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ON THE STABILITY OF DIFFERENTIAL SYSTEMS INVOLVING ψ-HILFER FRACTIONAL DERIVATIVE

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Abstract. This paper deals with the stability of solutions to ψ-Hilfer fractional differential systems. We derive the fundamental solution for the system by using the generalized Laplace transform and the Mittag-Leffler function with two parameters. In addition, we obtained some necessary conditions on the stability of the solutions to linear fractional differential systems for homogeneous, non-homogeneous and non-autonomous cases. Numerical examples are also given to illustrate the behavior of solutions.

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1. Introduction

The theories of fractional differential systems are widely acknowledged due to the physical properties which are powerful gadgets to describe the real-world phenomenon. By the characteristics we mentioned above, fractional calculus caught a lot of attention in application with other research fields, such as chemistry, engineering, modelling, viscoelastic and the others [6, 15, 17]. In the last decades, there are many definitions of fractional derivatives presented by expert mathematicians. There are two senses that caught the most attention in differential equation which are Caputo fractional derivative and Riemann-Liouville fractional derivative. These two common definitions leads to several generalization of fractional derivatives such as fractional derivatives of a function with respect to another function [3, 7, 14], variable-order fractional derivatives [2, 9, 18] and Hilfer derivative [6] which interposed between fractional derivatives in the sense of Riemann-Liouville and Caputo.

On the other hand, the fundamental solutions of linear differential systems with integer order are usually written in the form of an exponential function. However, researchers combined the definition of fractional calculus with the differential systems which is known as fractional differential systems and caught much interest. For fractional order systems, fundamental solutions are displayed in a more general form of Mittag-Leffler function which interpolates between normal exponential function and the function related to power-law also know as Lorentzian function. These make properties and stability of solutions to fractional differential systems are the main contemplate considered in several pieces of research.

Many researchers agree that stability theories are considered as the important tools in control theory due to the physical properties which visualized that delicate perturbation does not produce disruptive results in the systems such as spring damping and small oscillating pendulum. Firstly, the stability of autonomous Caputo fractional differential systems is originally been proposed by Matignon [10] in the year 1996. Then other researchers investigate and present more works in the field of stability to fractional differential systems such as stability theorems for Riemann-Liouville fractional differential systems [12], linear fractional differential equations with constant coefficients [4] and so on.

In 2015, the stability result of Hilfer fractional differential systems where $y \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)_{n\times 1}$ are matrices such that $0 < \alpha_i < 1$, $0 \le \beta \le 1$ and $\gamma_i = \alpha_i + \beta - \alpha_i\beta$.

$$
\begin{aligned}
{}_{0^+}D_t^{\alpha,\beta} y(t) &= Ay(t), \\
{}_{0^+}I_t^{\mathbf{1}-\gamma} y(0^+) &= y_0
\end{aligned}
$$

with the fundamental solution

$$y(t) = y_0 t^{\gamma-1} E_{\alpha,\gamma}(At^{\alpha}) \tag{1.1}$$

was studied in [13]. In 2018, Jarad et al. [7] studies generalized fractional derivatives and Laplace transform for solving the following Cauchy problem involving a generalized Riemann-Liouville fractional derivative of the form:

$$\begin{aligned} &{}_aD^{\alpha}_{\psi}y(t) - \lambda y(t) = f(t) \quad t > a, \quad \alpha \in (0,1], \quad \lambda \in \mathbb{R}, \\ &{}_aI^{1-\alpha}_{\psi}y(a^+) = c, \quad c \in \mathbb{R} \end{aligned}$$

and the Cauchy problem of generalized Caputo fractional derivative of the form:

$$\begin{aligned} &{}^C_aD^{\alpha}_{\psi}y(t) - \lambda y(t) = f(t) \quad t > a, \quad \alpha \in (0,1], \quad \lambda \in \mathbb{R}, \\ &y(a^+) = c, \quad c \in \mathbb{R}. \end{aligned}$$

Obviously, the functions

$$y(t) = (\psi(t) - \psi(a))^{\alpha-1} E_{\alpha,\alpha}(A(\psi(t) - \psi(a))^{\alpha})c$$

and

$$y(t) = E_{\alpha}(A(\psi(t) - \psi(a))^{\alpha})c$$

are the fundamental solutions of generalized Riemann-Liouville fractional differential and generalized Caputo fractional differential systems, respectively.

Motivated by these works, the purpose of our research is to extend the stability result of Hilfer fractional differential systems to the generalized Hilfer fractional derivative with respect to another function (ψ-Hilfer derivative) given by

$$\begin{aligned} &{}_aD^{\alpha,\beta}_{\psi}y(t) = Ay(t) + B(t), \\ &{}_aI^{(1-\beta)(\mathbf{1}-\alpha)}_{\psi}y(a) = C, \end{aligned} \tag{1.2}$$

where $y \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is a matrix and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a vector such that $0 < \alpha_i < 1$, $0 \le \beta \le 1$ and $B(t) : [a,\infty) \to \mathbb{R}^{n\times n}$ is continuous matrix function. Also for $i = 1, 2, ..., n$, the system is said to be commensurate order system if $\alpha_1 = \alpha_2 = ... = \alpha_n$. Our study gives a generalization to those results presented in the literature. It should be noted that our fundamental solution can be reduced to the solutions of the corresponding Riemann-Liouville sense and Caputo sense when we take $\beta = 0$ and $\beta = 1$, respectively. In particular, fundamental solution and stability of solutions to linear differential with ψ-Caputo derivative in a recent paper [1] can also be obtained from our work for the case of $0 < \alpha < 1$. We point out that our fundamental solutions are written in terms of Mittag-Leffler function with two parameters α and β, and the fractional order α for the differential system is an $n \times 1$ vector which leads to a more general fractional differential system. Also, if $\psi(t) = t$ and $a = 0$, our fundamental solution can be reduced to (1.1).

This paper is organized in the following way. In section 2, the concepts of ψ-Hilfer differential system will be introduced. Fundamental solutions and stability of systems will be proved in Section 3. Examples will be provided in Section 4 to illustrate the results.

2. Preliminaries

In this section, fundamental information and important notation will be briefed.

Definition 2.1. ([8]) Let $\psi(t) : [a,\infty) \to \mathbb{R}$, $\psi'(t) \neq 0$ and $\alpha > 0$. The left generalized Riemann-Liouville fractional integral of order α for function $f(t)$ is defined by

$$ {}_aI_\psi^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\psi(t) - \psi(\tau))^{\alpha-1} \psi'(\tau) f(\tau) \mathrm{d}\tau. $$

Definition 2.2. ([8]) Let $\alpha > 0$. The left ψ-Riemann-Liouville fractional derivative of order α for function $f(t)$ is defined by

$$ {}_aD_\psi^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_a^t (\psi(t) - \psi(\tau))^{n-\alpha-1} f(\tau) \psi'(\tau) \mathrm{d}\tau. $$

Definition 2.3. ([1]) Let $\alpha > 0$. The left ψ-Caputo fractional derivative of order α is defined by

$$ {}_a^C D_\psi^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (\psi(t) - \psi(\tau))^{n-\alpha-1} (\psi'(\tau))^{1-n} f^{(n)}(\tau) \mathrm{d}\tau, $$

where $n - 1 < \alpha \leq n$ and $n \in \mathbb{N}$.

Definition 2.4. ([16]) Let $n - 1 < \alpha < n$, $\beta \in [0, 1]$ and the fuctions $f, \psi \in C^n([a, b], \mathbb{R})$ such that $\psi'(t) \neq 0$ for $t \in [a, b]$ and ψ be a increasing function. Then the ψ-Hilfer fractional derivative is defined by

$$ {}_aD_\psi^{\alpha,\beta} f(t) = {}_aI_\psi^{\alpha(n-\beta)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n {}_aI_\psi^{(1-\beta)(n-\alpha)} f(t). $$

We can see that when $\beta = 0$, ψ-Hilfer fractional derivative conforms to ψ-Riemann-Liouville fractional derivative and harmonizes with ψ-Caputo fractional derivative when $\beta = 1$.

Definition 2.5. ([8]) Let $\alpha, \beta > 0$. The Mittag-Leffler function with two parameters is defined by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad z \in \mathbb{C}. $$

Definition 2.6. ([7]) Let $f, \psi : [a, \infty) \to \mathbb{R}$ such that $\psi'(t) \in [0, \infty)$. The generalized Laplace transform of f is defined by

$$\mathcal{L}_\psi\{f(t)\} = \int_a^\infty e^{-s(\psi(t)-\psi(a))} f(t)\psi'(t)dt.$$

Theorem 2.7. ([7]) Let $Re(\alpha) > 0$, $\beta > 0$ and $\left|\frac{\lambda}{s^\alpha}\right| < 1$. Then

$$\mathcal{L}_\psi\{(\psi(t) - \psi(a))^{\beta-1} E_{\alpha,\beta}(\lambda(\psi(t) - \psi(a))^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda}.$$

Theorem 2.8. ([5]) Let $n - 1 < \alpha < n$, $\beta \in [0, 1]$. Then

$$\mathcal{L}_\psi\{{}_aD_\psi^{\alpha,\beta} f(t)\} = s^\alpha \mathcal{L}_\psi\{f(t)\} - \sum_{i=0}^{n-1} s^{n(1-\beta)+\alpha\beta-i-1}({}_aI_\psi^{(1-\beta)(n-\alpha)-i} f(a)),$$

where

$${}_aI_\psi^{(1-\beta)(n-\alpha)} f(a) = \lim_{t\to a^+} {}_aI_\psi^{(1-\beta)(n-\alpha)} f(t).$$

Lemma 2.9. ([13]) (Gronwall's inequality). Suppose that $\xi(t) \geq 0$ and $\varphi(t)$ are continuous functions in $[a, b]$ and $\delta, \varepsilon \geq 0$. If

$$\varphi(t) \leq \delta + \int_a^t [\xi(s)\varphi(s) + \varepsilon]ds,$$

then

$$\varphi(t) \leq (\delta + \varepsilon(b - a)) \exp\left(\int_a^t \xi(s)ds\right), \quad t \in [a, b].$$

Lemma 2.10. ([3]) (Gronwall's inequality respect to another function). Let $\xi(t), \sigma(t) \geq 0$ and $\varphi(t) \geq 0$ be nondecreasing continuous functions in $[a, b]$ with $\psi'(t) \neq 0$ and $\sigma(t)$ is nondecreasing. If

$$\xi(t) \leq \sigma(t) + \varphi(t) \int_a^t (\psi(t) - \psi(s))^{\alpha-1} \xi(s)\psi'(s)ds,$$

then

$$\xi(t) \leq \sigma(t) E_\alpha(\varphi(t)\Gamma(\alpha)(\psi(t) - \psi(\tau))^\alpha), \quad t \in [a, b], \quad \tau \in [a, t].$$

3. Solution to ψ-Hilfer fractional differential system

Firstly, we discuss about fundamental solutions to the ψ-Hilfer fractional differential system (1.2) where $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$ such that $\gamma_i = \alpha_i + \beta - \alpha_i\beta$.

Theorem 3.1. *The fractional differential system* (1.2) *has the solution given by*

$$\begin{aligned} y(t) =& C(\psi(t)-\psi(a))^{\gamma-1}E_{\alpha,\gamma}(A(\psi(t)-\psi(a))^{\alpha}) \\ &+\int_a^t (\psi(t)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(A(\psi(t)-\psi(s))^{\alpha})B(s)\psi'(s)ds. \end{aligned} \tag{3.1}$$

Proof. Applying Theorem 2.7 and Theorem 2.8 on (1.2) we get

$$\begin{aligned} & s^{\alpha}\mathcal{L}_{\psi}\{y(t)\} - s^{\alpha\beta-\beta}({}_aI_{\psi}^{(1-\beta)(1-\alpha)}y(a)) = A\mathcal{L}_{\psi}\{y(t)\} + \mathcal{L}_{\psi}\{B(t)\}, \\ & (s^{\alpha}I-A)\mathcal{L}_{\psi}\{y(t)\} = \mathcal{L}_{\psi}\{B(t)\} + s^{\alpha\beta-\beta}C \end{aligned}$$

and

$$\mathcal{L}_{\psi}\{y(t)\} = (s^{\alpha}I-A)^{-1}\mathcal{L}_{\psi}\{B(t)\} + s^{\alpha\beta-\beta}(s^{\alpha}I-A)^{-1}C.$$

Applying inverse Laplace transform, we have

$$\begin{aligned} y(t) =& C(\psi(t)-\psi(a))^{\gamma-1}E_{\alpha,\gamma}(A(\psi(t)-\psi(a))^{\alpha}) \\ &+\int_a^t (\psi(t)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(A(\psi(t)-\psi(s))^{\alpha})B(s)\psi'(s)ds. \end{aligned}$$

The proof is completed. □

Remark 3.2. If $B(t) = 0$ in (1.2), then the fractional systems become a homogeneous system

$$\begin{aligned} &{}_aD_{\psi}^{\alpha,\beta}y(t) = Ay(t), \\ &{}_aI_{\psi}^{(1-\beta)(\mathbf{1}-\alpha)}y(a) = C. \end{aligned} \tag{3.2}$$

Consequently, we see that the function

$$U(t) = C(\psi(t)-\psi(a))^{\gamma-1}E_{\alpha,\gamma}(A(\psi(t)-\psi(a))^{\alpha})$$

is the fundamental solution of (3.2).

Corollary 3.3. *If A has distinct eigenvalues $\lambda_i \in \mathbb{C}$, $i = 1,2,...,n$ associated with the corresponding eigenvectors V_i, $i = 1,2,...,n$, then the fundamental solution of* (3.2) *can be expressed as*

$$U(t) = (\psi(t)-\psi(a))^{\gamma-1}\sum_{i=1}^{n} C_iV_iE_{\alpha,\gamma}(\lambda_i(\psi(t)-\psi(a))^{\alpha}).$$

In addition, the solution to nonhomogeneous differential systems (1.2) can be written as

$$U^*(t) = U(t) + \int_a^t (\psi(t)-\psi(s))^{\alpha-1}\sum_{i=1}^{n} V_iE_{\alpha,\alpha}(\lambda_i(\psi(t)-\psi(s))^{\alpha})B_i(s)\psi'(s)ds.$$

We move further to solution of homogeneous case when A has repeated eigenvalues. We have

$$U_k = (\psi(t) - \psi(a))^{\gamma-1} V_k E_{\alpha,\gamma}(\lambda(\psi(t) - \psi(a))^{\alpha}).$$

Corollary 3.4. *If A has repeated eigenvalues λ with multiplicity n and V_1 is only corresponding eigenvector, then the fundamental solution of* (3.2) *can be expressed as*

$$U(t) = (\psi(t) - \psi(a))^{\gamma-1} \sum_{i=1}^{n} V_i (\psi(t) - \psi(a))^{\alpha(n-i)} E_{\alpha,\gamma}^{(n-i)}(\lambda(\psi(t) - \psi(a))^{\alpha}),$$

where $E_{\alpha,\gamma}^{(j)}(z) = \frac{d^j}{dz^j} E_{\alpha,\gamma}(z)$.

In additon, solution of nonhomogeneous systems (1.2) can be written by

$$U^*(t) = U(t) + \int_a^t \sum_{i=1}^{n} (\psi(t) - \psi(s))^{\alpha(n+1-i)-1} V_i \times E_{\alpha,\alpha}^{(n-i)}(\lambda(\psi(t) - \psi(s))^{\alpha}) B_i(s)\psi'(s)ds.$$

4. Stability of solutions to ψ-Hilfer fractional differential systems

In this section, we investigate the stability of solutions to ψ-Hilfer differential systems for homogeneous, non-homogeneous and non-autonomous cases.

We first collect some important lemmas which are useful in stability analysis.

Lemma 4.1. ([11]) *Let $F(s)$ be the Laplace transform of function $f(t)$. If all poles of $sF(s)$ are in the open left-half plane, then*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

Lemma 4.2. ([13]) *Let $\alpha \in (0,1)$, $\beta \in \mathbb{C}$ and $\mu \in \mathbb{R}$ be such that*

$$\frac{\alpha\pi}{2} < \mu < \min\{\pi, \alpha\pi\}.$$

Then, for integers $p \geq 1$, $|\arg(z)| \leq \mu$ and $|z| \to \infty$ we have

$$E_{\alpha,\beta}(z) = \frac{z^{(1-\beta)/\alpha}}{\alpha} \exp(z^{\frac{1}{\alpha}}) - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - k\alpha)} + O(|z|^{-1-p}),$$

where $O(|z|^{-1-p})$ is defined by

$$\sum_{k=0}^{\infty} a_k |z|^{-1-p-k}, \quad a_0 \neq 0, \quad a_k \in \mathbb{R}.$$

And when $\mu \leq |\arg(z)| \leq \pi$ and $|z| \to \infty$ we have

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - k\alpha)} + O(|z|^{-1-p}) \tag{4.1}$$

and

$$E_{\alpha,\alpha}(z) = -\sum_{k=2}^{p} \frac{z^{-k}}{\Gamma(\alpha - k\alpha)} + O(|z|^{-1-p}). \tag{4.2}$$

Next, we state our main result for ψ-Hilfer fractional differential systems. These results are generalized of [13].

4.1. **Stability of fractional order homogeneous differential system.**

Theorem 4.3. *The zero solution of the fractional order homogeneous differential system* (3.2) *is asymptotically stable if ψ and all of the eigenvalues λ_i satisfied*

$$\lim_{t\to\infty} \psi(t) = \infty, \quad |\arg(\lambda_i)| > \frac{\alpha_i \pi}{2}, \quad i = 1, 2, ..., n$$

Proof. From Theorem 3.1 we get

$$y(t) = C(\psi(t) - \psi(a))^{\gamma-1} E_{\alpha,\gamma}(A(\psi(t) - \psi(a))^{\alpha}).$$

Now, suppose that A is diagonalizable, so that we can write

$$D = T^{-1}AT = diag(\lambda_i)_{n\times n}.$$

We get

$$\begin{aligned} E_{\alpha,\gamma}(A(\psi(t) - \psi(a))^{\alpha}) &= TE_{\alpha,\gamma}(D(\psi(t) - \psi(a))^{\alpha})T^{-1} \\ &= T diag(E_{\alpha,\gamma}(\lambda_i(\psi(t) - \psi(a))^{\alpha}))_{n\times n} T^{-1}. \end{aligned}$$

Then

$$\|E_{\alpha,\gamma}(D(\psi(t) - \psi(a))^{\alpha})\| = \|diag(E_{\alpha,\gamma}(\lambda_i(\psi(t) - \psi(a))^{\alpha}))_{n\times n}\| \to 0$$

as $t \to \infty$ for all $1 \leq i \leq n$.

Suppose that A is similar to Jordan canonical form. Then there exists an invertible matrix T such that $J = T^{-1}AT = diag(J_i)_{r\times r}$, where

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}_{n_i \times n_i}, \quad i = 1, 2, ..., r$$

is the Jordan block with $\sum_{i=1}^{r} n_i = n$. Obviously, we gain

$$E_{\alpha,\gamma}(A(\psi(t) - \psi(a))^{\alpha}) = TE_{\alpha,\gamma}(diag(J_i)_{r\times r}(\psi(t) - \psi(a))^{\alpha})T^{-1}.$$

Let C_k^j where $1 \leq j \leq n_i - 1$ are the binomial coefficients. Then

$$\begin{aligned} & E_{\alpha,\gamma}(J_i(\psi(t)-\psi(a))^{\alpha}) \\ &= \sum_{k=0}^{\infty} \frac{((\psi(t)-\psi(a))^{\alpha})^k}{\Gamma(k\alpha+\gamma)} J_i^k \\ &= \sum_{k=0}^{\infty} \frac{((\psi(t)-\psi(a))^{\alpha})^k}{\Gamma(k\alpha+\gamma)} \begin{bmatrix} \lambda_i^k & C_k^1\lambda_i^{k-1} & \cdots & C_k^{n_i-1}\lambda_i^{k-n_i+1} \\ & \lambda_i^k & \ddots & \vdots \\ & & \ddots & C_k^1\lambda_i^{k-1} \\ & & & \lambda_i^k \end{bmatrix}. \end{aligned}$$

Obviously, we get

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{((\psi(t)-\psi(a))^{\alpha})^k}{\Gamma(k\alpha+\gamma)} C_k^{n_i-1}\lambda_i^{k-n_i+1} \\ &= \frac{1}{(n-1)!}\left(\frac{d}{d\lambda_i}\right)^{n_i-1} E_{\alpha,\gamma}(\lambda_i(\psi(t)-\psi(a))^{\alpha}). \end{aligned}$$

This implies

$$|E_{\alpha,\gamma}(\lambda_i(\psi(t)-\psi(a))^{\alpha})| \to 0 \quad as \quad t \to \infty,$$

which leads

$$\left|\frac{1}{j!}\left(\frac{d}{d\lambda_i}\right)^j E_{\alpha,\gamma}(\lambda_i(\psi(t)-\psi(a))^{\alpha})\right| \to 0, \quad 1 \leq j \leq n_i - 1, \quad as \quad t \to \infty.$$

So for any non-zero initial value $y(a)$ when $t \to \infty$ it follows that

$$\|y(t)\| = \|C(\psi(t)-\psi(a))^{\gamma-1}E_{\alpha,\gamma}(A(\psi(t)-\psi(a))^{\alpha})\| \to 0,$$

which completes the proof. □

Remark 4.4. For commensurate order homogeneous system, If $|\arg(\lambda)| > \frac{\alpha\pi}{2}$, the system (3.2) is asymptotically stable.

Remark 4.5. For α_i are rational numbers. let $\omega = \frac{1}{M}$ where M is the lowest common multiple of denominators u_i of α_i where $\alpha_i = \frac{v_i}{u_i}$, $gcd(u_i, v_i) = 1$, $u_i, v_i \in \mathbb{N}$ and all roots of $det(diag(\lambda^{M\alpha_i})_{n\times n} - A) = 0$ satisfied $|\arg(\lambda)| > \frac{\omega\pi}{2}$, The system (3.2) is asymptotically stable.

Theorem 4.6. *If A has an eigenvalue λ_* such that $|\arg(\lambda_*)| < \frac{\alpha\pi}{2}$, then the zero solution of the homogeneous system* (3.2) *is unstable.*

Proof. Suppose A is diagonalizable matrix. Then by using Lemma 4.2, we obtain

$$E_{\alpha,\gamma}(\lambda_*(\psi(t)-\psi(a))^{\alpha})$$

$$
\begin{aligned}
&= \frac{(\lambda_*(\psi(t)-\psi(a))^\alpha)^{(1-\gamma)/\alpha}}{\alpha} \exp(\lambda_*(\psi(t)-\psi(a))^\alpha)^{\frac{1}{\alpha}}) \\
&\qquad - \sum_{k=1}^{p} \frac{(\lambda_*(\psi(t)-\psi(a))^\alpha)^{-k}}{\Gamma(\gamma-k\alpha)} + O((|\lambda_*(\psi(t)-\psi(a))^\alpha)|^{-1-p}) \\
&\to \infty, \quad \text{as} \quad t \to \infty.
\end{aligned}
$$

Also, we have

$$
\lim_{t\to\infty} \|y(t)\| = \infty.
$$

Next, we concern in the case that A is similar to Jordan canonical form. We have

$$
\begin{aligned}
&\frac{1}{(n-1)!}\left(\frac{d}{d\lambda_i}\right)^{n-1} E_{\alpha,\gamma}(\lambda_i(\psi(t)-\psi(a))^\alpha) \\
&= \frac{1}{(n-1)!}\Bigg\{ \frac{\prod_{j=0}^{n-2}(1-\gamma-j\alpha)}{\alpha^n} \lambda_*^{(1-\gamma-(n-1)\alpha)/\alpha}(\psi(t)-\psi(a))^{1-\gamma} + \cdots \\
&\qquad + \frac{(n-1)n-(n-1)\gamma-(n-1)(n-2)\alpha}{2\alpha^n} \\
&\qquad \times \lambda_*^{(n-1-\gamma-(n-1)\alpha)/\alpha}(\psi(t)-\psi(a))^{n-1-\gamma} \\
&\qquad + \frac{1}{\alpha^n}\lambda_*^{(n-\gamma-(n-1)\alpha)/\alpha}(\psi(t)-\psi(a))^{n-\gamma} \Bigg\} \\
&\qquad \times \exp(\lambda_*^{1/\alpha}(\psi(t)-\psi(a))) \\
&\quad - \frac{1}{(n-1)!}\sum_{k=1}^{p} \frac{\prod_{j=0}^{n-2}(-k-j)}{\Gamma(\gamma-\alpha k)\lambda_*^{k+n-1}(\psi(t)-\psi(a))^{\alpha k}} \\
&\quad + O(|\lambda_*|^{-n-p}(\psi(t)-\psi(a))^{-p\alpha-\alpha}).
\end{aligned}
$$

For large t, we obtain

$$
\begin{aligned}
&\frac{(\psi(t)-\psi(a))^{\gamma-1}}{(n-1)!}\left(\frac{d}{d\lambda_i}\right)^{n-1} E_{\alpha,\gamma}(\lambda_i(\psi(t)-\psi(a))^\alpha) \\
&\geq \frac{1}{(n-1)!}\Bigg\{ \left| \frac{1}{\alpha^n}\lambda_*^{(n-\gamma-(n-1)\alpha)/\alpha}(\psi(t)-\psi(a))^{n-1} \right| \\
&\qquad - \left| \frac{\prod_{j=0}^{n-2}(1-\gamma-j\alpha)}{\alpha^n} \lambda_*^{(1-\gamma-(n-1)\alpha)/\alpha} \right| \\
&\qquad - \cdots - \left| \frac{(n-1)n-(n-1)\gamma-(n-1)(n-2)\alpha}{2\alpha^n} \right| \\
&\qquad \times \left| \lambda_*^{(n-1-\gamma-(n-1)\alpha)/\alpha}(\psi(t)-\psi(a))^{n-2} \right| \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\quad\times \exp\left(|\lambda_*|^{1/\alpha}\cos\left(\frac{\arg(\lambda_*)}{\alpha}\right)(\psi(t)-\psi(a))\right)\\
&-\frac{1}{(n-1)!}\sum_{k=1}^{p}\frac{\left|\prod_{j=0}^{n-1}(-k-j)\right|}{|\Gamma(\gamma-\alpha k)|\,|\lambda_*|^{k+n}\,(\psi(t)-\psi(a))^{\alpha k-n+1}}\\
&+O(|\lambda_*|^{-n-p}(\psi(t)-\psi(a))^{-p\alpha-\alpha+\gamma-1})\\
&\to\infty,\quad \text{as}\quad t\to\infty.
\end{aligned}
$$

Since $|\arg(\lambda_*)|<\frac{\alpha\pi}{2}$, the zero solution of the homogeneous system is unstable. □

Theorem 4.7. *If A has zero eigenvalue, then the zero solution of the homogeneous system* (3.2) *is unstable.*

Proof. We have

$$
\left(\frac{d}{d\lambda}\right)^{n-1}E_{\alpha,\gamma}(\lambda(\psi(t)-\psi(a))^{\alpha})=\sum_{k=0}^{\infty}\frac{(k+n-1)!\lambda^{k}(\psi(t)-\psi(a))^{\alpha(k+n-1)}}{k!\Gamma(\alpha k+\alpha(n-1)+\gamma)}.
$$

And for $\lambda=0$, we have

$$
\left(\frac{d}{d\lambda}\right)^{n-1}E_{\alpha,\gamma}(\lambda(\psi(t)-\psi(a))^{\alpha})=\frac{(n-1)!(\psi(t)-\psi(a))^{\alpha(n-1)}}{\Gamma(\alpha(n-1)+\gamma)}.
$$

Multiply $\frac{(\psi(t)-\psi(a))^{\gamma-1}}{(n-1)!}$ on above equality, we obtain

$$
\begin{aligned}
y(t)&=\frac{(\psi(t)-\psi(a))^{\gamma-1}}{(n-1)!}\left(\frac{d}{d\lambda}\right)^{n-1}E_{\alpha,\gamma}(\lambda(\psi(t)-\psi(a))^{\alpha})\\
&=\frac{(\psi(t)-\psi(a))^{\alpha(n-1)+\gamma-1}}{\Gamma(\alpha(n-1)+\gamma)}.
\end{aligned}
$$

For $n\geq 1$, it is obvious that

$$
\lim_{t\to\infty}\|y(t)\|=\lim_{t\to\infty}\frac{(\psi(t)-\psi(a))^{\alpha(n-1)+\gamma-1}}{\Gamma(\alpha(n-1)+\gamma)}=\infty.
$$

Hence, the system is unstable. □

Theorem 4.8. *If $\lim_{t\to\infty}\psi(t)=\infty$ and A has all eigenvlues satisfy $|\arg(\lambda_i)|\geq\frac{\alpha\pi}{2}$, $i=1,2,...,n$ which critical eigenvalues λ_c satisfy $|\arg(\lambda_c)|=\frac{\alpha\pi}{2}$ have the same algebraic and geometric multiplicities, then the homogeneous system* (3.2) *is stable.*

Proof. From Lemma 4.2, we have

$$\begin{aligned} &E_{\alpha,\gamma}(\lambda_c(\psi(t)-\psi(a))^\alpha) \\ &= \frac{(\lambda_c(\psi(t)-\psi(a))^\alpha)^{(1-\gamma)/\alpha}}{\alpha} \exp(\lambda_c(\psi(t)-\psi(a))^\alpha)^{\frac{1}{\alpha}}) \\ &\quad - \sum_{k=1}^{p} \frac{(\lambda_c(\psi(t)-\psi(a))^\alpha)^{-k}}{\Gamma(\gamma-k\alpha)} + O((|\lambda_c(\psi(t)-\psi(a))^\alpha)|^{-1-p}). \end{aligned}$$

We set $\lambda_c = re^{i\frac{\alpha\pi}{2}}$ where r is the modulus of λ_c, we obtain

$$\begin{aligned} &(\psi(t)-\psi(a))^{\gamma-1} E_{\alpha,\gamma}(\lambda_c(\psi(t)-\psi(a))^\alpha) \\ &= \frac{r^{(1-\gamma)/\alpha}}{\alpha} \exp\left(\ln\left(i\exp\left(-i\frac{\gamma\pi}{2}\right)\right) + ir^{1/\alpha}(\psi(t)-\psi(a))\right) \\ &\quad - \sum_{k=1}^{p} \frac{r^{-k}(\psi(t)-\psi(a))^{\gamma-\alpha k-1}\exp\left(-i\frac{\alpha k\pi}{2}\right)}{\Gamma(\gamma-\alpha k)} \\ &\quad + O((\psi(t)-\psi(a))^{\gamma-\alpha p-\alpha-1}). \end{aligned}$$

Obviously,

$$\lim_{t\to\infty} \|y(t)\| = \frac{r^{(1-\gamma)/\alpha}}{\alpha},$$

hence the system is stable but not asymptotically stable. □

4.2. **Stability of fractional order nonhomogeneous and nonautonomous differential systems.** Next, we move forward to stability of nonhomogeneous ψ-Hilfer fractional differential system. By setting

$$\Omega(t) := C(\psi(t)-\psi(a))^{\gamma-1} E_{\alpha,\gamma}(\lambda_i(\psi(t)-\psi(a))^\alpha),$$

we get from (3.1) that

$$y(t) = \Omega(t) + \int_a^t (\psi(t)-\psi(s))^{\alpha-1} E_{\alpha,\alpha}(A(\psi(t)-\psi(s))^\alpha) B(s)\psi'(s)ds.$$

Theorem 4.9. *Suppose that all conditions in Theorem* 4.3 *hold and* $\|B(t)\| \le M_B$ *for some* $M_B > 0$. *Then the solution of the system* (1.2) *converges to* 0 *as* $t \to \infty$.

Proof. From Theorem 3.1 and (4.2), the solution to nonhomogeneous system leads to

$$\begin{aligned}\|y(t)\| &\leq \|\Omega(t)\| + \int_a^t \|(\psi(t)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(\lambda_i(\psi(t)-\psi(s))^{\alpha})\|\|B(s)\|\psi'(s)ds \\ &\leq \|\Omega(t)\| + M_B\int_a^t \|(\psi(t)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(\lambda_i(\psi(t)-\psi(s))^{\alpha})\|\psi'(s)ds \\ &\leq \|\Omega(t)\| + M_B\left(\sum_{k=2}^{p}\frac{|\lambda_i|^{-k}(\psi(t)-\psi(a))^{-k\alpha+\alpha}}{\Gamma(\alpha-k\alpha+1)}\right. \\ &\qquad\left. + O\left(\frac{1}{|\lambda_i|^{p+1}(\psi(t)-\psi(a))^{p\alpha}}\right)\right) \\ &\to 0, \quad t\to\infty.\end{aligned}$$

The proof is completed. □

From Theorem 4.9, we only obtain global behavior of solution. Lastly, we consider stability of following nonautonomous system

$$\begin{aligned} {}_aD_{\psi}^{\alpha,\beta}y(t) &= Ay(t) + B(t)y(t), \quad t>a, \\ {}_aI_{\psi}^{(1-\beta)(1-\alpha)}y(a) &= C, \end{aligned} \tag{4.3}$$

where $B(t):[a,\infty)\to\mathbb{R}^{n\times n}$ is continuous matrix.

Theorem 4.10. *Suppose ψ is a strictly increase function on $[a,\infty)$, $\|B(t)\|\leq M_B$ for some $M_B>0$, $\|E_{\alpha,\alpha}(\lambda_i(\psi(t)-\psi(s))^{\alpha})\|\leq M_E(t)$ and all λ, α satisfied $|\arg(\lambda_i)|>\frac{\alpha\pi}{2}$. Then the solution of system* (4.3) *is asymptotically stable.*

Proof. From Lemma 2.10 and Theorem 3.1, we get

$$\begin{aligned}\|y(t)\| &\leq \|\Omega(t)\| + \int_a^t \|(\psi(t)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(\lambda_i(\psi(t)-\psi(s))^{\alpha})\| \\ &\qquad \times\|B(s)\|\|y(s)\|\psi'(s)ds \\ &\leq \|\Omega(t)\| + M_BM_E(t)\int_a^t \|(\psi(t)-\psi(s))^{\alpha-1}\|\|y(s)\|\psi'(s)ds \\ &\leq \|\Omega(t)\|E_{\alpha}(M_BM_E(t)\Gamma(\alpha)(\psi(t)-\psi(\tau))^{\alpha}), \quad \tau\in[a,t].\end{aligned}$$

Further $\|\Omega(t)\|\to 0$ as $t\to\infty$, we have

$$\lim_{t\to\infty}\|y(t)\| = 0,$$

which gives the assertion of the theorem. □

Theorem 4.11. *Suppose ψ is a strictly increase function on $[a,\infty)$ and all eigenvlues satisfy $|\arg(\lambda_i)|\geq\frac{\alpha\pi}{2}$ which critical eigenvalues λ_c satisfy $|\arg(\lambda_c)| =$*

$\frac{\alpha\pi}{2}$ *with the same algebraic and geometric multiplicities. If* $\int_0^\infty \|B(t)\|\psi'(t)dt$ *is bounded, then the solution of system* (4.3) *is stable.*

Proof. From Theorem 4.8 there exists a positive number M_Ω such that $\|\Omega(t)\| \leq M_\Omega$. Also, for $\beta = 0$ there exists a positive number N_Ω such that

$$\|(\psi(t)-\psi(a))^{\alpha-1}E_{\alpha,\alpha}(A(\psi(t)-\psi(a))^\alpha)\| \leq N_\Omega.$$

Applying the Gronwall's inequality in Lemma 2.9, we have

$$\begin{aligned}\|y(t)\| &\leq M_\Omega + \int_a^t N_\Omega \|B(s)\|\|y(s)\|\psi'(s)ds \\ &\leq M_\Omega \exp\left(N_\Omega \int_a^t \|B(s)\|\psi'(s)ds\right).\end{aligned}$$

This completes the proof. □

5. Examples

To demonstrate our main results, this section provides examples and numerical illustration for the stability of solutions to ψ-Hilfer differential systems. By choosing various parameters α, β and $\psi(t)$, we compare the solutions when the fractional derivative reduces to ordinary derivative, ψ-Riemann-Liouville derivative, ψ-Caputo derivative and ψ-Hilfer derivative.

Example 5.1. Let $\psi(t) = \sqrt{t}$. Consider the system

$$\begin{bmatrix} {}_0D^{\alpha,\beta}_{\psi(t)}w(t) \\ {}_0D^{\alpha,\beta}_{\psi(t)}x(t) \\ {}_0D^{\alpha,\beta}_{\psi(t)}y(t) \\ {}_0D^{\alpha,\beta}_{\psi(t)}z(t) \end{bmatrix} = A \begin{bmatrix} w(t) \\ x(t) \\ y(t) \\ z(t) \end{bmatrix}, \quad \begin{bmatrix} w(0) \\ x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix},$$

where

$$A = \begin{bmatrix} -5 & 1 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & -4 & -3 \end{bmatrix}.$$

Then the eigenvalues of matrix A are

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{-1, -6, -3-6i, -3+6i\}$$

with the corresponding eigenvectors, $V_1 = (\frac{1}{4}, 1, 0, 0)^T, V_2 = (-1, 1, 0, 0)^T, V_3 = (0, 0, \frac{3i}{2}, 1)^T$ and $V_4 = (0, 0, \frac{-3i}{2}, 1)^T$. Therefore, the solution of the system (3.2) where $\gamma_i = \alpha_i + \beta - \alpha_i\beta$ for $i = 1, 2, 3, 4$ is

$$U(t) = (\sqrt{t})^{\gamma-1}\left(C_1 \begin{pmatrix} \frac{1}{4} \\ 1 \\ 0 \\ 0 \end{pmatrix} E_{\alpha,\gamma}(-(\sqrt{t})^{\alpha}) + C_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} E_{\alpha,\gamma}(-6(\sqrt{t})^{\alpha}) \right.$$

$$\left. + C_3 \begin{pmatrix} 0 \\ 0 \\ \frac{3i}{2} \\ 1 \end{pmatrix} E_{\alpha,\gamma}((-3-6i)(\sqrt{t})^{\alpha}) + C_4 \begin{pmatrix} 0 \\ 0 \\ \frac{-3i}{2} \\ 1 \end{pmatrix} E_{\alpha,\gamma}((-3+6i)(\sqrt{t})^{\alpha}) \right).$$

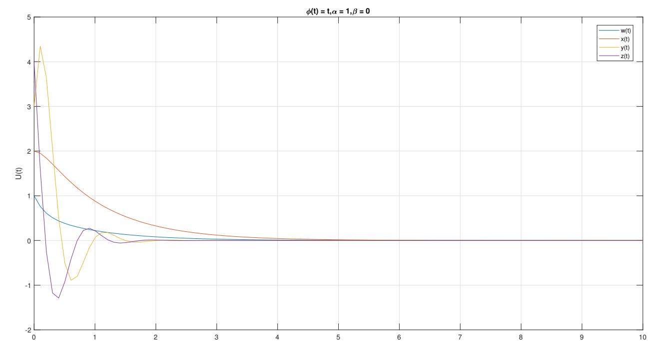
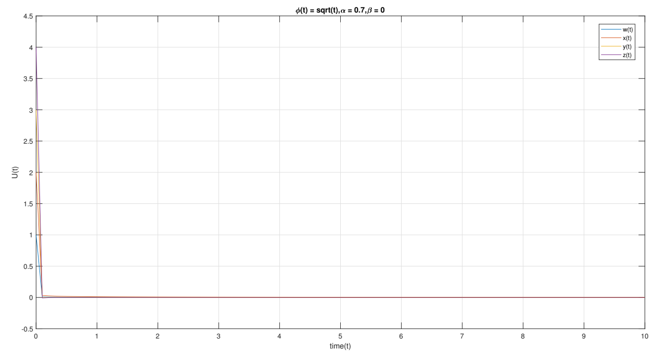
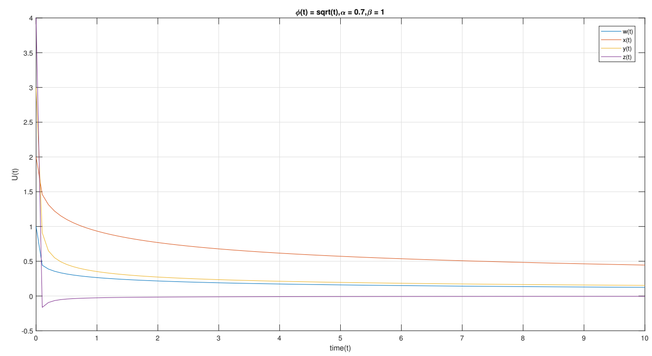


FIGURE 1. Solutions to ordinary differential system ($\psi(t) = t, \alpha = 1, \beta = 0$)

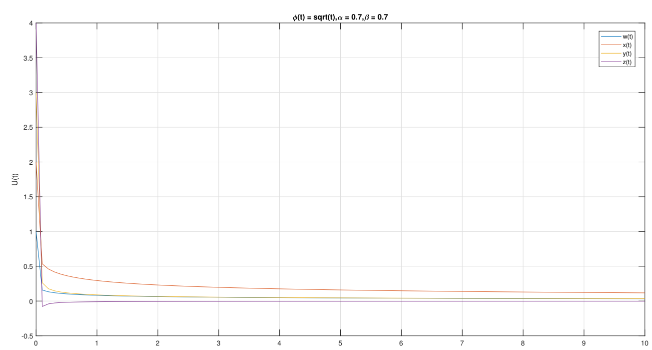
Unlike fractional differential systems, the complex eigenvalues with negative real part motivate small exquisite oscillation of solution to ordinary differential system (when $\alpha = 1, \beta = 0$ and $\psi(t) = t$) shown in Figure 1. From Figure 2 we can see that all the zero solutions of the differential system (3.2) are stable since stability condition in Theorem 4.3 is satisfied. The picture demonstrates that the stability of solution to fractional differential systems are more slick than the solution of the ordinary one. Furthermore, we can see that the solution to ψ-Riemann-Liouville fractional differential system (Figure 2(A)) converges rapidly to zero whereas solution to ψ-Caputo fractional differential system (Figure 2(B)) converges to zero with the gradual rate and solution to ψ-Hilfer differential system interpolates between the solutions to the systems mentioned above shown in Figure 2(C).



(A) ψ-Riemann-Liouville system ($\psi(t) = \sqrt{t}, \alpha = 0.7, \beta = 0$)



(B) ψ-Caputo system ($\psi(t) = \sqrt{t}, \alpha = 0.7, \beta = 1$)



(C) ψ-Hilfer system ($\psi(t) = \sqrt{t}, \alpha = 0.7, \beta = 0.7$)

FIGURE 2. Solutions to fractional differential systems with $\psi(t) = \sqrt{t}$

Example 5.2. Let $\psi(t) = t^2$. Consider the system

$$\begin{bmatrix} {}_0D^{\alpha,\beta}_{\psi(t)} w(t) \\ {}_0D^{\alpha,\beta}_{\psi(t)} x(t) \\ {}_0D^{\alpha,\beta}_{\psi(t)} y(t) \\ {}_0D^{\alpha,\beta}_{\psi(t)} z(t) \end{bmatrix} = A \begin{bmatrix} w(t) \\ x(t) \\ y(t) \\ z(t) \end{bmatrix}, \qquad \begin{bmatrix} w(0) \\ x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

where

$$A = \begin{bmatrix} -2 & -5 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & -2 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Then the eigenvalues of matrix A are

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{i, i, -i, -i\}$$

with the corresponding eigenvectors $V_1 = (-2+i, 1, 0, 0)^T, V_2 = (0, 0, -2+i, 1)^T, V_3 = (-2-i, 1, 0, 0)^T$ and $V_4 = (0, 0, -2-i, 1)^T$.

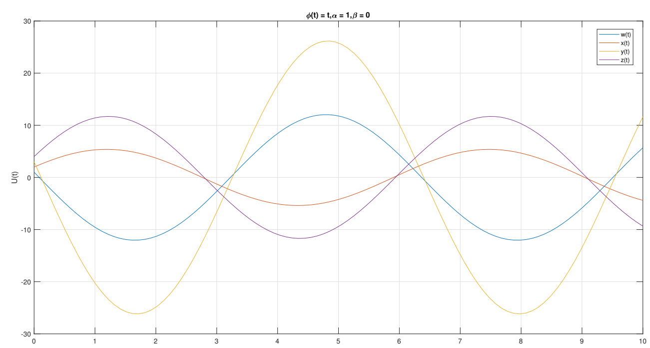
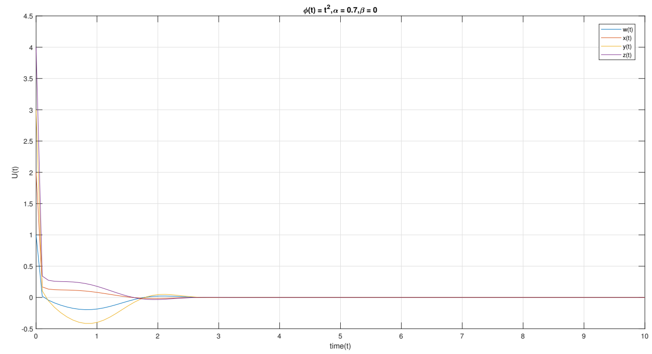
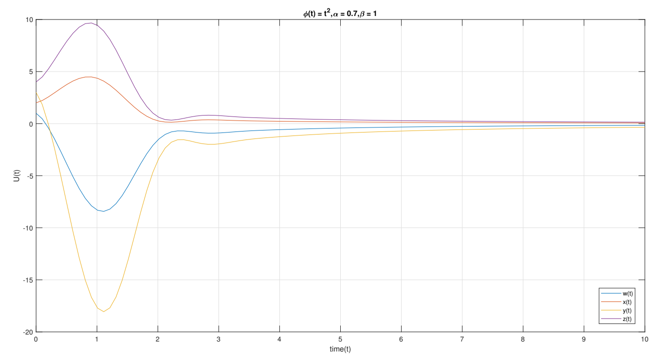


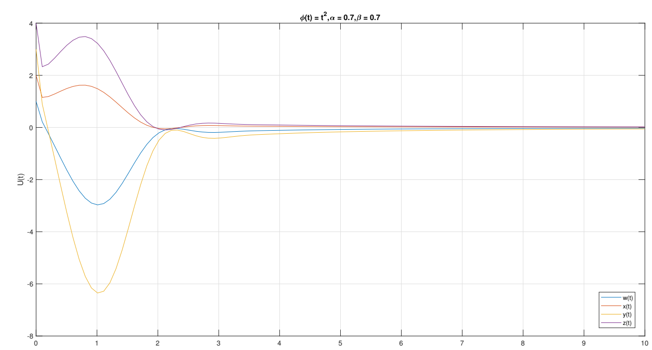
FIGURE 3. Solutions to ordinary differential system ($\psi(t) = t, \alpha = 1, \beta = 0$)



(A) ψ-Riemann-Liouville system ($\psi(t) = t^2, \alpha = 0.7, \beta = 0$)



(B) ψ-Caputo system ($\psi(t) = t^2, \alpha = 0.7, \beta = 1$)



(C) ψ-Hilfer system ($\psi(t) = t^2, \alpha = 0.7, \beta = 0.7$)

FIGURE 4. Solutions to fractional differential systems with $\psi(t) = t^2$

Therefore, the solution of the system (3.2) where $\gamma_i = \alpha_i + \beta - \alpha_i\beta$ for $i = 1, 2, 3, 4$ is

$$U(t) = t^{2\gamma-2}\left(C_1 \begin{pmatrix} -2+i \\ 1 \\ 0 \\ 0 \end{pmatrix} E_{\alpha,\gamma}(it^{2\alpha}) + C_2 \begin{pmatrix} 0 \\ 0 \\ -2+i \\ 1 \end{pmatrix} E_{\alpha,\gamma}(it^{2\alpha}) \right.$$

$$\left. + C_3 \begin{pmatrix} -2-i \\ 1 \\ 0 \\ 0 \end{pmatrix} E_{\alpha,\gamma}(-it^{2\alpha}) + C_4 \begin{pmatrix} 0 \\ 0 \\ -2-i \\ 1 \end{pmatrix} E_{\alpha,\gamma}(-it^{2\alpha}) \right).$$

The complex eigenvalues with zero real part conduct solution to ordinary differential equation (when $\alpha = 1, \beta = 0$ and $\psi(t) = t$) to be periodic as shown in Figure 3. In contrast, the solutions to fractional differential equation are stable as the result of α establish wider argument range of eigenvalues that satisfied the condition of stability theory in Theorem 4.3. Similar to Figure 2, solution to ψ-Riemann-Liouville fractional differential system (Figure 4(A)) is fleetly stable whereas solution to ψ-Caputo fractional differential system (Figure 4(B)) is stable with the slower rate and solution to ψ-Hilfer differential system interpolates between the solutions to the systems mentioned above as shown in Figure 4(C).

6. Conclusion

We obtained the fundamental solution to ψ-Hilfer fractional linear differential systems, solution of nonhomogeneous and nonautonomous systems. The stabilities of homogeneous, nonhomogeneous and nonautonomous systems involving ψ-Hilfer fractional derivative with order $0 < \alpha < 1$ are studied in this paper. The given examples illustrate that parameters β motive solution to ψ-Hilfer differential system to interpose between differential system ψ-Caputo derivative and ψ-Riemann-Liouville derivative.

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