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SOME BEST PROXIMITY POINT RESULTS OF SEVERAL α-ψ INTERPOLATIVE PROXIMAL CONTRACTIONS

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Abstract. In this paper, we introduce several types α-ψ interpolative proximal contractions and provide some sufficient conditions to prove the existence of best proximity points for these contractions in metric spaces. In the case of proximal contraction of the first kind, these metric spaces are not necessarily complete. Meanwhile, some new results can derive from our results. Finally, some examples are provided to show the validity of our results.

1. Introduction and Preliminaries

Fixed point theory is one of the important branches of nonlinear analysis. Indeed, we find that many mathematics questions can be transformed into the problem of finding fixed points of mappings. In order to obtain the fixed

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points of certain given mapping, we must utilize some suitable conditions. However, when a non-self mapping in a metric space has no fixed points, then it could be interesting to study the existence and uniqueness of some points that minimize the distance between the point and its corresponding image. These points are known as best proximity points. The best proximity points were introduced by Fan [8] and modified by Basha in [4]. Many authors obtained some recent best proximity points and fixed point results, refer to [1, 2, 7, 10, 13, 14, 15, 16, 17, 18, 19, 20, 23, 24].

For the sake of completeness, we collect some notations and notions related to the best proximity point theory, which will be used throughout the rest of this work.

Let A and B be two nonempty subsets of a metric space (X, d). We will use the following notations:

$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\};$$
$$A_0 = \{x \in A : d(x,y) = d(A,B), \text{ for some } y \in B\};$$
$$B_0 = \{y \in B : d(x,y) = d(A,B), \text{ for some } x \in A\}.$$

Definition 1.1. Let $T : A \to B$ be a given non-self mapping. If there exists an element x^* such that $d(x^*, Tx^*) = d(A,B)$, then T has a best proximity point x^*.

Remark 1.2. Note that if non-self mapping reduces to self mapping in Definition 1.1, that is, $A = B$. In this case, if there exists an element x^* such that $d(x^*, Tx^*) = d(A,B) = 0$, then the best proximity point of T becomes the fixed point of T.

In 1922, Since Banach [3] introduced the famous Banach contraction principle, many authors have made generalization and improvement based on it. Hence, a lot of interesting results are obtained. These results including the relaxation of contraction inequalities, all kinds of new contraction conditions, the generalization of metric spaces. As a pioneering work of fixed point theory, Banach contraction principle is of great significance, and its specific results are as follows.

Theorem 1.3. ([3]) *Let (X, d) be a complete metric space and $T : X \to X$ be a Banach contraction, that is,*

$$d(Tx, Ty) \le kd(x,y),$$

for all $x, y \in X$, where $k \in (0,1)$. Then T has a unique fixed point.

In 2010, as an extension of Banach contraction, the case of Banach contraction under non-self mapping was proposed by Basha [5].

Definition 1.4. ([5]) Let (X, d) be a metric space and A, B be two nonempty subsets of X. $T : A \to B$ is said to be a proximal contraction, if there exists $k \in (0, 1)$ such that

$$\left.\begin{array}{l} d(x_1, Ty_1) = d(A, B) \\ d(x_2, Ty_2) = d(A, B) \end{array}\right\} \Rightarrow d(x_1, x_2) \le kd(y_1, y_2), \tag{1.1}$$

for all $x_1, x_2, y_1, y_2 \in A$.

Remark 1.5. It is easy to observe that a proximal contraction mapping is exactly a contraction mapping when a non-self mapping reduces to a self mapping. However, a non-self mapping that proximal contraction need not be a contraction. More importantly, a non-self proximal contraction mapping is not necessarily continuous.

Example 1.6. ([5]) Let $A = [0, 1]$, $B = [2, 3]$ and $d = |x - y|$. Define a mapping T by

$$T(x) = \begin{cases} 3 - x, & \text{if } x \text{ is rational,} \\ 2 + x, & \text{otherwise.} \end{cases}$$

Then T is a proximal contraction, but T is not continuous except at $x = \frac{1}{2}$.

Definition 1.7. ([5]) Let A, B be two nonempty subsets of X. If each sequence $\{y_n\}$ in B satisfies that $d(x, y_n) \to d(x, B)$ for some $x \in A$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y \in B$, then we say that B is approximately compact with respect to A.

It is obvious that every compact subset of X is approximately compact with respect to any subsets. In addition, each compact subset of X is approximately compact with respect to itself.

The next result is a fundamental theorem of the best proximity point theory.

Theorem 1.8. ([5]) *Let (X, d) be a complete metric space. Suppose that A, B are two nonempty closed subsets of X such that B is approximately compact with respect to A. Moreover, Assume that A_0 and B_0 are nonempty, $T : A \to B$ is a proximal contraction such that $T(A_0) \subset B_0$. Then T has a unique best proximity point in A.*

On the other hand, we know that a mapping satisfying Banach contraction inequality must be continuous. In 1968, Kannan [11] presented a famous contraction and a key metric fixed point theorem, which is different from the Banach contraction principle, named Kannan fixed point theorem. This kind of contractions give a positive answers to the open question: whether there exist discontinuous mappings at fixed points satisfying certain contraction conditions in complete metric spaces.

Theorem 1.9. ([11]) *Let (X, d) be a complete metric space and $T : X \to X$ be a Kannan contraction mapping, that is, for all $x, y \in X$, the following inequality holds:*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)],$$

where $k \in [0, \frac{1}{2})$. Then T has a unique fixed point.

Inspired by Kannan's contraction and proximal contraction, Basha [6] considered the following proximal contraction.

Definition 1.10. ([6]) Let (X, d) be a metric space and A, B be two non-empty subsets of X. A mapping $T : A \to B$ is said to be a K-proximal contraction, if there exists some $k \in (0, \frac{1}{2})$ such that

$$\left.\begin{array}{l} d(x_1, Ty_1) = d(A, B) \\ d(x_2, Ty_2) = d(A, B) \end{array}\right\} \Rightarrow d(x_1, x_2) \leq k[d(x_1, y_1) + d(x_2, y_2)], \tag{1.2}$$

for all $x_1, x_2, y_1, y_2 \in A$.

In the same year, Karapınar [12] revisited Kannan contraction and introduced a kind of interesting contraction by the interpolative method. A self mapping $T : X \to X$ defined on a metric space (X, d) is said to be an interpolative Kannan contraction, if the following inequality holds:

$$d(Tx, Ty) \leq k[d(x, Tx)]^\tau [d(y, dy)]^{1-\tau},$$

for all $x, y \in \{z \in X : d(z, Tz) > 0\}$, where $k \in [0, 1)$, $\tau \in (0, 1)$. The main result in [12] is given below:

Theorem 1.11. ([12]) *Let (X, d) be a complete metric space. If $T : X \to X$ is an interpolative Kannan contraction, then T has a fixed point in X.*

In 2012, Samet et al. [21] presented α-ψ contraction mapping by α-admissible mapping.

Definition 1.12. ([21]) Let $T : X \to X$ be a mapping and $\alpha : X \times X \to \mathbb{R}_+$ be a function. Then T is said to be α-admissible if

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1, \text{ for all } x, y \in X.$$

In 2013, based on α-admissible, Jleli et al. [9] proposed the notion of α-proximal contraction and got some best proximity results of α-ψ proximal contraction.

Definition 1.13. ([9]) Let $T : A \to B$ be a non-self mapping. Define a function $\alpha : A \times A \to [0, \infty)$. We say that T is α-proximal admissible, if

$$\left.\begin{array}{l} \alpha(x_1, x_2) \geq 1 \\ d(y_1, Tx_1) = d(A, B) \\ d(y_2, Tx_2) = d(A, B) \end{array}\right\} \Rightarrow \alpha(y_1, y_2) \geq 1, \tag{1.3}$$

for all $x_1, x_2, y_1, y_2 \in A$.

Note that if $A = B$, then T is α-proximal admissible implies that T is α-admissible.

Let Ψ represents all non-decreasing functions and $\Sigma_{n=1}^{\infty}\psi^n(t) < 0$ for all $t > 0$, where ψ^n is the n-th iteration of ψ. If $\psi \in \Psi$, then $\psi(t) < t$, for all $t > 0$.

Definition 1.14. ([9]) Let $T : A \to B$ be a mapping and $\alpha : A \times A \to [0, \infty)$ be a function and $\psi \in \Psi$. We say that T is an α-ψ-proximal contraction mapping, if the following inequality holds:

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)), \forall x, y \in A.$$

To introduce the Theorem 1.16, we recall the notion of the P-property.

Definition 1.15. ([22]) Let A, B be two nonempty subsets of a metric space (X, d). Then (A, B) is said to be have the P-property if and only of

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \Rightarrow d(x_1, x_2) = d(y_1, y_2), \tag{1.4}$$

where $x_1, x_2 \in A_0$, $y_1, y_2 \in B_0$.

Theorem 1.16. ([9]) *Let (X, d) be a complete metric space and A, B be two nonempty closed subsets of X such that A_0 is nonempty. Define a non-self mapping $T : A \to B$. If the following conditions hold:*

(i) *$T(A_0) \subset B_0$ and (A, B) has the P-property;*
(ii) *T is α-proximal admissible;*
(iii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1;$$

(iv) *T is a continuous α-ψ-proximal contraction,*

then T has a best proximity point in A_0.

Recently, Sahin et al. [20] introduced the notions of T-best orbitally complete and best orbitally continuous in the best proximity point theory.

Definition 1.17. ([20]) Let (X, d) be a metric space, A, B be nonempty subsets of X, $T : A \to B$ be a mapping and $x \in A$. Then, the set of iterative sequences

$$O_T(x) = \{\{x_n\} \subseteq A : x_0 = x, d(x_{n+1}, Tx_n) = d(A, B), \text{for all } n \in \mathbb{N}\}$$

is called the orbit of x.

Note that, when $A = B = X$, we have

$$O_T(x) = \{T^n x\}.$$

Definition 1.18. ([20]) Let (X, d) be a metric space, A, B be nonempty subsets of X, $T : A \to B$ be a mapping. Then T is said to be best orbitally continuous at a point x^* in A, if for each $x \in A$ and $\{x_n\} \subset O_T(x)$ the implication

$$x_{n_i} \to x^* \quad \text{implies that} \quad Tx_{n_i} \to Tx^*,\ i \to +\infty.$$

holds for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$. If the mapping T is best orbitally continuous at each point in A, then the mapping T is said to be best orbitally continuous on A.

Remark 1.19. Take $A = B = X$ in Definition 1.18. Then the best orbitally continuity of the mapping T becomes the orbitally continuity of T in the sense of Ćirić.

Definition 1.20. ([20]) Let (X, d) be a metric space, A, B be nonempty subsets of X. Suppose that $T : A \to B$ and $g : A \to \mathbb{R}$ are two mappings. If for each $x \in A$ and $\{x_n\} \subset O_T(x)$,

$$x_{n_i} \to x^* \quad \text{implies that} \quad g(x^*) \leq \liminf_{i\to+\infty} g(x_{n_i}),$$

holds for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, then f is said to be best orbitally lower semi-continuous at x^* in A. If the mapping g is best orbitally lower semi-continuous at each point in A, then it is said to be best orbitally lower semi-continuous on A.

It easily follows that if we take $A = B = X$ in the above definition, then the best orbitally lower semi-continuity of g becomes the orbitally lower semi-continuity of g.

Definition 1.21. ([20]) Let (X, d) be a metric space, A, B be nonempty subsets of X and $T : A \to B$ be a mapping. The set A is said to be T-best orbitally complete, if for all $x \in A$ and $\{x_n\}$ in $O_T(x)$, every Cauchy subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges to a point in A_0.

Lemma 1.22. ([20]) *Let (X, d) be a metric space, A, B be two nonempty subsets of X. Suppose that $T : A \to B$ is a given mapping. If T is best orbitally lower semi-continuous on A, then $g : A \to \mathbb{R}$ defined as $g(x) = d(x, Tx)$ is best orbitally lower semi-continuous on A.*

Inspired by the results of Altun et al. [1], Sahin et al. [20] Karapınar [12] and Jleli [9], we introduce several type of interpolative proximal contractions and provide some sufficient conditions to prove the existence of best proximity points for these contractions in a metric space. Moreover, some interesting results can be deduced from our main results. Finally, some examples are given to show the validity of our results.

2. Main results

In this section, by combining these known contraction mappings, we firstly present some new type of contraction non-self mappings. We begin with the following definitions, that are crucial to introduce our main results.

Definition 2.1. Let (X, d) be a metric space and A, B be two nonempty subsets of X, $T : A \to B$ be a mapping. If there exist $\tau_1, \tau_2 \in [0, 1)$ such that the following inequality holds:

$$\alpha(x_1, x_2)d(x_1, x_2) \leq \psi([d(y_1, y_2)]^{\tau_1}[d(y_1, x_1)]^{\tau_2}[d(y_2, x_2)]^{1-\tau_1-\tau_2}), \qquad (2.1)$$

for all $x_1, x_2, y_1, y_2 \in A$ with $x_i \neq y_i$ for $i = 1, 2$ satisfying the condition

$$d(x_1, Ty_1) = d(x_2, Ty_2) = d(A, B),$$

then we say that T is an α-ψ interpolative Reich-Rus-Ćirić type proximal contraction of the first kind.

Definition 2.2. Let (X, d) be a metric space and A, B be two nonempty subsets of X, $T : A \to B$ be a mapping. If there exists $\tau_1 \in [0, 1)$ such that the following inequality holds:

$$\alpha(x_1, x_2)d(x_1, x_2) \leq \psi([d(x_1, y_1)]^{\tau_1}[d(x_2, y_2)]^{1-\tau_1}), \qquad (2.2)$$

for all $x_1, x_2, y_1, y_2 \in A$ with $x_i \neq y_i$ for $i = 1, 2$ satisfying the condition

$$d(x_1, Ty_1) = d(x_2, Ty_2) = d(A, B),$$

then we say that T is an α-ψ interpolative Kannan type proximal contraction of the first kind.

Definition 2.3. Let (X, d) be a metric space and A, B be two nonempty subsets of X, $T : A \to B$ be a mapping. If there exist $\tau_1, \tau_2 \in [0, 1)$ such that the following inequality holds:

$$\alpha(x_1, x_2)d(Tx_1, Tx_2) \leq \psi([d(Ty_1, Ty_2)]^{\tau_1}[d(Ty_1, Tx_1)]^{\tau_2}[d(Ty_2, Tx_2)]^{1-\tau_1-\tau_2}), \qquad (2.3)$$

for all $x_1, x_2, y_1, y_2 \in A$ with $Tx_i \neq Ty_i$ for $i = 1, 2$ satisfying the condition

$$d(x_1, Ty_1) = d(x_2, Ty_2) = d(A, B),$$

then we say that T is an α-ψ interpolative Reich-Rus-Ćirić type proximal contraction of the second kind.

Definition 2.4. Let (X, d) be a metric space and A, B be two nonempty subsets of X, $T : A \to B$ be a mapping. If there exists $\tau_1 \in [0, 1)$ such that the following inequality holds:

$$\alpha(x_1, x_2)d(Tx_1, Tx_2) \leq \psi([d(Tx_1, Ty_1)]^{\tau_1}[d(Tx_2, Ty_2)]^{1-\tau_1}), \qquad (2.4)$$

for all $x_1, x_2, y_1, y_2 \in A$ with $Tx_i \neq y_i$ for $i = 1, 2$ satisfying the condition

$$d(x_1, Ty_1) = d(x_2, Ty_2) = d(A, B),$$

then we say that T is an α-ψ interpolative Kannan type proximal contraction of the second kind.

The following theorem is one of our main results.

Theorem 2.5. *Let (X, d) be a metric space and A, B be the nonempty subsets of X such that B is approximately compact with respect to A. Define a mapping $T : A \to B$. If the following conditions hold:*

(i) *$T(A_0) \subset B_0$ and T is α-proximal admissible;*

(ii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1;$$

(iii) *T is an α-ψ interpolative Reich-Rus-Ćirić type proximal contraction of the first kind;*

(iv) *A is T-best orbitally complete and $g(x) = d(x, Tx)$ is best orbitally lower semi-continuous on A,*

then T has a best proximity point in A_0.

Proof. From (ii), there exist $x_0, x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A, B) \text{ and } \alpha(x_0, x_1) \geq 1. \tag{2.5}$$

Consider $Tx_1 \in T(A_0) \subset B_0$, so there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B). \tag{2.6}$$

Take (2.5) and (2.6) into account, since T is an α-proximal admissible, we can get

$$\alpha(x_1, x_2) \geq 1.$$

Using $T(A_0) \subset B_0$ again, clearly $Tx_2 \in T(A_0) \subset B_0$ still holds. Since T is α-proximal admissible, we have

$$d(x_3, Tx_2) = d(A, B) \text{ and } \alpha(x_2, x_3) \geq 1.$$

Continuing this process, we can produce a sequence $\{x_n\} \subset A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A, B) \text{ and } \alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}. \tag{2.7}$$

Now, if $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then from (2.7), we have $d(x_n, Tx_n) = d(A, B)$, that is, x_n is a best proximity point of T, the proof is completed. Therefore, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, that is,, $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. By the construction of sequence $\{x_n\}$, we known that

$$d(x_n, Tx_{n-1}) = d(A, B)$$

and

$$d(x_{n+1}, Tx_n) = d(A, B),$$

for all $n \geq 1$. Since T is an α-ψ interpolative Reich-Rus-Ćirić type contraction of the first kind, we apply $x_1 = x_n$, $x_2 = x_{n+1}$, $y_1 = x_{n-1}$ and $y_2 = x_n$ to (2.1), by (2.7), we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}) \\ &\leq \psi([d(x_{n-1}, x_n)]^{\tau_1} [d(x_{n-1}, x_n)]^{\tau_2} [d(x_n, x_{n+1})]^{1-\tau_1-\tau_2}) \\ &= \psi([d(x_{n-1}, x_n)]^{\tau_1+\tau_2} [d(x_n, x_{n+1})]^{1-\tau_1-\tau_2}). \end{aligned} \tag{2.8}$$

Suppose that, for some $n \in \mathbb{N}$, $d(x_n, x_{n+1}) \geq d(x_{n-1}, x_n)$. In this case, (2.8) is equal to

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \psi([d(x_{n-1}, x_n)]^{\tau_1+\tau_2} [d(x_n, x_{n+1})]^{1-\tau_1-\tau_2}) \\ &\leq \psi([d(x_n, x_{n+1})]^{\tau_1+\tau_2} [d(x_n, x_{n+1})]^{1-\tau_1-\tau_2}) \\ &= \psi(d(x_n, x_{n+1})) \\ &< d(x_n, x_{n+1}), \end{aligned}$$

which is a contradiction. So it must be $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. The fact that (2.8) can be rewritten as

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \psi([d(x_{n-1}, x_n)]^{\tau_1+\tau_2} [d(x_n, x_{n+1})]^{1-\tau_1-\tau_2}) \\ &\leq \psi([d(x_{n-1}, x_n)]^{\tau_1+\tau_2} [d(x_{n-1}, x_n)]^{1-\tau_1-\tau_2}) \\ &= \psi(d(x_{n-1}, x_n)) \\ &< d(x_{n-1}, x_n). \end{aligned} \tag{2.9}$$

This implies that the sequence $\{d(x_n, x_{n+1})\}$ is a strictly decreasing. So there exists $\xi \geq 0$ such that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = \xi$. From (2.9), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \psi(d(x_{n-1}, x_n)) \\ &\leq \psi^2(d(x_{n-2}, x_{n-1})) \\ &\;\;\vdots \\ &\leq \psi^n(d(x_0, x_1)). \end{aligned} \tag{2.10}$$

Take limit on the two sides of (2.10), by the property of ψ, we get

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{2.11}$$

Now, for all $m, n \in \mathbb{N}$ with $n < m$, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m).$$

Letting $n \to +\infty$, then by (2.11), we can get that $\lim_{n\to\infty} d(x_n, x_m) = 0$. It means that $\{x_n\}$ is a Cauchy sequence in A. Since A is a T-best orbitally complete, there exists $x^* \in A$ such $x_n \to x^*$ as $n \to +\infty$. On the other hand, by the definition of sequence $\{x_n\}$, we know that $d(x_{n+1}, Tx_n) = d(A, B)$, for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} d(x^*, B) &\leq d(x^*, Tx_n) \\ &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &= d(x^*, x_{n+1}) + d(A, B) \\ &\leq d(x^*, x_{n+1}) + d(x^*, B). \end{aligned}$$

Hence, $d(x^*, Tx_n) \to d(x^*, B)$ as $n \to +\infty$. Since B is approximately compact with respect to A, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \to y^*$ as $k \to +\infty$, for some $y^* \in B$. According to $d(x_{n+1}, Tx_n) = d(A, B)$, we can get

$$d(x^*, y^*) = d(A, B).$$

Finally, from (iv), we know that $g(x) = d(x, Tx)$ is best orbitally lower semi-continuous on A, so

$$\begin{aligned} d(A, B) &\leq d(x^*, Tx^*) \\ &= g(x^*) \\ &\leq \liminf_{k\to+\infty} g(x_{n_k}) \\ &= \liminf_{k\to+\infty} d(x_{n_k}, Tx_{n_k}) \\ &= d(x^*, y^*) \\ &= d(A, B). \end{aligned}$$

Hence, we obtain that $d(x^*, Tx^*) = d(A, B)$, so T has a best proximity point in A_0. □

If the best orbitally continuity of T instead of best orbitally lower semi-continuity of g, then we can remove the approximately compactness of B. Hence, we give the following result.

Theorem 2.6. *Let (X, d) be a metric space and A, B be the nonempty subsets of X. Define a mapping $T : A \to B$. If the following conditions hold:*

(i) *$T(A_0) \subseteq B_0$ and T is α-proximal admissible;*

(ii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1;$$

(iii) *T is an α-ψ interpolative Reich-Rus-Ćirić type proximal contraction of the first kind;*

(iv) *A is T-best orbitally complete and T is best orbitally continuous on A,*

then T has a best proximity point in A_0.

Proof. Let $x_0 \in A_0$, according to the proof of Theorem 2.5, we can get the same sequence $\{x_n\}$ in $O_T(x_0)$, Meanwhile, $\{x_n\}$ is also a Cauchy sequence in A. Since A is T-best orbitally complete, there exists $x^* \in A_0$ such that $x_n \to x^*$ as $n \to +\infty$. By the best orbitally continuity of T, when $n \to +\infty$, we have $Tx_n \to Tx$. So we get

$$d(x^*, Tx^*) = \lim_{n\to+\infty} d(x_{n+1}, Tx_n) = d(A, B).$$

Hence, T has a best proximity point in A. □

Next, we give the case of the Kannan type contraction of the first kind.

Theorem 2.7. *Let (X, d) be a metric space and A, B be nonempty subsets of X such that B is approximate compact with respect to A. Define a mapping $T : A \to B$. If the following conditions hold:*

(i) *$T(A_0) \subseteq B_0$ and T is α-proximal admissible;*

(ii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1;$$

(iii) *T is an α-ψ interpolative Kannan type proximal contraction of the first kind;*

(iv) *A is T-best orbitally complete and $g(x) = d(x, Tx)$ is best orbitally lower semi-continuous on A,*

then T has a best proximity point in A.

Proof. Similar to the proof of Theorem 2.5, we can get the similar sequence $\{x_n\}$ in A_0, for all $n \geq 1$, we obtain $\alpha(x_n, x_{n+1}) \geq 1$,

$$d(x_n, Tx_{n-1}) = d(A, B)$$

and

$$d(x_{n+1}, Tx_n) = d(A, B).$$

Since T an α-ψ interpolative Kannan type proximal contraction of the first kind, by (2.2), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}) \\ &\leq \psi([d(x_n, x_{n-1})]^{\tau_1} [d(x_{n+1}, x_n)]^{1-\tau_2}). \end{aligned} \tag{2.12}$$

Suppose that there exists some $n \in \mathbb{N}$ such that $d(x_n, x_{n+1}) \geq d(x_{n-1}, x_n)$. In this case, (2.12) is equal to

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \psi([d(x_n, x_{n-1})]^{\tau_1}[d(x_{n+1}, x_n)]^{1-\tau_1}) \\ &= \psi([d(x_{n-1}, x_n)]^{\tau_1}[d(x_n, x_{n+1})]^{1-\tau_1}) \\ &\leq \psi(d(x_n, x_{n+1})) \\ &< d(x_n, x_{n+1}), \end{aligned}$$

which is a contradiction. So $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$, for all $n \geq 1$. Then (2.12) can be rewritten as

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n). \tag{2.13}$$

Hence, $\{d(x_n, x_{n+1})\}$ is a strictly decreasing positive sequence. So there exists $\eta \geq 0$ such that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = \eta$. by (2.13), we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \psi(d(x_{n-1}, x_n)) \\ &\leq \psi^2(d(x_{n-2}, x_{n-1})) \\ &\ \ \vdots \\ &\leq \psi^n(d(x_0, x_1)). \end{aligned} \tag{2.14}$$

Take limit on the two sides of (2.14), by the property of ψ, when $n \to +\infty$, we get

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{2.15}$$

Now, for $\forall m, n \in \mathbb{N}$ with $n < m$, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m).$$

Let $n \to +\infty$, then by (2.15), we can deduce that $d(x_n, x_m) \to 0$, that is, $\{x_n\}$ is a Cauchy sequence in A. Following the similar proof of Theorem 2.5, we can also prove that $d(x^*, Tx^*) = d(A, B)$, that is, T has a best proximity point in A. □

In the case of Kannan type, the following result also holds naturally. We can also use the best orbitally continuity of T to replace the best orbitally semi-continuity of g, thus eliminate the approximate compactness of B.

Theorem 2.8. *Let (X, d) be a metric space and A, B be nonempty subsets of X. Define a mapping $T : A \to B$. If the following conditions hold:*

(i) *$T(A_0) \subset B_0$ and T is α-proximal admissible;*

(ii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1;$$

(iii) *T is an α-ψ interpolative Kannan type proximal contraction of the first kind;*
(iv) *A is T-best orbitally complete and T is best orbitally continuous on A,*

then T has a best proximity point in A.

Next, we discuss the cases of proximal contractions of the second kind. For the second kind of contractions, we remove the T-best orbitally completeness of A and the best orbitally lower semi-continuity of $g(x) = d(x, Tx)$, add the continuity of T, the completeness of metric space. In this case, we get the following results.

Theorem 2.9. *Let (X, d) be a complete metric space and A, B be two nonempty subsets of X with B is closed such that A is approximately compact with respect to B. Define a mapping $T : A \to B$. If the following conditions hold:*

(i) *$T(A_0) \subset B_0$ and T is α-proximal admissible;*
(ii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1;$$

(iii) *T is a continuous α-ψ interpolative Reich-Rus-Ćirić type proximal contraction of the second kind,*

then T has a best proximity point in A.

Proof. As shown in Theorem 2.5, we can easily find a sequence $\{x_n\} \subset A_0$ such that, for all $n \in \mathbb{N}$, $\alpha(x_n, x_{n+1}) \geq 1$ and

$$d(x_{n+1}, Tx_n) = d(A, B). \tag{2.16}$$

Now, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Otherwise, the proof is completed. Since T is an α-ψ interpolative Reich-Rus-Ćirić type proximal contraction of the second kind, we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq \alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}) \\ &\leq \psi([d(Tx_{n-1}, Tx_n)]^{\tau_1} [d(Tx_{n-1}, Tx_n)]^{\tau_2} [d(Tx_n, Tx_{n+1})]^{1-\tau_1-\tau_2}) \\ &= \psi([d(Tx_{n-1}, Tx_n)]^{\tau_1+\tau_2} [d(Tx_n, Tx_{n+1})]^{1-\tau_1-\tau_2}). \end{aligned}$$

For all $n \geq 1$, we have

$$d(Tx_n, Tx_{n+1}) \geq d(Tx_{n-1}, Tx_n).$$

Hence, the above equation can be rewritten as

$$d(Tx_n, Tx_{n+1}) \leq \psi([d(Tx_n, Tx_{n+1})]) < [d(Tx_n, Tx_{n+1})],$$

which is a contradiction. So

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq \psi([d(Tx_{n-1}, Tx_n)]) \\ &\leq \psi^2([d(Tx_{n-2}, Tx_{n-1})]) \\ &\ \vdots \\ &\leq \psi^n([d(Tx_0, Tx_1)]). \end{aligned}$$

Take limit on two sides of above inequality, when $n \to +\infty$, we can attain $d(Tx_n, Tx_{n+1}) \to 0$.

Next, for all $m, n \in \mathbb{N}$ with $n < m$, by the triangle inequality of metric, we have

$$d(Tx_n, Tx_m) \leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \cdots + d(Tx_{m-1}, Tx_m).$$

Take $n \to +\infty$, $m \to +\infty$, it easily follows that $d(Tx_n, Tx_m) \to 0$. So $\{Tx_n\}$ is a Cauchy sequence in B. Since (X, d) is a complete metric space and B is a closed subset of X, there exists $y^* \in B$ such that $Tx_n \to y^*$. Moreover, by (2.16), we can get

$$\begin{aligned} d(y^*, A) &\leq d(y^*, x_{n+1}) \\ &\leq d(y^*, Tx_n) + d(Tx_n, x_{n+1}) \\ &= d(y^*, Tx_n) + d(A, B) \\ &\leq d(y^*, Tx_n) + d(y^*, A). \end{aligned}$$

Therefore, when $n \to +\infty$, there must be $d(y^*, x_n) \to d(y^*, A)$. Since A is approximately compact with respect to B, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \to z^* \in A$ as $k \to +\infty$. Consider the continuity of T, by (2.16), we have

$$d(z^*, Tz^*) = \lim_{k \to +\infty} d(x_{n_k+1}, Tx_{n_k}) = d(A, B).$$

Hence T has a best proximity point in A. □

Using the similar method of Theorem 2.9, in the case of Kannan type of the second kind, we can also obtain the following theorem.

Theorem 2.10. *Let (X, d) be a complete metric space. A, B are two nonempty subsets of X with B is closed such that A is approximately compact with respect to B. Define a mapping $T : A \to B$. If the following conditions hold:*

(i) *$T(A_0) \subset B_0$ and T is α-proximal admissible;*

(ii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1;$$

(iii) *T is a continuous α-ψ interpolative Kannan type proximal contraction of the second kind,* *then T has a best proximity point in A.*

According to the particularity of the α-admissible mapping and the diversity of functions ψ. Take $\alpha(x, y) = 1$, $\psi(t) = kt$ with $k \in (0, 1)$, then there are some results can be derived directly from our main theorems.

Corollary 2.11. *Let (X, d) be a metric space and A, B be two nonempty subsets of X such that B is approximately compact with respect to A. Define a mapping $T : A \to B$. If the following conditions hold:*

- (i) *$T(A_0) \subset B_0$ and $A_0 \neq \varnothing$;*
- (ii) *T is an interpolative Reich-Rus-Ćirić type proximal contraction of the first kind;*
- (iii) *A is T-best orbitally complete and $g(x) = d(x, Tx)$ is best orbitally lower semi-continuous in A,*

then T has a best proximity point in A.

Corollary 2.12. *Let (X, d) be a metric space. A, B are two non-empty subsets of X such that B is approximately compact with respect to A. Define a mapping $T : A \to B$. If the following conditions hold:*

- (i) *$T(A_0) \subset B_0$ and $A_0 \neq \varnothing$;*
- (ii) *T is an interpolative Kannan type proximal contraction of the first kind;*
- (iii) *A is T-best orbitally complete and $g(x) = d(x, Tx)$ is best orbitally lower semi-continuous in A,*

then T has a best proximity point in A.

Remark 2.13. Take $\alpha(x, y) = 1$, $\psi(t) = kt$ with $k \in (0, 1)$.

- (1) We can easily show that Corollary 2.11 and Corollary 2.12 satisfy the all conditions of Theorem 2.5 and Theorem 2.6, respectively;
- (2) The condition (ii) can be omitted naturally, so we must require that $A_0 \neq \varnothing$.

In the case of proximal contraction of the second kind, the following results easily deduce from our main results.

Corollary 2.14. *Let (X, d) be a complete metric space and A, B be two nonempty subsets of X with B is closed such that A is approximately compact with respect to B. Define a mapping $T : A \to B$. If the following conditions hold:*

- (i) *$T(A_0) \subset B_0$ and $A_0 \neq \varnothing$;*
- (ii) *T is a continuous interpolative Reich-Rus-Ćirić type proximal contraction of the second kind,*

then T has a best proximity point in A.

Corollary 2.15. *Let (X, d) be a complete metric space. A, B are two non-empty subsets of X with B is closed such that A is approximately compact with respect to B. Define a mapping $T : A \to B$. If the following conditions hold:*

(i) *$T(A_0) \subset B_0$ and $A_0 \neq \varnothing$;*
(ii) *T is a continuous interpolative Kannan type proximal contraction of the second kind,*

then T has a best proximity point in A.

Now, in order to verify the validity of our theorems, we give the following examples.

Example 2.16. Let $X = [0, 3)^2$ and d be the Euclidean metric on $\mathbb{R}^2$. Set

$$A = \{(u, v) : u^2 + v^2 = 4\};$$

$$B = \{(u, v) : u^2 + v^2 = 1\}.$$

Then, clearly $A_0 = A$, $B_0 = B$ and $d(A, B) = 1$. Define a mapping $T : A \to B$ by

$$Tx = T(u, v) = \begin{cases} (\frac{u}{2}, \frac{v}{2}), & u \geq 0, \\ (-1, 0), & u < 0. \end{cases}$$

Take $\alpha(x_1, x_2) = 1$, $\psi(t) = \frac{t}{2}$. Firstly, we prove that T is a α-ψ interpolative Reich-Rus-Ćirić type proximal contraction mapping of the first kind. We notice that if $y = (w, z) \in A$ with $w \geq 0$, $x = (u, v) \in A$ with $u \geq 0$, we can get

$$d(x, Ty) = d((u, v), (\frac{w}{2}, \frac{z}{2})) = 1 \quad \Rightarrow \quad T(u, v) = (\frac{w}{2}, \frac{z}{2}).$$

Therefore, $(u, v) = (w, z)$, that is, $x = y$. Consider $y_1 = (w_1, z_1)$, $y_2 = (w_2, z_2) \in A$, where $w_1 < 0$ and $w_2 < 0$. Now, we have

$$d(x_1, Ty_1) = d((u_1, v_1), (-1, 0)) = 1 = d(A, B),$$

$$d(x_2, Ty_2) = d((u_2, v_2), (-1, 0)) = 1 = d(A, B).$$

It implies that $x_1 = x_2 = (-2, 0)$, so we have $d(x_1, x_2) = 0$. Therefore, by (2.1), T is an α-ψ interpolative Reich-Rus-Ćirić type proximal contraction mapping of the first kind. It is easy to verify that other conditions of Theorem 2.5 hold. Then T has a best proximity point in A_0.

Example 2.17. Let $X = \mathbb{R}^2$ and d be the Euclidean metric on $\mathbb{R}^2$. Set

$$A = \{(u, 0) : u \in \mathbb{R}\};$$

$$B = \{(u, 2) : u \in \mathbb{R}\}.$$

Then $A_0 = A$, $B_0 = B$ and $d(A, B) = 2$. Define a mapping $T : A \to B$ as

$$Tx = T(u, 0) = \begin{cases} (0, 2), & u < 0, \\ (u, 2), & u \geq 0. \end{cases}$$

Take $\alpha(x_1, x_2) = 1$, $\psi(t) = \frac{t}{2}$. We show that T is an α-ψ interpolative Kannan type proximal contraction of the first kind. Now, if $x = (u_1, v_2) \in A$ with $u_1 \geq 0$, $y = (u_2, v_2) \in A$ with $u_2 \geq 0$, Then the following implication holds:

$$d(y, Tx) = d(A, B) = 2 \quad \Rightarrow \quad x = y.$$

In order to get the inequality (2.2), we consider the following case. If $y_1 = (u_1, 0)$, $y_2 = (u_2, 0) \in A$ with $u_1 < 0$ and $u_2 < 0$, then

$$d(x_1, Ty_1) = d(A, B)$$

and

$$d(x_2, Ty_2) = d(A, B),$$

which implies that$x_1 = x_2 = (0, 0)$, so we get $d(x_1, x_2) = 0$. Therefore T is an α-ψ interpolative Kannan proximal contraction of the first kind. we can easily prove the other conditions of Theorem 2.6 hold. Then T has a best proximity point in A_0.

3. Conclusion

In this paper, we introduce several new types proximal contractions by some known mappings and provide some sufficient condition to ensure that the existence and uniqueness of the best proximity points of these contraction mappings in metric spaces. Some new results can be derived directly from our results. In the end, we utilize two non-trivial examples to verify our results.

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