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# REMARKS ON "EQUILIBRIUM PROBLEMS IN HADAMARD MANIFOLDS" BY V. COLAO ET AL.

**Sehie Park**

The National Academy of Sciences, Republic of Korea, Seoul 137–044
and
Department of Mathematical Sciences, Seoul National University
Seoul 151–747, Korea
e-mail: shpark@math.snu.ac.kr

**Abstract.** In 2012, Colao, Lopez, Marino, and Martin-Marquez [J. Math. Anal. Appl. 388 (2012) 61–77] developed an equilibrium theory in Hadamard manifolds. In this paper, we show that three of their key results (the KKM lemma, the Ky Fan type minimax inequality, and Nash equilibrium theorem) on Hadamard manifolds can be extended to hyperbolic spaces and are particular ones for abstract convex spaces in the sense of ours in [7,8]. Similarly, most of main theorems in the KKM theory on abstract convex spaces can be applied to hyperbolic spaces and Hadamard manifolds.

## 1. Introduction

In 1990, Reich and Shafrir [9] introduced hyperbolic spaces in order to try to develop a theory of nonexpansive iterations in more general infinite-dimensional manifolds than normed vector spaces. This class of metric spaces contains all normed vector spaces and Hadamard manifolds, as well as the Hilbert ball and the Cartesian product of Hilbert balls.

In 1992, we began to study the KKM theory and, in 2006, to extend it to abstract convex spaces. Since 2008, we found that any hyperbolic spaces are G-convex spaces [4] and also particular cases of $c$-spaces [5-8]. Actually, in 2010 [7,8], we indicated but not concretely that most of key results in the KKM theory can be applied to hyperbolic spaces.

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Recently, in 2012, Colao, Lopez, Marino, and Martin-Marquez [1] developed an equilibrium theory in Hadamard manifolds. It is clear that their theory is closely related to the KKM theory on hyperbolic spaces.

In this paper, we show that three of their key results can be extended to hyperbolic spaces and are particular ones for abstract convex spaces in the sense of ours in [7,8]. Similarly, most of main theorems in the KKM theory on abstract convex spaces can be applied to hyperbolic spaces and Hadamard manifolds.

Section 2 devotes to review some preliminary facts on our abstract convex spaces as in [7,8]. In Section 3, we are concerned with definitions and examples of hyperbolic spaces and we show that any of such spaces are KKM spaces, which means that most results in [7,8] are applicable to them. Section 4 deals with a KKM type theorem on hyperbolic spaces. In Section 5, a minimax inequality on hyperbolic spaces is deduced from a general version on abstract convex spaces. It can applied to get the existence of solutions to an equilibrium problem under mild conditions on the bifunction $F$ as in [1]. Section 6 deals with the Nash type equilibrium theorems. We introduce two forms of them previously obtained by us. From one of them, we deduce a Nash equilibrium theorem on hyperbolic spaces generalizing the corresponding one on Hadamard manifolds in [1].

Note that [1] contains more results on nonexpansive maps on Hadamard manifolds as in [9] which are beyond of our reach in this paper.

## 2. Abstract convex spaces

We follow our recent works [7,8] and the references therein.

**Definition 2.1.** An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space $E$, a nonempty set $D$, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$, where $\langle D \rangle$ is the set of all nonempty finite subsets of $D$.

For any $D' \subset D$, the $\Gamma$-*convex hull* of $D'$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset $X$ of $E$ is called a $\Gamma$-*convex subset* of $(E, D; \Gamma)$ relative to $D'$ if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset $X$ of $E$ is said to be $\Gamma$-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, $X$ is $\Gamma$-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

**Definition 2.2.** Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all } A \in \langle D \rangle,$$

then $G$ is called a *KKM map*.

**Definition 2.3.** The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle.

**Example 2.4.** The following are typical examples of KKM spaces. For details, see [8] and the references therein.

(1) A *convex space* $(X, D) = (X, D; \Gamma)$ is a triple where $X$ is a subset of a vector space such that $\operatorname{co} D \subset X$, and each $\Gamma_A$ is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde for $X = D$.

(2) An abstract convex space $(X, D; \Gamma)$ is called an H-*space* by Park if $\Gamma = \{\Gamma_A\}$ is a family of contractible (or, more generally, $\omega$-connected) subsets of $X$ indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = X$, $(X; \Gamma)$ is called a *c*-space by Horvath.

(3) A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

   Here, $\Delta_n$ is the standard $n$-simplex with vertices $\{e_i\}_{i=0}^n$, and $\Delta_J$ the face of $\Delta_n$ corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \operatorname{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

(4) A *space having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a $\phi_A$-*space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

   consists of a topological space $X$, a nonempty set $D$, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular $n$-simplexes) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Convex space $\Longrightarrow$ H-space
$\Longrightarrow$ G-convex space $\Longrightarrow$ $\phi_A$-space $\Longrightarrow$ KKM space

$$\Longrightarrow \text{ Partial KKM space } \Longrightarrow \text{ Abstract convex space.}$$

Recall that, in 2010 [7], we derived generalized forms of the Ky Fan minimax inequality, the von Neumann–Sion minimax theorem, the von Neumann–Fan intersection theorem, the Fan type analytic alternative, and the Nash equilibrium theorem for partial KKM spaces. Consequently, our results in [7] unify and generalize most of previously known particular cases of the same nature.

Moreover, in [8], we clearly derived a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, [8] unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

## 3. Hyperbolic spaces

In 1990, Reich and Shafrir [9] introduced hyperbolic spaces in order to try to develop a theory of nonexpansive iterations in more general infinite-dimensional manifolds than normed vector spaces:

**Definition 3.1.** ([9]) Let $(X, \rho)$ be a metric space and $\mathbb{R}$ the real line. We say that a map $c : \mathbb{R} \to X$ is a *metric embedding* of $\mathbb{R}$ into $X$ if

$$\rho(c(s), c(t)) = |s - t|$$

for all real $s$ and $t$. The image of a metric embedding is called a *metric line*. The image of a real interval $[a, b] := \{t \in \mathbb{R} \mid a \leq t \leq b\}$ under such a map is called a *metric segment*.

Assume that $(X, \rho)$ contains a family $M$ of metric lines, such that for each pair of distinct points $x$ and $y$ in $X$ there is a unique metric line in $M$ which passes through $x$ and $y$. This metric line determines a unique metric segment denoted by $[x, y]$ joining $x$ and $y$. For each $0 \leq t \leq 1$ there is a unique point $z$ in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \quad \text{and} \quad \rho(z, y) = (1 - t)\rho(x, y).$$

This point is denoted by $(1 - t)x \oplus ty$.

We say that $X$, or more precisely $(X, \rho, M)$, is a *hyperbolic space* if

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \leq \frac{1}{2}\rho(y, z)$$

for all $x,\ y$ and $z$ in $X$.

**Example 3.2.** ([9]) The following are examples of hyperbolic spaces:

(1) All normed vector spaces.
(2) All Hadamard manifolds, that is, all finite-dimensional connected, simply connected, complete Riemannian manifolds of constant curvature.
(3) The Hilbert ball equipped with the hyperbolic metric.
(4) Arbitrary product of hyperbolic spaces.

**Definition 3.3.** ([9]) A subset $C$ of a hyperbolic space $X$ is said to be *convex* if, for each pair of points $x$ and $y$ in $C$, the metric segment $[x, y]$ is also contained in $C$. The *closed convex hull* of a subset $D$ of $X$ is the intersection of all closed convex subsets of $X$ which contains $D$.

In our previous works, we noted that any hyperbolic spaces are G-convex spaces [3] and also particular cases of $c$-spaces [5-8]. This can be strengthened as follows:

**Definition 3.4.** The *convex hull* $\operatorname{co} D$ of a subset $D$ of a hyperbolic space $X$ is the intersection of all convex subsets of $X$ which contains $D$.

**Lemma 3.5.** *Any convex subset $Y$ of a hyperbolic space $X = (X, \rho, M)$ can be made into a c-space $(X; \Gamma)$ and hence a KKM space.*

*Proof.* For any $A \in \langle Y \rangle$, let $\Gamma_A = \Gamma(A) = \operatorname{co} A$. Then it is easily seen to be contractible. Therefore $(X; \Gamma)$ is a $c$-space in the sense of Horvath and hence, a KKM space by our KKM theory. □

In view of Lemma 3.5, all results in [7,8] hold for any convex subset of a hyperbolic spaces. In the following sections, we give some examples of this fact in [1].

## 4. The KKM theorem on hyperbolic spaces

In 2012, V. Colao, G. Lopez, G. Marino, and V. Martin-Marquez [1] established an equilibrium theory for Hadamard manifolds. Actually, in their abstract, they claimed as follows: "The existence of equilibrium points for a bifunction is proved under suitable conditions, and applications to variational inequality, fixed point and Nash equilibrium problems are provided. The convergence of Picard iteration for firmly nonexpansive mappings along with the definition of resolvents for bifunctions in this setting is used to devise an algorithm to approximate equilibrium points."

From now on, we show that some key results in [1] can be extended to hyperbolic spaces instead of Hadamard manifolds by applying our KKM theory of abstract convex spaces in [7,8].

The following is given as [8, Theorem 3]:

**Theorem 4.1.** (Generalized partial KKM principle) *Let $(E, D; \Gamma)$ be a partial KKM space and $G : D \multimap E$ a map such that*

(1) *$G$ is closed-valued;*

(2) *$G$ is a KKM map (that is, $\Gamma_A \subset G(A)$ for all $A \in \langle D \rangle$); and*

(3) *there exists a nonempty compact subset $K$ of $E$ such that one of the following holds:*

(i) *$K = E$;*

(ii) *$K = \bigcap\{G(z) \mid z \in M\}$ for some $M \in \langle D \rangle$; or*

(iii) *for each $N \in \langle D \rangle$, there exists a compact $\Gamma$-convex subset $L_N$ of $E$ relative to some $D' \subset D$ such that $N \subset D'$ and*

$$L_N \cap \bigcap_{z \in D'} G(z) \subset K.$$

*Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.*

Since any hyperbolic space is a partial KKM space, Theorem 4.1 is applicable to hyperbolic spaces. Hence, we immediately have the following form of the KKM theorem in the setting of hyperbolic spaces:

**Theorem 4.2.** *Let $M$ be a hyperbolic space and $K \subset M$ a convex subset. Let $G : K \multimap K$ be a multimap such that, for each $x \in K$, $G(x)$ is closed. Suppose that*

(i) *there exists $x_0 \in K$ such that $G(x_0)$ is compact;*

(ii) *$\forall x_1, \dots, x_m \in K$, $\operatorname{co}\{x_1, \dots, x_m\} \subset \bigcup_{i=1}^m G(x_i)$.*

*Then $\bigcap_{x \in K} G(x) \neq \emptyset$.*

**Remark 4.3.** In [1, Lemma 3.1], Theorem 4.2 was provided in the setting of Hadamard manifolds with almost two page proof.

## 5. A minimax inequality on hyperbolic spaces

From the partial KKM principle we deduced the following very general version of the Ky Fan minimax inequality in [7, Theorem 4.1]:

**Theorem 5.1.** *Let $(X, D; \Gamma)$ be a partial KKM space, $f : D \times X \to \overline{\mathbb{R}}$, $g : X \times X \to \overline{\mathbb{R}}$ extended real functions, and $\gamma \in \overline{\mathbb{R}}$ such that*

(1) *for each $z \in D$, $\{y \in X \mid f(z, y) \leq \gamma\}$ is closed;*

(2) *for each $y \in X$, $\operatorname{co}_\Gamma\{z \in D \mid f(z, y) > \gamma\} \subset \{x \in X \mid g(x, y) > \gamma\}$; and*

(3) *the coercivity condition (3) of Theorem 4.1 holds for $G(z) := \{y \in X \mid f(z, y) \leq \gamma\}$.*

*Then either* (i) *there exists a $\widehat{x} \in X$ such that $f(z, \widehat{x}) \leq \gamma$ for all $z \in D$; or*

(ii) *there exists an $x_0 \in X$ such that $g(x_0, x_0) > \gamma$.*

*Moreover, if $\gamma := \sup_{x \in X} g(x, x)$, then we have*

$$\inf_{y\in X}\sup_{z\in D} f(z,y) \le \sup_{x\in X} g(x,x).$$

This is a correct formulation of [8, Theorem 4] and applicable to hyperbolic spaces.

From Theorem 4.2, as for [1, Theorem 3.2], we are able to get the existence of solutions to an equilibrium problem under mild conditions on the bifunction $F$. However, we can apply Theorem 5.1:

**Theorem 5.2.** *Let $F : K \times K \to \mathbb{R}$ be a bifunction on a convex subset $K$ of a hyperbolic space $M$ such that*

(i) *for any $x \in K$, $F(x,x) \ge 0$;*

(ii) *for every $x \in K$, the set $\{y \in K \mid F(x,y) < 0\}$ is convex;*

(iii) *for every $y \in K$, $x \mapsto F(x,y)$ is upper semicontinuous;*

(iv) *there exists a compact set $L \subset M$ and a point $y_0 \in L \cap K$ such that $F(x,y_0) < 0$ for all $x \in K \setminus L$.*

*Then there exists a point $x_0 \in L \cap K$ satisfying $F(x_0,y) \ge 0$ for all $y \in K$.*

*Proof.* Let

$$G(y) := \{x \in K \mid F(x,y) \ge 0\}.$$

(1) Since $F(\cdot,y)$ is upper semicontinuous by (iii), $G(y)$ is closed for all $y \in K$.

(2) For each $x \in K$, $\{y \in K \mid F(x,y) < 0\}$ is convex by (ii).

(3) By condition (iv) there exists a point $y_0 \in K$ for which $G(y_0) \subset L$, so $G(y_0)$ is compact. Hence (iv) is a coercivity condition (3) of Theorem 4.1.

Then, in view of condition (i), by Theorem 5.1, there exists a point $x_0 \in K$ such that $F(x_0,y) \ge 0$ for all $y \in K$. Note that $x_0 \in G(y_0) \subset L \cap K$. □

**Remark 5.3.** 1. If $K$ is a closed convex subset of an Hadamard manifold, then Theorem 5.2 reduces to [1, Theorem 3.2], whose proof can be modified to show that Theorem 4.2 implies Theorem 5.2.

2. We may assume $L \subset K$ instead of $L \subset M$ in Theorem 5.2.

By setting $L = K$ in Theorem 5.2, we have the following:

**Corollary 5.4.** *Let $K$ be a compact convex subset of a hyperbolic space $M$ and $F : K \times K \to \mathbb{R}$ such that*

(i) *for any $x \in K$, $F(x,x) \ge 0$;*

(ii) *for every $x \in K$, the set $\{y \in K \mid F(x,y) < 0\}$ is convex;*

(iii) *for every $y \in K$, $x \mapsto F(x,y)$ is upper semicontinuous.*

*Then there exists a point $x_0 \in K$ satisfying $F(x_0,y) \ge 0$ for all $y \in K$.*

**Remark 5.5.** If $K$ is a subset of an Hadamard manifold, then Corollary 5.4 reduces to [1, Corollary 3.3].

## 6. A Nash equilibrium theorem on hyperbolic spaces

Let $\{X_i\}_{i\in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X = \prod_{j\in I} X_j, \quad X^i = \prod_{j\in I\setminus\{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I \setminus \{i\}$, let $x^i_j$ denote the $j$th coordinate of $x^i$. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: its $i$th coordinate is $x_i$ and, for $j \neq i$ the $j$th coordinate is $x^i_j$. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where $x^i$ denotes the projection of $x$ in $X^i$.

In [7, Theorem 9.1], we obtained the following form of the Nash-Fan type equilibrium theorems:

**Theorem 6.1.** *Let $\{(X_i; \Gamma_i)\}_{i=1}^n$ be a finite family of compact abstract convex spaces such that $(X; \Gamma) = (\prod_{i=1}^n X_i; \Gamma)$ is a partial KKM space and, for each $i$, let $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ be real functions such that*

*(0) $f_i(x) \leq g_i(x)$ for each $x \in X$;*

*(1) for each $x^i \in X^i$, $x_i \mapsto g_i[x^i, x_i]$ is quasiconcave on $X_i$;*

*(2) for each $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ is u.s.c. on $X_i$; and*

*(3) for each $x_i \in X_i$, $x^i \mapsto f_i[x^i, x_i]$ is l.s.c. on $X^i$.*

*Then there exists a point $\widehat{x} \in X$ such that*

$$g_i(\widehat{x}) \geq \max_{y_i\in X_i} f_i[\widehat{x}^i, y_i] \quad \textit{for all} \quad i = 1, 2, \dots, n.$$

This can be applicable for hyperbolic spaces $X_i$ since any product of hyperbolic spaces is also hyperbolic and so satisfies the KKM principle. However, this is not comparable to the following generalized Nash-Ma type theorem [2, Theorem 5], [7, Theorem 9.2] where $I$ can be infinite:

**Theorem 6.2.** *Let $\{(X_i; \Gamma_i)\}_{i\in I}$ be a family of compact Hausdorff G-convex spaces and, for each $i \in I$, let $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ be real functions satisfying $(0) - (3)$ in Theorem 6.1. Then there exists a point $\widehat{x} \in X$ such that*

$$g_i(\widehat{x}) \geq \max_{y_i\in X_i} f_i[\widehat{x}^i, y_i] \quad \textit{for all} \quad i \in I.$$

From Theorem 6.2, we have the following for hyperbolic spaces:

**Theorem 6.3.** *For any $i \in I$, let $K_i$ be a compact convex subset of a hyperbolic space $M_i$ and $f_i : K \to \mathbb{R}$ a continuous function such that it is convex in the $i$-th variable. Then there exists a Nash equilibrium point.*

Recall that the Nash equilibrium problem associated to $\{K_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$ consists of finding $x = (x_i)_{i\in I} \in K$ such that $f_i(x) \geq \max_{y_i\in X_i} f_i[\widehat{x}^i, y_i]$ for all $i \in I$. In other words, no player can reduce his loss by varying his strategy alone.

*Proof.* Note that $K_i$ is a subset of a metric space and hence Hausdorff. Moreover, it is a $c$-space and hence $G$-convex. Therefore, we can apply Theorem 6.2 for $K_i$ instead of $X_i$. □

**Remark 6.4.** ([1, Theorem 3.12]) is a particular case of Theorem 6.3 for a finite family of Hadamard manifolds $M_i$.

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