Nonlinear Functional Analysis and Applications
Vol. 17, No. 1 (2012), pp. 89-98

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REGULARIZATION FOR THE PROBLEM OF FINDING A COMMON FIXED POINT OF A FINITE FAMILY OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we study some regularization methods for the problem of finding a common fixed point of a finite family of nonexpansive mappings $T_i : C_i \longrightarrow C_i, \ i = 1, 2, ..., N$ from a closed convex subset C_i of an uniformly convex and uniformly smooth Banach spaces into itself.

1. Introduction

Let E be a Banach space with its dual space E^*. For the sake of simplicity, the norm of E and E^* are denoted by the symbol $\|.\|$. We write $\langle x, x^* \rangle$ instead of $x^*(x)$ for $x^* \in E^*$ and $x \in E$. We use the symbols $\rightharpoonup$ and $\longrightarrow$ to denote the weak convergence and strong convergence, respectively.

The problem of finding a fixed point of a nonexpansive mapping $T : E \longrightarrow E$ is equivalent to the problem of finding a zero of operator $A = I - T$.

One of the methods to solve the problem $0 \in A(x)$, with A is maximal monotone in Hilbert space H, is the proximal point algorithm. This algorithm is proposed by Rockafellar [10]. Starting from any initial guess $x_0 \in H$, this algorithm generates a sequence $\{x_n\}$ given by

$$x_{n+1} = J^A_{c_n}(x_n + e_n), \tag{1.1}$$

where $J^A_r = (I + rA)^{-1}$ is the resolvent of A on the space H for all $r > 0$. Rockafellar [10] proved the weak convergence of his algorithm (1.1) provided

Received March 18, 2011. Revised February 21, 2012.
2010 Mathematics Subject Classification: 47H06, 47H09, 47H10, 47J25.
Keywords: Accretive operators, uniformly smooth and uniformly convex Banach space, sunny nonexpansive retraction, weak sequential continuous mapping, regularization.

that the regularization sequence $\{c_n\}$ remains bounded away from zero and the error sequence $\{e_n\}$ satisfies the condition $\sum_{n=0}^{\infty} \|e_n\| < \infty$. However, Güler's example [7] shows that in infinite dimensional Hilbert space, proximal point algorithm (1.1) has only weak convergence. An example of the authors Bauschke, Matoušková and Reich [4] also show that the proximal algorithm only converges weakly but not in norm.

Ryazantseva [11] extended the proximal point algorithm (1.1) for the case that A is a $m-$accretive mapping in a properly Banach space E and proved the weak convergence the sequence of iterations of (1.1) to a solution of the equation $0 \in A(x)$ which is assumed to be unique. Then, to obtain the strong convergence for algorithm (1.1), Ryazantseva [12] combined the proximal algorithm with the regularization, named regularization proximal algorithm, in the form

$$c_n(A(x_{n+1}) + \alpha_n x_{n+1}) + x_{n+1} = x_n, \; x_0 \in E. \qquad (1.2)$$

Under some conditions on c_n and α_n, the strong convergence of $\{x_n\}$ of (1.2) is guaranteed only when the dual mapping j is weak sequential continuous and strong continuous, and the sequence $\{x_n\}$ is bounded.

Attouch and Alvarez [3] considered an extension of the proximal point algorithm (1.1) in the form

$$c_n A(u_{n+1}) + u_{n+1} - u_n = \gamma_n(u_n - u_{n-1}), \; u_0, \; u_1 \in H, \qquad (1.3)$$

which is called an inertial proximal point algorithm, where $\{c_n\}$ and $\{\gamma_n\}$ are two sequences of positive numbers. With this algorithm we also only obtained weak convergence of the sequence $\{x_n\}$ to a solution of problem $A(x) \ni 0$ in Hilbert space when the sequences $\{c_n\}$ and $\{\gamma_n\}$ are chosen suitable [3].

The purpose of this paper is to construct an operator version of the Tikhonov regularization method and give a regularization inertial proximal point algorithm to solve the problem of finding a common fixed point of a finite family of nonexpansive self - mappings $T_i : \; C_i \longrightarrow C_i, \; i = 1, 2, ..., N$ on a closed convex subset C_i of an uniformly convex and uniformly smooth Banach space E. Next, in the final section we give an application for the convex feasibility problems.

2. Preliminaries

Definition 2.1. A Banach space E is said to be uniformly convex if for any $\varepsilon \in (0, 2]$ the inequalities $\|x\| \leq 1, \; \|y\| \leq 1, \; \|x - y\| \geq \varepsilon$ imply there exists a $\delta = \delta(\varepsilon) \geq 0$ such that

$$\frac{\|x + y\|}{2} \leq 1 - \delta.$$

The function

$$\delta_E(\varepsilon) = \inf\{1 - 2^{-1}\|x + y\| : \; \|x\| = \|y\| = 1, \; \|x - y\| = \varepsilon\} \qquad (2.1)$$

is called the modulus of convexity of the space E, it defined on the interval $[0, 2]$ is continuous, increasing and $\delta_E(0) = 0$. The space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0,\ \forall \varepsilon \in (0, 2]$.
The function

$$\rho_E(\tau) = \sup\{2^{-1}\big(\|x + y\| + \|x - y\|\big) - 1 :\ \|x\| = 1,\ \|y\| = \tau\}, \tag{2.2}$$

is called the modulus of smoothness of the space E, it defined on the interval $[0, +\infty)$ is convex, continuous, increasing and $\rho_E(0) = 0$.

Definition 2.2. A Banach space E is said to be uniformly smooth if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0. \tag{2.3}$$

It is well known that every uniformly convex and uniformly smooth Banach space is reflexive.

Definition 2.3. A mapping j from E onto E^* satisfying the condition

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\} \tag{2.4}$$

is called the normalized duality mapping of E.

In any smooth Banach space $J(x) = 2^{-1}\text{grad}\|x\|^2$, and if E is a Hilbert space, then $J = I$, where I is the identity mapping. It is well known that if E^* is stricly convex or E is smooth, then J is single valued. Suppose that J is single valued, then J is said to be weakly sequentially continuous if for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x,\ J(x_n) \overset{*}{\rightharpoonup} J(x)$. We denote the single valued normalized duality mapping by j.

Definition 2.4. An operator $A :\ D(A) \subseteq E \longrightarrow 2^E$ is called accretive if for all $x, y \in D(A)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \geq 0,\ \forall u \in A(x),\ v \in A(y). \tag{2.5}$$

Definition 2.5. An operator $A :\ D(A) \subseteq E \longrightarrow 2^E$ is called $m-$accretive if it is an accretive operator and the range $R(\lambda A + I) = E$ for all $\lambda > 0$.

If A is a $m-$accretive operator, then it is a demiclosed operator, i.e., if the sequence $\{x_n\} \subset D(A)$ satisfies $x_n \rightharpoonup x$ and $A(x_n) \longrightarrow f$, then $A(x) = f$ [2].

Definition 2.6. A mapping $T :\ C \longrightarrow E$ is called nonexpansive mapping on a closed convex subset C of a Banach space E if

$$\|Tx - Ty\| \leq \|x - y\|,\ \forall x, y \in C. \tag{2.6}$$

If $T :\ C \longrightarrow E$ is a nonexpansive mapping, then $I - T$ is accretive operator. In the case $C \equiv E$, we have $I - T$ is $m-$accretive operator.

Definition 2.7. Let C be a nonempty closed convex subset of E. A mapping $Q_C :\ E \longrightarrow C$ is said to be

(i) a retraction onto C if $Q_C^2 = Q_C$;

(ii) a nonexpansive retraction if it also satisfies the inequality

$$\|Q_C x - Q_C y\| \le \|x - y\|, \ \forall x, y \in E; \tag{2.7}$$

(iii) a sunny retraction if for all $x \in E$ and for all $t \in [0, +\infty)$,

$$Q_C(Q_C x + t(x - Q_C x)) = Q_C x. \tag{2.8}$$

A closed convex subset C of E is said to be a nonexpansive retract of E, if there exists a nonexpansive retraction from E onto C and is said to be a sunny nonexpansive retract of E, if there exists a sunny nonexpansive retraction from E onto C.

Proposition 2.8. [1] *Let C be a nonempty closed convex subset of a smooth Banach E. A mapping $Q_C : E \longrightarrow C$ is a sunny nonexpansive retraction if and only if*

$$\langle x - Q_C x, j(\xi - Q_G x)\rangle \le 0, \ \forall x \in E, \ \forall \xi \in C. \tag{2.9}$$

Reich [9] showed that if E is uniformly smooth and D is the fixed point set of a nonexpansive mapping from C into itself, then there is a sunny nonexpansive retraction from C onto D and it can be constructed as follows.

Lemma 2.9 (Reich [9])**.** *Let E be a uniformly smooth Banach space and let $T : C \longrightarrow C$ be a nonexpansive mapping with a fixed point. For each $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \longmapsto tu + (1-t)Tx$ converges strongly as $t \longrightarrow 0$ to a fixed point of T. Define $Q : C \longrightarrow Fix(T)$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $Fix(T)$; that is, Q satisfies the property*

$$\langle u - Qu, j(z - Qu)\rangle \le 0, \ u \in C, \ z \in Fix(T). \tag{2.10}$$

3. Main results

We need the following lemmas in the proof of our results.

Lemma 3.1. [1] *Let E be an uniformly convex and uniformly smooth Banach space. If $A = I - T$ with a nonexpansive mapping $T : D(A) \subset E \longrightarrow E$ then for all $x, y \in D(T)$, the domain of T,*

$$\langle Ax - Ay, j(x - y)\rangle \ge L^{-1} R^2 \delta_E\Big(\frac{\|Ax - Ay\|}{4R}\Big), \tag{3.1}$$

where $\|x\| \le R, \ \|y\| \le R$ and $1 < L < 1.7$ is Figiel constant.

Lemma 3.2. [8] *Let $\{a_n\}, \{b_n\}, \{\sigma_n\}$ be the sequences of positive numbers satisfying the conditions*

(i) $a_{n+1} \le (1 - b_n)a_n + \sigma_n, \ b_n < 1$;

(ii) $\sum_{n=0}^{\infty} b_n = +\infty,\ \lim_{n\to\infty} \sigma_n/b_n = 0.$

Then $\lim_{n\to\infty} a_n = 0.$

We consider the problem

$$\text{Find an element } x^* \in S = \cap_{i=1}^N F(T_i), \tag{3.2}$$

where $F(T_i)$ is the set of fixed points of nonexpansive mappings $T_i :\ C_i \longrightarrow C_i$ and C_i is a closed convex subset of an uniformly convex and uniformly smooth Banach space E, $i = 1, 2, ..., N$.

Theorem 3.3. *Suppose that E is a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a closed convex nonexpansive retract of E and let $T_i :\ C_i \longrightarrow C_i,\ i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N F(T_i) \neq \emptyset$.*

(i) *For each $\alpha_n > 0$ the equation*

$$\sum_{i=1}^{N} A_i(x_n) + \alpha_n x_n = 0, \tag{3.3}$$

has a unique solution x_n, where $A_i = I - T_iQ_{C_i}$ and $Q_{C_i} :\ E \longrightarrow C_i$ is a nonexpansive retraction form E onto $C_i,\ i = 1, 2, ..., N$;

(ii) *If in addition, $\alpha_n \longrightarrow 0$, then $x_n \longrightarrow Q_S\theta$, where $Q_S :\ E \longrightarrow S$ is a sunny nonexpansive retraction from E onto S and θ is origin of E.*

Moreover, we have the following estimate

$$\|x_{n+1} - x_n\| \leq \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n} R_0, \tag{3.4}$$

where $R_0 = 2\|Q_S\theta\|$.

Proof. (i) First, it is clear that $T_iQ_{C_i}$ is a nonexpansive mapping on E and $F(T_i) = F(T_iQ_{C_i}),\ i = 1, 2, ..., N$, so $S = \cap_{i=1}^N F(T_iQ_{C_i})$. Since the operator $\sum_{i=1}^N A_i$ is Lipschitz continuous and accretive on E, it is $m-$accretive [5]. Therefore equation (3.3) has a unique solution x_n.

(ii) For each $x^* \in S$, we have

$$\langle \sum_{i=1}^{N} A_i(x_n), j(x_n - x^*)\rangle + \alpha_n\langle x_n, j(x_n - x^*)\rangle = 0. \tag{3.5}$$

By the accretiveness of $\sum_{i=1}^N A_i$, we obtain

$$\langle x_n, j(x_n - x^*)\rangle \leq 0. \tag{3.6}$$

The obtained inequatlity yields the estimates

$$\|x_n - x^*\|^2 \leq \langle x^*, j(x_n - x^*)\rangle \leq \|x^*\|\|x_n - x^*\|. \tag{3.7}$$

Hence, $\|x_n\| \leq 2\|x^*\|$, i.e., the sequence $\{x_n\}$ is bounded. Every bounded set in a reflexive Banach space is relatively weakly compact. This means that there exists some subsequence $\{x_{n_k}\} \subset \{x_n\}$ and an element $\overline{x} \in E$ such that $x_{n_k} \rightharpoonup \overline{x}$ as $k \longrightarrow \infty$.

We will show that $\overline{x} \in S$. Indeed, for each $i \in \{1, 2, ..., N\}$, $x^* \in S$ and $R > 0$ satisfies $R \geq \max\{\sup \|x_n\|, \|x^*\|\}$, and by using Lemma 3.1, we have

$$\begin{aligned}\delta_E\left(\frac{\|A_i(x_n)\|}{4R}\right) &\leq \frac{L}{R^2}\langle A_i(x_n), j(x_n - x^*)\rangle \\ &\leq \frac{L}{R^2}\langle \sum_{k=1}^{N} A_k(x_n), j(x_n - x^*)\rangle \\ &\leq \frac{L\alpha_n}{R^2}\|x_n\|.\|x_n - x^*\| \\ &\leq \frac{L\alpha_n}{R^2} 2\|x^*\|^2 \longrightarrow 0,\ n \longrightarrow \infty.\end{aligned}$$

By the continuity of the function $\delta_E(.)$ and the uniformly convexity of Banach space E, we obtain $A_i(x_n) \longrightarrow 0,\ n \longrightarrow \infty$. Every $m-$accretive operator is demiclosed, hence $A_i(\overline{x}) = 0$. Since $i \in \{1, 2, ..., N\}$ is arbitrary element, so $\overline{x} \in S$.

In inequality (3.7) replacing x_n by x_{n_k} and x^* by $\overline{x}$, using the weak continuity of j we obtain $x_{n_k} \longrightarrow \overline{x}$. From inequality (3.6) we get

$$\langle \overline{x}, j(\overline{x} - x^*)\rangle \leq 0,\ \forall x^* \in S. \tag{3.8}$$

Now, we show that the inequality (3.8) has unique solution. Suppose that $\overline{x}_1 \in S$ is also its solution. Then

$$\langle \overline{x}_1, j(\overline{x}_1 - x^*)\rangle \leq 0,\ \forall x^* \in S. \tag{3.9}$$

In inequalities (3.8) and (3.9) replacing x^* by $\overline{x}_1$ and $\overline{x}$, respectively, we obtain

$$\begin{aligned}&\langle \overline{x}, j(\overline{x} - \overline{x}_1)\rangle \leq 0, \\ &\langle -\overline{x}_1, j(\overline{x} - \overline{x}_1)\rangle \leq 0.\end{aligned}$$

Their combination gives $\|\overline{x} - \overline{x}_1\|^2 \leq 0$, thus $\overline{x} = \overline{x}_1 = Q_S\theta$ and the sequence $\{x_n\}$ converges weakly to $\overline{x} = Q_S\theta$, because $Q_S\theta$ satisfies the inequality (3.8). Finally, from the first inequality in (3.7), implies that $x_n \longrightarrow Q_S\theta$.

Now, we will prove inequality (3.4). In equation (3.3), replacing n by $n+1$, we obtain

$$\sum_{i=1}^{N} A_i(x_{n+1}) + \alpha_{n+1}x_{n+1} = 0. \tag{3.10}$$

From equations (3.10) and (3.3) and by the accretiveness of the operator $\sum_{i=1}^{N} A_i$, we get

$$\langle \alpha_{n+1} x_{n+1} - \alpha_n x_n, j(x_{n+1} - x_n) \rangle \leq 0. \tag{3.11}$$

Therefore,

$$\begin{aligned}\alpha_n \|x_{n+1} - x_n\|^2 &\leq (\alpha_{n+1} - \alpha_n)\langle -x_{n+1}, j(x_{n+1} - x_n)\rangle \\ &\leq |\alpha_{n+1} - \alpha_n|.\|x_{n+1}\|.\|x_{n+1} - x_n\| \\ &\leq 2\|Q_S\theta\|.|\alpha_{n+1} - \alpha_n|.\|x_{n+1} - x_n\|.\end{aligned}$$

Hence,

$$\|x_{n+1} - x_n\| \leq \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n} R_0, \ \forall n \geq 0,$$

where $R_0 = 2\|Q_S\theta\|$. □

Next, we consider a regularization inertial proximal point algorithm in the form

$$c_n\left(\sum_{i=1}^{N} A_i(u_{n+1}) + \alpha_n u_{n+1}\right) + u_{n+1} = u_n + \gamma_n(u_n - u_{n-1}), \ u_0, \ u_1 \in E \tag{3.12}$$

to solve problem (3.2).

Theorem 3.4. *Suppose that E is a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a closed convex nonexpansive retract of E and let $T_i: \ C_i \longrightarrow C_i, \ i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. If the sequences $\{c_n\}, \ \{\alpha_n\}$ and $\{\gamma_n\}$ satisfy*

(i) $0 < c_0 < c_n, \ \alpha_n > 0, \alpha_n \longrightarrow 0, \ \dfrac{|\alpha_{n+1} - \alpha_n|}{\alpha_n^2} \longrightarrow 0, \ \sum_{n=0}^{\infty} \alpha_n = +\infty$;

(ii) $\gamma_n \geq 0, \ \gamma_n \alpha_n^{-1} \|u_n - u_{n-1}\| \longrightarrow 0$,

then the sequence $\{u_n\}$ defined by equation (3.12) converges strongly to $Q_S\theta$, where $Q_S: \ E \longrightarrow S$ is a sunny nonexpansive retraction from E onto S.

Proof. First, we show that equation (3.12) has unique solution u_{n+1}. Indeed, since the operator $\sum_{i=1}^{N} A_i$ is Lipschitz continuous and accretive on E, it is $m-$accretive [5]. Therefore, equation (3.12) has a unique solution u_{n+1}.
Now, we rewrite equations (3.3) and (3.12) in their equivalent forms

$$d_n \sum_{i=1}^{N} A_i(x_n) + x_n = \beta_n x_n, \tag{3.13}$$

$$d_n \sum_{i=1}^{N} A_i(u_{n+1}) + u_{n+1} = \beta_n(u_n + \gamma_n(u_n - u_{n-1})), \tag{3.14}$$

where $\beta_n = \dfrac{1}{1+c_n\alpha_n}$ and $d_n = c_n\beta_n$.
From equations (3.13) and (3.14) and by virtue of the property of $\sum_{i=1}^N A_i$, we get

$$\|u_{u+1} - x_n\| \le \beta_n\|u_n - x_n\| + \beta_n\gamma_n\|u_n - u_{n-1}\|.$$

Therefore,

$$\begin{aligned}\|u_{n+1} - x_{n+1}\| &\le \|u_{n+1} - x_n\| + \|x_{n+1} - x_n\| \\ &\le \beta_n\|u_n - x_n\| + \beta_n\gamma_n\|u_n - u_{n-1}\| + \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n}R_0,\end{aligned} \tag{3.15}$$

or equivalent to

$$\|u_{n+1} - x_{n+1}\| \le (1 - b_n)\|u_n - x_n\| + \sigma_n,\ b_n = \frac{c_n\alpha_n}{1+c_n\alpha_n}, \tag{3.16}$$

where $\sigma_n = \beta_n\gamma_n\|u_n - u_{n-1}\| + \dfrac{|\alpha_{n+1} - \alpha_n|}{\alpha_n}R_0$.
Under the assumption, we have

$$\begin{aligned}\frac{\sigma_n}{b_n} &= \frac{1}{c_n}\alpha_n^{-1}\gamma_n\|u_n - u_{n-1}\| + (\frac{1}{c_n} + \alpha_n)\frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n^2}R_0 \\ &\le \frac{1}{c_0}\alpha_n^{-1}\gamma_n\|u_n - u_{n-1}\| + (\frac{1}{c_0} + \alpha_n)\frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n^2}R_0 \longrightarrow 0.\end{aligned}$$

Furthermore, by $\sum_{n=0}^\infty \alpha_n = +\infty$ hence $\sum_{n=0}^\infty b_n = +\infty$.
By Lemma 3.2, $\|u_n - x_n\| \longrightarrow 0$. Since $x_n \longrightarrow Q_S\theta$ as $n \longrightarrow \infty$, $u_n \longrightarrow Q_S\theta$ as $n \longrightarrow \infty$. □

4. An application

Consider the following convex feasibility problem:

$$\text{Finding an element } x^* \in S = \cap_{i=1}^N S_i \ne \emptyset, \tag{4.1}$$

where $S_i,\ i = 1, 2, ..., N$ are closed convex sunny nonexpansive retracts of an uniformly convex and uniformly smooth Banach space E.

In this section, we give an application of regularization algorithms (3.3) and (3.12) to find a solution of (4.1).

Let Q_{S_i} denote the sunny nonexpansive retraction from E onto $S_i,\ i = 1, 2, ..., N$. It is clear that $F(Q_{S_i}) = S_i,\ i = 1, 2, ..., N$. Thus, the problem (4.1) is equivalent to the problem of finding a common fixed point of finite family of nonexpansive mappings $T_i = Q_{S_i},\ i = 1, 2, ..., N$. By Theorem 3.3 and Theorem 3.4, we have the following results:

Theorem 4.1. *If the positive sequence* $\{\alpha_n\}$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$*, then the sequence* $\{x_n\}$ *is defined by*

$$\sum_{i=1}^{N} B_i(x_n) + \alpha_n x_n = 0,\ n \geq 0, \tag{4.2}$$

converges strongly to a solution $Q_S\theta$ *of* (4.1)*, where* $B_i = I - Q_{S_i},\ i = 1, 2, ..., N,$ Q_S *is a sunny nonexpansive retraction from* E *onto* S*.*

Theorem 4.2. *If the sequences* $\{c_n\},\ \{\alpha_n\}$ *and* $\{\gamma_n\}$ *satisfy*

(i) $0 < c_0 < c_n,\ \alpha_n > 0, \alpha_n \longrightarrow 0,\ \dfrac{|\alpha_{n+1} - \alpha_n|}{\alpha_n^2} \longrightarrow 0,\ \sum_{n=0}^{\infty} \alpha_n = +\infty;$

(ii) $\gamma_n \geq 0,\ \gamma_n \alpha_n^{-1} \|u_n - u_{n-1}\| \longrightarrow 0,$

then the sequence $\{u_n\}$ *is defined by* $u_0,\ u_1 \in E$ *and*

$$c_n\Big(\sum_{i=1}^{N} B_i(u_{n+1}) + \alpha_n u_{n+1}\Big) + u_{n+1} = u_n + \gamma_n(u_n - u_{n-1}),\ n \geq 1 \tag{4.3}$$

converges strongly to a solution $Q_S\theta$ *of* (4.1)*, where* $B_i = I - Q_{S_i},\ i = 1, 2, ..., N,$ Q_S *is a sunny nonexpansive retraction from* E *onto* S*.*

Finally, we consider a special case of problem (4.1), it is the problem of finding a solution of a general system of linear equations.

Let S denote the set of solutions of the general system of linear equations

$$\sum_{j=1}^{k} a_{ij} x_j = b_i,\ i = 1, 2, ..., N, \tag{4.4}$$

and we suppose $S \neq \emptyset$, and $\sum_{j=1}^{k} a_{ij}^2 > 0,\ \forall i = 1, 2, ..., N$. An element $x^* \in S$ is called the normal solution of system (4.4) if $\|x^*\| \leq \|x\|,\ \forall x \in S$.

Let

$$S_i = \{(x_1, x_2, ..., x_k) \mid \sum_{j=1}^{k} a_{ij} x_j = b_i\},\ i = 1, 2, ..., N. \tag{4.5}$$

Then, S_i is a hyperplane in $\mathbb{R}^k$.

It is well - known that, the sunny nonexpansive retraction Q_{S_i} from $\mathbb{R}^k$ onto S_i is also the orthogonal projection from $\mathbb{R}^k$ onto $S_i,\ i = 1, 2, ..., N$. Moreover,

$$Q_{S_i}(x) = \left(x_l - a_{il}\frac{\sum\limits_{j=1}^{k} a_{ij} x_j - b_i}{\sum\limits_{j=1}^{n} a_{ij}^2}\right)_{l=1}^{k},\ i = 1, 2, ..., N, \tag{4.6}$$

for all $x = (x_1, ..., x_k) \in \mathbb{R}^k$.

We have two corollarys of Theorem 4.1 and Theorem 4.2, respectively:

Corollary 4.3. *If the positive sequence* $\{\alpha_n\}$ *satisfies* $\lim_{n\to\infty}\alpha_n = 0$, *then the sequence* $\{x^{(n)}\}$ *is defined by*

$$\sum_{i=1}^{N} B_i(x^{(n)}) + \alpha_n x^{(n)} = 0,\ n \geq 0, \tag{4.7}$$

converges strongly to the normal solution of system (4.4).

Corollary 4.4. *If the sequences* $\{c_n\}$, $\{\alpha_n\}$ *and* $\{\gamma_n\}$ *satisfy*

i) $0 < c_0 < c_n,\ \alpha_n > 0, \alpha_n \longrightarrow 0,\ \dfrac{|\alpha_{n+1}-\alpha_n|}{\alpha_n^2} \longrightarrow 0,\ \sum_{n=0}^{\infty}\alpha_n = +\infty$;

ii) $\gamma_n \geq 0,\ \gamma_n\alpha_n^{-1}\|u_n - u_{n-1}\| \longrightarrow 0$,

then the sequence $\{u^{(n)}\}$ *is defined by* $u^{(0)},\ u^{(1)} \in \mathbb{R}^k$ *and*

$$c_n\Big(\sum_{i=1}^{N} B_i(u^{(n+1)})+\alpha_n u^{(n+1)}\Big)+u^{(n+1)} = u^{(n)}+\gamma_n(u^{(n)}-u^{(n-1)}),\ n \geq 1 \tag{4.8}$$

converges strongly to the normal solution of system (4.4).

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