



## CONVERGENCE THEOREMS FOR TWO NONLINEAR MAPPINGS IN $CAT(0)$ SPACES

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**Abstract.** In this paper, we construct an iteration scheme involving a hybrid pair of the Suzuki generalized nonexpansive single-valued and multi-valued mappings in a complete  $CAT(0)$  space. In process, we remove a restricted condition (called end-point condition) in Akkasriworn and Sokhuma's results [2] in Banach spaces and utilize the same to prove some convergence theorems. The results in this paper, are analogs of the results of Akkasriworn et al. [3] in Banach spaces.

### 1. INTRODUCTION

Fixed point theory in a  $CAT(0)$  space was first studied by Kirk [12, 13]. He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete  $CAT(0)$  space always has a fixed point. Since then the existence problem of fixed point and the  $\Delta$ -convergence problem of iterative sequences to a fixed point for nonexpansive mappings, Suzuki generalized nonexpansive mappings in a  $CAT(0)$  space have been rapidly developed and have appeared in many papers [4, 10, 11, 18].

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Let  $(X, d)$  be a geodesic metric space. We denote by  $2^K$  the family of nonempty subsets of  $K$ , by  $FB(K)$  the collection of all nonempty closed bounded subsets of  $K$ , by  $KC(K)$  the collection of all nonempty compact convex subsets of  $K$ .

A subset  $K$  of  $X$  is called proximal if for each  $x \in X$ , there exists an element  $k \in K$  such that

$$d(x, k) = \text{dist}(x, K) = \inf\{d(x, y) : y \in K\}.$$

We denote by  $PB(K)$ , the collection of all nonempty bounded proximal subsets of  $K$ .

Let  $H$  be the Hausdorff metric with respect to  $d$ , that is,

$$H(A, B) = \max \left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A) \right\}, \quad A, B \in FB(X),$$

where  $\text{dist}(x, B) = \inf\{d(x, y) : y \in B\}$  is the distance from the point  $x$  to the subset  $B$ .

A mapping  $t : K \rightarrow K$  is said to be *nonexpansive* if

$$d(tx, ty) \leq d(x, y) \text{ for all } x, y \in K.$$

A mapping  $t : K \rightarrow K$  is said to be *Suzuki generalized nonexpansive* if

$$\frac{1}{2}d(x, tx) \leq d(x, y) \Rightarrow d(tx, ty) \leq d(x, y) \text{ for all } x, y \in K.$$

A point  $x$  is called a fixed point of  $t$  if  $tx = x$ .

A multi-valued mapping  $T : K \rightarrow FB(K)$  is said to be *nonexpansive* if

$$H(Tx, Ty) \leq d(x, y) \text{ for all } x, y \in K.$$

In 2010, Abkar and Eslamian [1] mentioned the Suzuki generalized multi-valued nonexpansive mapping as follows:

A multi-valued mapping  $T : K \rightarrow FB(K)$  is said to be a *Suzuki generalized multi-valued nonexpansive mapping* if

$$\frac{1}{2}\text{dist}(x, Tx) \leq d(x, y) \Rightarrow H(Tx, Ty) \leq d(x, y) \text{ for all } x, y \in K.$$

Let  $T : K \rightarrow PB(K)$  be a multi-valued mapping and define the mapping  $P_T$  for each  $x$  by

$$P_T(x) := \{y \in Tx : d(x, y) = \text{dist}(x, Tx)\}.$$

A point  $x$  is called a fixed point for a multivalued mapping  $T$  if  $x \in Tx$ .

We use the notation  $Fix(T)$  stands for the set of fixed points of a mapping  $T$  and  $Fix(t) \cap Fix(T)$  stands for the set of common fixed points of  $t$  and  $T$ . Precisely, a point  $x$  is called a common fixed point of  $t$  and  $T$  if  $tx = x \in Tx$ .

In 2009, Laokul and Panyanak [14] defined the iterative algorithm and proved the  $\Delta$ -convergence for nonexpansive mapping in  $CAT(0)$  spaces as follows:

**Theorem 1.1.** *Let  $K$  be a nonempty closed convex subset of a complete  $CAT(0)$  space and  $t : K \rightarrow K$  be a nonexpansive mapping with  $Fix(t) := \{x \in C : tx = x\} \neq \emptyset$ . Suppose the sequence  $\{x_n\}$  is generated iteratively by  $x_1 \in C$ ,*

$$\begin{cases} y_n = \beta_n tx_n \oplus (1 - \beta_n)x_n, \\ x_{n+1} = \alpha_n ty_n \oplus (1 - \alpha_n)x_n, \end{cases}$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are real sequences in  $[0, 1]$  such that one of the following two conditions is satisfied:

- (i)  $\alpha_n \in [a, b]$  and  $\beta_n \in [0, b]$  for some  $a, b$  with  $0 < a \leq b < 1$ ,
- (ii)  $\alpha_n \in [a, 1]$  and  $\beta_n \in [a, b]$  for some  $a, b$  with  $0 < a \leq b < 1$ .

Then the sequence  $\{x_n\}$  is  $\Delta$ -convergent to a fixed point of  $t$ .

In 2010, Sokhuma and Kaewkhao [21] proved the convergence theorem for a common fixed point in Banach spaces as follows:

**Theorem 1.2.** *Let  $K$  be a nonempty compact convex subset of a uniformly convex Banach space  $X$ , and  $t : K \rightarrow K$  and  $T : K \rightarrow KC(K)$  be a single-valued nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume in addition that  $Fix(t) \cap Fix(T) \neq \emptyset$  and  $Tw = \{w\}$  for all  $w \in Fix(t) \cap Fix(T)$ . Suppose that the iteration  $\{x_n\}$  is generated by  $x_1 \in K$ ,*

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n z_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n ty_n, \end{cases}$$

for all  $n \in \mathbb{N}$ , where  $z_n \in Tx_n$  and  $\{\alpha_n\}, \{\beta_n\}$  are sequences of positive numbers satisfying  $0 < a \leq \alpha_n, \beta_n \leq b < 1$ . Then the sequence  $\{x_n\}$  converges strongly to a common fixed point of  $t$  and  $T$ .

In 2013, Sokhuma [19] proved the convergence theorem for a common fixed point in  $CAT(0)$  spaces as follows:

**Theorem 1.3.** *Let  $K$  be a nonempty compact convex subset of a complete  $CAT(0)$  space  $X$ , and  $t : K \rightarrow K$  and  $T : K \rightarrow FC(K)$  a single-valued nonexpansive mapping and a multi-valued nonexpansive mapping, respectively, and  $Fix(t) \cap Fix(T) \neq \emptyset$  satisfying  $Tw = \{w\}$  for all  $w \in Fix(t) \cap Fix(T)$ . Let the iterative sequence  $\{x_n\}$  is generated by  $x_1 \in K$ ,*

$$\begin{cases} y_n = (1 - \beta_n)x_n \oplus \beta_n z_n, \\ x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n ty_n, \end{cases}$$

for all  $n \in \mathbb{N}$ , where  $z_n \in Tx_n$  and  $\{\alpha_n\}, \{\beta_n\}$  are sequences of positive numbers satisfying  $0 < a \leq \alpha_n, \beta_n \leq b < 1$ . Then the sequence  $\{x_n\}$  converges strongly to a common fixed point of  $t$  and  $T$ .

In 2015, Akkasriworn and Sokhuma [2] proved the convergence theorem for a common fixed point in a complete  $CAT(0)$  spaces as follow:

**Theorem 1.4.** *Let  $K$  be a nonempty bounded closed convex subset of a complete  $CAT(0)$  space  $X$ ,  $t : K \rightarrow K$  and  $T : K \rightarrow FB(K)$  an asymptotically nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume that  $t$  and  $T$  are commuting and  $Fix(t) \cap Fix(T) \neq \emptyset$  satisfying  $Tw = \{w\}$  for all  $w \in Fix(t) \cap Fix(T)$  and  $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ . Let  $\{x_n\}$  be the sequence of the modified Ishikawa iteration defined by*

$$\begin{cases} y_n = (1 - \beta_n)x_n \oplus \beta_n z_n, \\ x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n t^n y_n, \end{cases}$$

for all  $n \in \mathbb{N}$ , where  $z_n \in T(t^n x_n)$  and  $\{\alpha_n\}, \{\beta_n\} \in [0, 1]$ . Then  $\{x_n\}$  is  $\Delta$ -convergent to a common fixed point of  $t$  and  $T$ .

In 2019, Sokhuma [20] proved the convergence theorem for a common fixed point in  $CAT(0)$  spaces as follow:

**Theorem 1.5.** *Let  $K$  be a nonempty bounded closed convex subset of a complete  $CAT(0)$  space  $X$ ,  $t : K \rightarrow K$  be a single-valued asymptotically nonexpansive mapping, and  $T : K \rightarrow PB(K)$  be a multi-valued nonexpansive mapping and  $P_T(x) = \{y \in Tx : d(x, y) = dist(x, Tx)\}$ . For fixed  $x_1 \in K$ . The sequence  $\{x_n\}$  of the Ishikawa iteration is defined by*

$$\begin{cases} y_n = (1 - \beta_n)x_n \oplus \beta_n z_n, \\ x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n t^n y_n, \end{cases}$$

for all  $n \in \mathbb{N}$ , where  $z_n \in P_T(t^n x_n)$  and  $\{\alpha_n\}, \{\beta_n\} \in (0, 1)$ . Then  $\{x_n\}$  is  $\Delta$ -convergent to a common fixed point of  $t$  and  $T$ .

In 2011, Espinola et al. [9] proved the theorem for a common fixed point in  $CAT(0)$  spaces as follows:

**Theorem 1.6.** *Let  $X$  be a complete uniformly convex space with convex metric and  $K$  be a bounded closed convex subset of  $X$ . Suppose  $t : K \rightarrow K$  and  $T : K \rightarrow KC(X)$  be a Suzuki generalized nonexpansive single-valued and a multi-valued mapping, respectively. If  $t$  and  $T$  are commute, then there exists  $w \in K$  such that  $tw = w \in Tw$ .*

The purpose of this paper is to study the iterative process, called the Ishikawa iteration method with respect to the Suzuki generalized nonexpansive single valued and multivalued mapping in a complete  $CAT(0)$  space. We also establish the convergence theorem of a sequence from such process in a nonempty bounded closed convex subset of a complete  $CAT(0)$  spaces. We remove a restricted condition (called end-point condition) in Akkasriworn et al. [3] and Sokhuma [20] results.

Now, we introduce an iteration method modifying the above ones and call it the Ishikawa iteration method.

**Definition 1.7.** Let  $K$  be a nonempty bounded closed convex subset of a complete  $CAT(0)$  space  $X$ ,  $t : K \rightarrow K$  and  $T : K \rightarrow PB(K)$  be a Suzuki generalized nonexpansive single-valued and a multi-valued mapping, respectively and

$$P_T(x) = \{y \in Tx : d(x, y) = \text{dist}(x, Tx)\}.$$

For fixed  $x_1 \in K$ . The sequence  $\{x_n\}$  of the Ishikawa iteration is defined by

$$\begin{cases} y_n = (1 - \beta_n)x_n \oplus \beta_n z_n, \\ x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n t y_n, \end{cases} \tag{1.1}$$

for all  $n \in \mathbb{N}$ , where  $z_n \in P_T(tx_n)$  and  $\{\alpha_n\}, \{\beta_n\} \in (0, 1)$ .

## 2. PRELIMINARIES

With a view to make, our presentation self contained, we collect some relevant basic definitions, results and iterative methods which will be used frequently in the text later.

Let  $(X, d)$  be a metric space. A geodesic path joining  $x \in X$  to  $y \in X$  is a map  $c$  from a closed interval  $[0, s] \subset \mathbb{R}$  to  $X$  such that  $c(0) = x$ ,  $c(s) = y$ , and  $d(c(t), c(u)) = |t - u|$  for all  $t, u \in [0, s]$ . In particular,  $c$  is an isometry and  $d(x, y) = s$ . The image  $\alpha$  of  $c$  is called a geodesic (or metric) segment joining  $x$  and  $y$ . When it is unique this geodesic segment is denoted by  $[x, y]$ . The space  $(X, d)$  is said to be a geodesic space if every two points of  $X$  are joined by a geodesic, and  $X$  is said to be uniquely geodesic if there is exactly one geodesic joining  $x$  and  $y$  for each  $x, y \in X$ . A subset  $Y \subseteq X$  is said to be convex if  $Y$  includes every geodesic segment joining any two of its points.

A geodesic triangle  $\Delta(x_1, x_2, x_3)$  in a geodesic metric space  $(X, d)$  consists of three points  $x_1, x_2, x_3$  in  $X$  (the vertices of  $\Delta$ ) and a geodesic segment between each pair of vertices (the edges of  $\Delta$ ). A comparison triangle for the geodesic triangle  $\Delta(x_1, x_2, x_3)$  in  $(X, d)$  is a triangle  $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$  in the Euclidean plane  $\mathbb{E}^2$  such that  $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$  for  $i, j \in \{1, 2, 3\}$ .

A geodesic space is said to be a  $CAT(0)$  space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

Let  $\Delta$  be a geodesic triangle in  $X$  and let  $\bar{\Delta}$  be a comparison triangle for  $\Delta$ . Then  $\Delta$  is said to satisfy the  $CAT(0)$  inequality if for all  $x, y \in \Delta$  and all comparison points  $\bar{x}, \bar{y} \in \bar{\Delta}$ ,  $d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y})$ . If  $x, y_1, y_2$  are points in a  $CAT(0)$  space and

$$y_0 = \frac{1}{2}y_1 \oplus \frac{1}{2}y_2,$$

then the  $CAT(0)$  inequality implies that

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \quad (2.1)$$

This is the  $(CN)$  inequality of Bruhat and Tits [6]. In fact, a geodesic space is a  $CAT(0)$  space if and only if it satisfies the  $(CN)$  inequality [5].

The following results and methods deal with the concept of asymptotic centers. Let  $K$  be a nonempty closed convex subset of a  $CAT(0)$  space  $X$  and  $\{x_n\}$  be a bounded sequence in  $X$ . For  $x \in X$ , define the asymptotic radius of  $\{x_n\}$  at  $x$  as the number

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x_n, x).$$

Let

$$r \equiv r(K, \{x_n\}) := \inf \{r(x, \{x_n\}) : x \in K\}$$

and

$$A \equiv A(K, \{x_n\}) := \{x \in K : r(x, \{x_n\}) = r\}.$$

The number  $r$  and the set  $A$  are called the asymptotic radius and asymptotic center of  $\{x_n\}$  relative to  $K$ , respectively.

It is easy to know that if  $X$  is complete  $CAT(0)$  spaces and  $K$  is a closed convex subset of  $X$ , then  $A(K, \{x_n\})$  consists of exactly one point. A sequence  $\{x_n\}$  in  $CAT(0)$  space  $X$  is said to be  $\Delta$ -convergent to  $x \in X$  if  $x$  is the unique asymptotic center of every subsequence of  $\{x_n\}$ . A bounded sequence  $\{x_n\}$  is said to be regular with respect to  $K$  if for every subsequence  $\{x'_n\}$ , we get

$$r(K, \{x_n\}) = r(K, \{x'_n\}).$$

We now give the definition of  $\Delta$ -convergence.

**Definition 2.1.** ([13, 16]) A sequence  $\{x_n\}$  in a  $CAT(0)$  space  $X$  is said to be  $\Delta$ -convergent to  $x \in X$  if  $x$  is the unique asymptotic center of  $\{u_n\}$  for every subsequence  $\{u_n\}$  of  $\{x_n\}$ . In this case we write  $\Delta - \lim_{n \rightarrow \infty} x_n = x$  and call  $x$  the  $\Delta$ -limit of  $\{x_n\}$ .

We now collect some elementary facts about  $CAT(0)$  spaces which will be used in the proofs of our main results. The following lemma can be found in [7, 8, 13].

**Lemma 2.2.** ([7]) *If  $K$  is a closed convex subset of a complete  $CAT(0)$  space and  $\{x_n\}$  is a bounded sequence in  $K$ , then the asymptotic center of  $\{x_n\}$  is in  $K$ .*

**Lemma 2.3.** ([8]) *Let  $(X, d)$  be a  $CAT(0)$  space.*

- (i) *For  $x, y \in X$  and  $u \in [0, 1]$ , there exists a unique point  $z \in [x, y]$  such that*

$$d(x, z) = ud(x, y) \quad \text{and} \quad d(y, z) = (1 - u)d(x, y). \tag{2.2}$$

*We use the notation  $(1 - u)x \oplus uy$  for the unique point  $z$  satisfying (2.2).*

- (ii) *For  $x, y, z \in X$  and  $u \in [0, 1]$ , we have*

$$d((1 - u)x \oplus uy, z) \leq (1 - u)d(x, z) + ud(y, z).$$

**Lemma 2.4.** ([13]) *Every bounded sequence in a complete  $CAT(0)$  space has a  $\Delta$ -convergent subsequence.*

We now collect some basic properties of Suzuki generalized nonexpansive mapping. Although the proofs follow the idea of the proofs in [22]. The following two propositions are very easy to verify.

**Proposition 2.5.** *Let  $K$  be a nonempty subset of a  $CAT(0)$  space  $X$  and  $t : K \rightarrow K$  be a nonexpansive mapping. Then  $t$  is a Suzuki generalized nonexpansive mapping.*

**Proposition 2.6.** *Let  $K$  be a nonempty subset of a  $CAT(0)$  space  $X$ . Suppose  $t : K \rightarrow K$  is a Suzuki generalized nonexpansive mapping and has a fixed point. Then  $t$  is a quasi-nonexpansive mapping.*

**Lemma 2.7.** *Let  $K$  be a nonempty subset of a  $CAT(0)$  space  $X$ . Suppose  $t : K \rightarrow K$  is a Suzuki generalized nonexpansive mapping. Then*

$$d(x, ty) \leq 3d(tx, x) + d(x, y)$$

*holds for all  $x, y \in K$ .*

The existence of fixed points for generalized Suzuki nonexpansive mappings in  $CAT(0)$  spaces was proved by Nanjaras et al. [17] as the following result.

**Theorem 2.8.** *Let  $K$  be a nonempty bounded closed convex subset of a complete  $CAT(0)$  space  $X$ . Suppose  $t : K \rightarrow K$  is a Suzuki generalized nonexpansive mappings. Then  $t$  has a fixed point in  $K$ .*

**Lemma 2.9.** *Let  $K$  be a closed and convex subset of a complete  $CAT(0)$  space  $X$  and let  $t : K \rightarrow X$  be a generalized Suzuki nonexpansive mappings. Let  $\{x_n\}$  be a bounded sequence in  $K$  such that  $\lim_{n \rightarrow \infty} d(tx_n, x_n) = 0$  and  $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = w$ . Then  $tw = w$ .*

The following fact is well-known [15].

**Lemma 2.10.** *Let  $X$  be a complete  $CAT(0)$  space and let  $x \in X$ . Suppose  $\{\alpha_n\}$  is a sequence in  $[a, b]$  for some  $a, b \in (0, 1)$  and  $\{x_n\}, \{y_n\}$  are sequences in  $X$  such that  $\limsup_{n \rightarrow \infty} d(x_n, x) \leq r$ ,  $\limsup_{n \rightarrow \infty} d(y_n, x) \leq r$ , and  $\lim_{n \rightarrow \infty} d((1 - \alpha_n)x_n \oplus \alpha_n y_n, x) = r$  for some  $r \geq 0$ . Then  $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ .*

**Lemma 2.11.** *Let  $X$  be a  $CAT(0)$  space,  $K$  be a nonempty compact convex subset of  $X$  and  $\{x_n\}$  be a sequence in  $K$ . Then,*

$$\text{dist}(y, Ty) \leq d(y, x_n) + \text{dist}(x_n, Tx_n) + H(Tx_n, Ty),$$

where  $y \in K$  and  $T$  be a multi-valued mapping from  $K$  in to  $FB(K)$ .

### 3. MAIN RESULTS

We first prove the following lemmas which play very important roles in this section.

**Lemma 3.1.** *Let  $T : K \rightarrow PB(K)$  be a multi-valued mapping and*

$$P_T(x) = \{y \in Tx : d(x, y) = \text{dist}(x, Tx)\}.$$

*Then the followings are equivalent:*

- (1)  $x \in \text{Fix}(T)$ , that is  $x \in Tx$ ;
- (2)  $P_T(x) = \{x\}$ , that is  $x = y$  for each  $y \in P_T(x)$ ;
- (3)  $x \in \text{Fix}(P_T)$ , that is  $x \in P_T(x)$ .

*Further,  $\text{Fix}(T) = \text{Fix}(P_T)$ .*

*Proof.* (1)  $\implies$  (2). Since  $x \in Tx$ ,  $d(x, Tx) = 0$ . Therefore, for any  $y \in P_T(x)$ ,  $d(x, y) = \text{dist}(x, Tx) = 0$  and so  $x = y$ . That is,  $P_T(x) = \{x\}$ .

(2)  $\implies$  (3). Since  $P_T(x) = \{x\}$ ,  $x \in \text{Fix}(P_T)$  and we get  $x \in P_T(x)$ .

(3)  $\implies$  (1). Since  $x \in \text{Fix}(P_T)$ ,  $x \in P_T(x)$ . Therefore,

$$d(x, x) = \text{dist}(x, Tx) = 0$$

and so  $x \in Tx$  by the closedness of  $Tx$ . This implies that  $\text{Fix}(T) = \text{Fix}(P_T)$ .  $\square$



**Lemma 3.2.** *Let  $K$  be a nonempty bounded closed convex subset of a complete CAT(0) space  $X$ ,  $t : K \rightarrow K$  and  $T : K \rightarrow PB(K)$  be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with  $Fix(t) \cap Fix(T) \neq \emptyset$  such that  $P_T$  is nonexpansive. Let  $\{x_n\}$  be the sequence of Ishikawa iteration defined by (1.1). Then  $\lim_{n \rightarrow \infty} d(x_n, w)$  exists for all  $w \in Fix(t) \cap Fix(T)$ .*

*Proof.* Let  $x_1 \in K$  and  $w \in Fix(t) \cap Fix(T)$ , in view of Lemma 3.1 we have  $w \in P_T(w) = \{w\}$ . Since,  $\frac{1}{2}d(tw, w) = 0 \leq d(x_n, w)$ ,  $d(tx_n, tw) \leq d(x_n, w)$ . Similarly, we obtain  $\frac{1}{2}d(tw, w) = 0 \leq d(y_n, w)$ , and then we get  $d(ty_n, tw) \leq d(y_n, w)$ . Now consider,

$$\begin{aligned} d(y_n, w) &= d((1 - \beta_n)x_n \oplus \beta_n z_n, w) \\ &\leq (1 - \beta_n)d(x_n, w) + \beta_n d(z_n, w) \\ &= (1 - \beta_n)d(x_n, w) + \beta_n dist(z_n, P_T(w)) \\ &\leq (1 - \beta_n)d(x_n, w) + \beta_n H(P_T(tx_n), P_T(w)) \\ &\leq (1 - \beta_n)d(x_n, w) + \beta_n d(tx_n, w) \\ &= (1 - \beta_n)d(x_n, w) + \beta_n d(tx_n, tw) \\ &\leq (1 - \beta_n)d(x_n, w) + \beta_n d(x_n, w) \\ &= d(x_n, w). \end{aligned}$$

Hence,  $d(y_n, w) \leq d(x_n, w)$ . It implies that

$$\begin{aligned} d(x_{n+1}, w) &= d((1 - \alpha_n)x_n \oplus \alpha_n ty_n, w) \\ &\leq (1 - \alpha_n)d(x_n, w) + \alpha_n d(ty_n, tw) \\ &= (1 - \alpha_n)d(x_n, w) + \alpha_n d(y_n, w) \\ &\leq (1 - \alpha_n)d(x_n, w) + \alpha_n d(x_n, w) \\ &= d(x_n, w). \end{aligned}$$

Since  $\{d(x_n, w)\}$  is bounded below and decreasing sequence, we obtain the limit of  $\{d(x_n, w)\}$ . □

**Lemma 3.3.** *Let  $K$  be a nonempty bounded closed convex subset of a complete CAT(0) space  $X$ ,  $t : K \rightarrow K$  and  $T : K \rightarrow PB(K)$  be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with  $Fix(t) \cap Fix(T) \neq \emptyset$  such that  $P_T$  is nonexpansive. Let  $\{x_n\}$  be the sequence of Ishikawa iteration defined by (1.1). If  $0 < a \leq \alpha_n \leq b < 1$  for some  $a, b \in \mathbb{R}$ . Then  $\lim_{n \rightarrow \infty} d(ty_n, x_n) = 0$ .*

*Proof.* Let  $x_1 \in K$  and  $w \in Fix(t) \cap Fix(T)$ , in view of Lemma 3.1 we have  $w \in P_T(w) = \{w\}$ . From Lemma 3.2, we setting  $\lim_{n \rightarrow \infty} d(x_n, w) = c$ . Recall

that,  $d(ty_n, w) = d(ty_n, tw) \leq d(y_n, w) \leq d(x_n, w)$ . Then we have,

$$\limsup_{n \rightarrow \infty} d(ty_n, w) \leq \limsup_{n \rightarrow \infty} d(y_n, w) \leq \limsup_{n \rightarrow \infty} d(x_n, w) = c. \quad (3.1)$$

Since,  $c = \lim_{n \rightarrow \infty} d(x_{n+1}, w) = \lim_{n \rightarrow \infty} d((1 - \alpha_n)x_n \oplus \alpha_n ty_n, w)$ , it follows from the condition of  $\alpha_n$  and Lemma 2.10 that  $\lim_{n \rightarrow \infty} d(ty_n, x_n) = 0$ .  $\square$

**Lemma 3.4.** *Let  $K$  be a nonempty bounded closed convex subset of a complete CAT(0) space  $X$ ,  $t : K \rightarrow K$  and  $T : K \rightarrow PB(K)$  be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with  $\text{Fix}(t) \cap \text{Fix}(T) \neq \emptyset$  such that  $P_T$  is nonexpansive. Let  $\{x_n\}$  be the sequence of Ishikawa iteration defined by (1.1). If  $0 < a \leq \alpha_n \leq b < 1$  for some  $a, b \in \mathbb{R}$ . Then  $\lim_{n \rightarrow \infty} d(x_n, z_n) = 0$ .*

*Proof.* Let  $x_0 \in K$  and  $w \in \text{Fix}(t) \cap \text{Fix}(T)$ , in view of Lemma 3.1 we have  $w \in P_T(w) = \{w\}$ . Consider,

$$\begin{aligned} d(x_{n+1}, w) &= d((1 - \alpha_n)x_n \oplus \alpha_n ty_n, w) \\ &\leq (1 - \alpha_n)d(x_n, w) + \alpha_n d(ty_n, w) \\ &\leq (1 - \alpha_n)d(x_n, w) + \alpha_n d(y_n, w) \\ &= d(x_n, w) - \alpha_n d(x_n, w) + \alpha_n d(y_n, w) \\ &= d(x_n, w) + \alpha_n (d(y_n, w) - d(x_n, w)) \end{aligned}$$

and hence

$$\frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n} \leq d(y_n, w) - d(x_n, w).$$

Therefore, since  $0 < a \leq \alpha_n \leq b < 1$ ,

$$\left( \frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n} \right) + d(x_n, w) \leq d(y_n, w).$$

Thus,

$$\liminf_{n \rightarrow \infty} \left\{ \left( \frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n} \right) + d(x_n, w) \right\} \leq \liminf_{n \rightarrow \infty} d(y_n, w).$$

It follows that

$$c \leq \liminf_{n \rightarrow \infty} d(y_n, w).$$

Since, from (3.1),  $\limsup_{n \rightarrow \infty} d(y_n, w) \leq c$ , we have

$$c = \lim_{n \rightarrow \infty} d(y_n, w) = \lim_{n \rightarrow \infty} d((1 - \beta_n)x_n \oplus \beta_n z_n, w). \quad (3.2)$$

Recall that

$$d(z_n, w) = \text{dist}(z_n, P_T(w)) \leq H(P_T(tx_n), P_T(w)) \leq d(tx_n, w) \leq d(x_n, w).$$

Hence we have

$$\limsup_{n \rightarrow \infty} d(z_n, w) \leq \limsup_{n \rightarrow \infty} d(x_n, w) = c.$$

Since  $0 < a \leq \alpha_n \leq b < 1$  for some  $a, b \in \mathbb{R}$  and (3.2) we obtain

$$\lim_{n \rightarrow \infty} d(x_n, z_n) = 0.$$

□

**Lemma 3.5.** *Let  $K$  be a nonempty bounded closed convex subset of a complete CAT(0) space  $X$ ,  $t : K \rightarrow K$  and  $T : K \rightarrow PB(K)$  be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with  $Fix(t) \cap Fix(T) \neq \emptyset$  such that  $P_T$  is nonexpansive. Let  $\{x_n\}$  be the sequence of Ishikawa iteration defined by (1.1). If  $0 < a \leq \alpha_n, \beta_n \leq b < 1$  for some  $a, b \in \mathbb{R}$ . Then  $\lim_{n \rightarrow \infty} d(tx_n, x_n) = 0$ .*

*Proof.* By Lemma 2.7, we obtain

$$\begin{aligned} d(tx_n, x_n) &\leq d(tx_n, y_n) + d(y_n, x_n) \\ &\leq 3d(ty_n, y_n) + d(x_n, y_n) + d(x_n, y_n) \\ &= 3d(ty_n, y_n) + 2d(x_n, y_n) \\ &\leq 3d(ty_n, x_n) + 3d(x_n, y_n) + 2d(x_n, y_n) \\ &= 3d(ty_n, x_n) + 5d(x_n, y_n) \\ &= 3d(ty_n, x_n) + 5d(x_n, (1 - \beta_n)x_n \oplus \beta_n z_n) \\ &\leq 3d(ty_n, x_n) + 5(1 - \beta_n)d(x_n, x_n) + 5\beta_n d(x_n, z_n) \\ &= 3d(ty_n, x_n) + 5\beta_n d(x_n, z_n). \end{aligned}$$

Therefore, we have

$$\lim_{n \rightarrow \infty} d(tx_n, x_n) \leq \lim_{n \rightarrow \infty} 3d(ty_n, x_n) + \lim_{n \rightarrow \infty} 5\beta_n d(x_n, z_n).$$

Hence, by Lemma 3.3 and Lemma 3.4,  $\lim_{n \rightarrow \infty} d(tx_n, x_n) = 0$ .

□

**Theorem 3.6.** *Let  $K$  be a nonempty bounded closed convex subset of a complete CAT(0) space  $X$ ,  $t : K \rightarrow K$  and  $T : K \rightarrow PB(K)$  be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with  $Fix(t) \cap Fix(T) \neq \emptyset$  such that  $P_T$  is nonexpansive. Let  $\{x_n\}$  be the sequence of Ishikawa iteration defined by (1.1). If  $0 < a \leq \alpha_n, \beta_n \leq b < 1$  for some  $a, b \in \mathbb{R}$ , then  $\{x_n\}$  is  $\Delta$ -convergent to  $y$  in  $Fix(t) \cap Fix(T)$ .*

*Proof.* Since  $\{x_n\}$  is  $\Delta$ -convergent to  $y$ , from Lemma 3.5,  $\lim_{n \rightarrow \infty} d(tx_n, x_n) = 0$ . By Lemma 2.9,  $y \in K$  and  $ty = y$ , it follows that  $y \in Fix(t)$ . By Lemma 2.11,

which implies that

$$\begin{aligned} \text{dist}(y, P_T(y)) &\leq d(y, x_n) + \text{dist}(x_n, P_T(tx_n)) + H(P_T(tx_n), P_T(y)) \\ &\leq d(y, x_n) + d(x_n, z_n) + d(tx_n, y) \\ &\leq d(x_n, y) + d(x_n, z_n) + d(tx_n, x_n) + d(x_n, y) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

It follows that,  $y \in \text{Fix}(P_T)$  then  $y \in \text{Fix}(T)$ . Therefore  $y \in \text{Fix}(t) \cap \text{Fix}(T)$  as desired.  $\square$

**Theorem 3.7.** *Let  $K$  be a nonempty bounded closed convex subset of a complete  $CAT(0)$  space  $X$ ,  $t : K \rightarrow K$  and  $T : K \rightarrow PB(K)$  be a single-valued and a multi-valued of Suzuki generalized nonexpansive mapping, respectively, with  $\text{Fix}(t) \cap \text{Fix}(T) \neq \emptyset$  such that  $P_T$  is nonexpansive. Let  $\{x_n\}$  be the sequence of Ishikawa iteration defined by (1.1). If  $0 < a \leq \alpha_n, \beta_n \leq b < 1$  for some  $a, b \in \mathbb{R}$ , then  $\{x_n\}$  is  $\Delta$ -convergent to a common fixed point of  $t$  and  $T$ .*

*Proof.* Since Lemma 3.5 guarantees that  $\{x_n\}$  is bounded and  $\lim_{n \rightarrow \infty} d(tx_n, x_n) = 0$ . We now let  $\omega_w(x_n) := \bigcup A(\{u_n\})$  where the union is taken over all subsequences  $\{u_n\}$  of  $\{x_n\}$ . We claim that  $\omega_w(x_n) \subset \text{Fix}(t) \cap \text{Fix}(T)$ , then there exists a subsequence  $\{u_n\}$  of  $\{x_n\}$  such that  $A(\{u_n\}) = \{u\}$ . By Lemma 2.2 and Lemma 2.3 there exists a subsequence  $\{v_n\}$  of  $\{u_n\}$  such that  $\Delta - \lim_{n \rightarrow \infty} v_n = v \in K$ . Since  $\lim_{n \rightarrow \infty} d(tv_n, v_n) = 0$ , then  $v \in \text{Fix}(t)$ . Since,

$$\begin{aligned} \text{dist}(v, P_T(v)) &\leq \text{dist}(v, P_T(tv_n)) + H(P_T(tv_n), P_T(v)) \\ &\leq d(v, z_n) + d(tv_n, v) \\ &\leq d(v, v_n) + d(v_n, z_n) + d(tv_n, v) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

It follows that  $v \in \text{Fix}(P_T)$ , we get  $v \in \text{Fix}(T)$  by Lemma 3.1. Therefore  $v \in \text{Fix}(t) \cap \text{Fix}(T)$  as desired. We claim that  $u = v$ . If not, since  $t$  is a single-valued Suzuki generalized mapping and  $v \in \text{Fix}(t) \cap \text{Fix}(T)$  such that  $\lim_{n \rightarrow \infty} d(x_n, v)$  exists by Lemma 3.2, then by the uniqueness of asymptotic center,

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(v_n, v) &< \limsup_{n \rightarrow \infty} d(v_n, u) \\ &\leq \limsup_{n \rightarrow \infty} d(u_n, u) \\ &< \limsup_{n \rightarrow \infty} d(u_n, v) \\ &= \limsup_{n \rightarrow \infty} d(x_n, v) \\ &= \limsup_{n \rightarrow \infty} d(v_n, v), \end{aligned}$$

which is a contradiction, and hence  $u = v \in \text{Fix}(t) \cap \text{Fix}(T)$ .

To show that  $\{x_n\}$  is  $\Delta$ -convergent to a common fixed point, it suffices to show that  $\omega_w(x_n)$  consists of exactly one point. Let  $\{u_n\}$  be a subsequence of  $\{x_n\}$ . By Lemma 2.2 and Lemma 2.3 there exists a subsequence  $\{v_n\}$  of  $\{u_n\}$  such that  $\Delta - \lim_{n \rightarrow \infty} v_n = v \in K$ . Let  $A(\{u_n\}) = \{u\}$  and  $A(\{x_n\}) = \{x\}$ . We have seen that  $u = v$  and  $v \in \text{Fix}(t) \cap \text{Fix}(T)$ .

We can complete the proof by showing that  $x = v$ . If not, since  $\lim_{n \rightarrow \infty} d(x_n, v)$  exists, by the uniqueness of asymptotic center,

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(v_n, v) &< \limsup_{n \rightarrow \infty} d(v_n, x) \\ &\leq \limsup_{n \rightarrow \infty} d(x_n, x) \\ &< \limsup_{n \rightarrow \infty} d(x_n, v) \\ &= \limsup_{n \rightarrow \infty} d(v_n, v) \end{aligned}$$

which is a contradiction, and hence the conclusion follows. □

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REFERENCES

- [1] A. Abkar and M. Eslamian, *Fixed Point Theorems for Suzuki Generalized Nonexpansive Multivalued Mappings in Banach Space*, Fixed Point Theory Appl., (2010); doi:10.1155/2010/457935.
- [2] N. Akkasriworn and K. Sokhuma, *Convergence theorem for a pair of asymptotically and multivalued nonexpansive mapping*, Commun. Korean Math. Soc., **30** (2015), 177–189.
- [3] N. Akkasriworn, K. Sokhuma and K. Chuikamwong, *Ishikawa iterative process for a pair of Suzuki generalized nonexpansive single valued and multivalued mappings in Banach spaces*, Int. J. Math. Anal., **19** (2012), 923–932.
- [4] M. Asadi, Sh. Ghasemzadehdibagi, S. Haghayeghi and N. Ahmad, *Fixed point theorems for  $(\alpha, p)$ -nonexpansive mappings in CAT(0) spaces*, Nonlinear Funct. Anal. Appl., **26**(5) (2021), 1045-1057.
- [5] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer, Berlin, 1999.
- [6] F. Bruhat and J. Tits, *Groupes reductifs sur un corps local I. Donnees radicielles valuees*, Inst. Hautes Etudes Sci. Publ. Math., **41** (1972), 5–251.
- [7] S. Dhompongsa, W.A. Kirk and B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal., **8** (2007), 35–45.
- [8] S. Dhompongsa, W.A. Kirk and B. Sims, *Fixed points of uniformly Lipschitzian mappings*, Nonlinear Anal., **65** (2006), 762–772.
- [9] R. Espinola, P. Lorenzo and A. Nicolae, *Fixed points, selections and common fixed points for nonexpansive-type Mappings*, J. Math. Anal. Appl., **382** (2011), 503–515.
- [10] J.K. Kim, R.P. Pathak, S. Dashputre, S.D. Diwan and R.L. Gupta, *Demiclosedness principle and convergence theorems for Lipschitzian type nonself-mappings in CAT(0) spaces*, Nonlinear Funct. Anal. Appl., **23**(1) (2018), 73-95.

- [11] K.S. Kim, *Existence theorem of a fixed point for asymptotically nonexpansive nonself mapping in  $CAT(0)$  spaces*, Nonlinear Funct. Anal. Appl., **25**(2) (2020), 355-362.
- [12] W.A. Kirk, *Geodesic geometry and fixed point theory. In: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003)*. Colecc. Abierta, vol.64, Univ. Sevilla Secr. Publ., Seville, **6** (2003), 195–225.
- [13] W.A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal., **68** (2008), 3689–3696.
- [14] T. Laokul and B. Panyanak, *Approximating Fixed Points of Nonexpansive Mappings in  $CAT(0)$  Spaces*, Int. J. Math. Anal., **3** (2009), 1305–1315.
- [15] W. Laowang and B. Panyanak, *Approximating fixed points of nonexpansive nonself mappings in  $CAT(0)$  spaces*, Fixed Point Theory Appl., (2010); doi:10.1155/2010/367274.
- [16] T.C. Lim, *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc., **60** (1976), 179–182.
- [17] B. Nanjaras, B. Panyanaka and W. Phuengrattana, *Fixed point theorems and convergence theorems for Suzuki-generalized nonexpansive mappings in  $CAT(0)$  spaces*, Nonlinear Analysis: Hybrid Syst., **4** (2010), 25–31.
- [18] G.A. Okeke, M. Abbas and M. de la Sen, *Fixed point theorems for convex minimization problems in complex valued  $CAT(0)$  spaces*, Nonlinear Funct. Anal. Appl., **25**(4) (2020), 671-696.
- [19] K. Sokhuma,  *$\Delta$ -Convergence Theorems for a Pair of Single valued and Multivalued Nonexpansive Mappings in  $CAT(0)$  spaces*, J. Math. Anal., **4** (2013), 23–31.
- [20] K. Sokhuma, *An Ishikawa Iteration Scheme for two Nonlinear Mappings in  $CAT(0)$  Spaces*, Kyungpook Math. J., **59** (2019), 665–678.
- [21] K. Sokhuma and A. Kaewkhao, *Ishikawa Iterative Process for a Pair of Single-valued and Multivalued Nonexpansive Mappings in Banach Spaces*, Fixed Point Theory Appl., (2010); doi:10.1155/2010/618767.
- [22] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mapping*, J. Math. Anal. Appl., **340** (2008), 1088–1095.