j})\to(u'',v'') & \text{in } L^2(Q_T)\times L^2(Q_T) \text{ weak},\\ (u,v)\in W(M,T). \end{cases} \tag{3.70}$$

We also note that

$$\begin{aligned}&\|F_m-f[u,v]\|_{L^\infty(0,T;L^2)}\\ &\le K_N(M,f)\sqrt{\frac{R^2-1}{2}}\Big[\sqrt{R-1}\,\|(u_{m-1},v_{m-1})-(u,v)\|_{W_1(T)}\\ &\quad+\sum_{k=1}^{N-1}\frac{\left(2\sqrt{R-1}\right)^k}{k!}\|(u_m,v_m)-(u_{m-1},v_{m-1})\|_{W_1(T)}^k\Big].\end{aligned} \tag{3.71}$$

Hence, from (3.69) and (3.71), we obtain

$$F_m(t)\to f[u,v] \text{ strongly in } L^\infty(0,T;L^2). \tag{3.72}$$

Similarly, we have that

$$G_m\to g[u,v] \text{ strongly in } L^\infty(0,T;L^2). \tag{3.73}$$

Finally, passing to limit in (3.5), (3.6) as $m = m_j \to \infty$, it implies from (3.69), $(3.70)_{1,2,3}$, (3.72) and (3.73) that there exists $(u, v) \in W(M, T)$ satisfying the equations

$$\begin{cases} \langle u''(t), w\rangle + a(u(t), w) = \langle f[u,v](t), w\rangle, \\ \langle v''(t), \phi\rangle + b(v(t), \phi) = \langle g[u,v](t), \phi\rangle, \end{cases} \tag{3.74}$$

for all $(w, \phi) \in V_R \times V_R$, a.e., $t \in (0, T)$, and the initial conditions

$$\big(u(0), u'(0)\big) = (\tilde{u}_0, \tilde{u}_1)\,,\ \big(v(0), v'(0)\big) = (\tilde{v}_0, \tilde{v}_1)\,. \tag{3.75}$$

On the other hand, from the assumption (A_2), we obtain from $(3.70)_4$, (3.72), (3.73) and (3.74), that

$$u'' = -L_1 u + f[u, v] \in L^\infty(0, T; L^2)$$

and

$$v'' = -L_2 v + g[u, v] \in L^\infty(0, T; L^2).$$

Thus, we have the solution $(u, v) \in W_1(M, T)$. The existence proof is completed.

(b) Uniqueness of the solution:

By applying a similar argument, which is used in the proof of Theorem 3.1, the solution $(u, v) \in W_1(M, T)$ is unique.

(c) The estimate (3.46):

Passing to the limit in (3.68) as $p \to +\infty$ for fixed m, we get (3.46). □

Remark 3.3. In order to construct a N-order iterative scheme, we need the condition $f \in C^N([0, 1] \times \mathbb{R}_+ \times \mathbb{R}^2)$. Then, we obtain a convergent sequence at a rate of order N to a local weak solution of the problem. We note that, this condition of f can be relaxed if we only consider the existence of solution (for more detail, we refer to [9]-[14]).

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