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COMMON FIXED POINTS OF A PAIR OF MAPPINGS CONCERNING CONTRACTIVE INEQUALITIES OF INTEGRAL TYPE

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Abstract. Several common fixed point theorems for a pair of weakly compatible mappings satisfying contractive inequalities of integral type in a metric space are proved. The results obtained in this paper improve or differ from a few results existing in the literature.

1. Introduction and preliminaries

Throughout this paper, we assume that $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers and

$$\Phi_1 = \Big\{ \varphi \mid \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is Lebesgue integrable and summable on each compact subset of } \mathbb{R}^+ \text{ and } \int_0^{\varepsilon} \varphi(t)dt > 0, \forall \varepsilon > 0 \Big\};$$

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$\Phi_2 = \{\varphi \mid \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is nondecreasing continuous and } \varphi(t) = 0 \Leftrightarrow t = 0\}$;
$\Phi_3 = \{\varphi \mid \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is lower semi-continuous and } \varphi(t) > 0, \forall t > 0\}$;
$\Phi_4 = \{\varphi \mid \varphi \in \Phi_3 \text{ and } \varphi(0) = 0\}$;
$\Phi_5 = \{\varphi \mid \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is nondecreasing continuous such that } \varphi \text{ is positive on } (0, +\infty), \varphi(0) = 0 \text{ and } \lim_{t\to+\infty} \varphi(t) = +\infty\}$.

In 2002, Branciari [2] gave an integral version of the outstanding Banach contraction principle and became the first to research the existence of fixed points for the contractive mappings of integral type.

Theorem 1.1. ([2]) *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x, y \in X, \tag{1.1}$$

where $\varphi \in \Phi_1$ and $c \in [0, 1)$ is a constant. Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

Later on, the researchers [1,5,8-13,15] and others extended the result of Branciari and gained a lot of fixed point and common fixed point theorems for various contractive mappings of integral type in metric spaces. In particular, Altun et al. [1] proved a common fixed point theorem of weakly compatible mappings concerning a general contractive condition of integral type. Kumar et al. [5] gave a common fixed point theorem for a pair of compatible mappings satisfying a contractive inequality of integral type.

Theorem 1.2. ([5]) *Let (X, d) be a complete metric space and $f, g : X \to X$ be compatible mappings such that*

$$f(X) \subset g(X), \ g \text{ is continuous}$$

and

$$\int_0^{d(fx,fy)} \varphi(t)dt \le c \int_0^{d(gx,gy)} \varphi(t)dt, \quad \forall x, y \in X, \tag{1.2}$$

where $\varphi \in \Phi_1$ and $c \in [0, 1)$ is a constant. Then f and g have a unique common fixed point in X.

In 2001, Rhoades [14] introduced the concept of φ-weakly contractive mappings and proved the following fixed point theorem, which extends the Banach contraction principle.

Theorem 1.3. ([14]) *Let f be a mapping from a complete metric space (X, d) into itself satisfying*

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)), \quad \forall x, y \in X, \tag{1.3}$$

where $\varphi \in \Phi_5$. Then f has a unique fixed point in X.

Liu et al. [11] proved the following results for contractive mappings of integral type.

Theorem 1.4. ([11]) *Let f be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq \int_0^{d(x,y)} \varphi(t)dt - \int_0^{\psi(d(x,y))} \varphi(t)dt, \quad \forall x, y \in X, \tag{1.4}$$

where $\varphi \in \Phi_1$ and $\psi \in \Phi_4$. Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

Theorem 1.5. ([11]) *Let f be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq \int_0^{M(x,y)} \varphi(t)dt - \int_0^{\psi(M(x,y))} \varphi(t)dt, \quad \forall x, y \in X, \tag{1.5}$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\right\}, \tag{1.6}$$

$\varphi \in \Phi_1$ and $\psi \in \Phi_4$. Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

Remark 1.6. Clearly, $\Phi_4 \subseteq \Phi_3$, $\Phi_5 \subseteq \Phi_2 \cap \Phi_4$ and Theorem 1.4 extends Theorem 1.3.

Inspired by the results in [1-15], we introduce four classes of mappings satisfying the contractive inequalities of integral type as follows:

$$\phi\left(\int_0^{d(fx,fy)} \varphi(t)dt\right) \leq \phi\left(\int_0^{M_i(x,y)} \varphi(t)dt\right) - \int_0^{\psi(M_i(x,y))} \varphi(t)dt, \quad \forall x, y \in X, \tag{1.7}$$

where $i \in \{1,2,3,4\}$, $(\varphi,\phi,\psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$,

$$\begin{aligned} M_1(x,y) = \max\Bigg\{ & d(gx,gy), d(fx,gx), d(fy,gy), \frac{1}{2}[d(fx,gy)+d(gx,fy)], \\ & \frac{d(fx,gy)d(gx,fy)}{1+d(fx,fy)}, \frac{[1+d(fx,gx)]d(fy,gy)}{1+d(fx,fy)}, \\ & \frac{d(fx,gx)[1+d(fy,gy)]}{1+d(fx,fy)}, \frac{[1+d(fx,gy)]d(gx,fy)}{2+d(fx,fy)}, \\ & \frac{d(fx,gy)[1+d(gx,fy)]}{2+d(fx,fy)}, \frac{[1+d(fx,gy)]d(gx,fy)}{2+d(gx,gy)}, \\ & \frac{d(fx,gy)[1+d(gx,fy)]}{2+d(gx,gy)} \Bigg\}, \end{aligned} \tag{1.8}$$

$$\begin{aligned} M_2(x,y) = \max\Bigg\{ & d(gx,gy), d(fx,gx), d(fy,gy), \frac{1}{2}[d(fx,gy)+d(gx,fy)], \\ & \frac{d(fx,gy)d(gx,fy)}{1+d(fx,fy)}, \frac{d(fx,gy)d(gx,fy)}{1+d(gx,gy)} \Bigg\}, \end{aligned} \tag{1.9}$$

$$M_3(x,y) = \max\left\{ d(gx,gy), d(fx,gx), d(fy,gy), \frac{1}{2}[d(fx,gy)+d(gx,fy)] \right\} \tag{1.10}$$

and

$$M_4(x,y) = d(gx,gy), \tag{1.11}$$

some of which include (1.3)-(1.5) as special cases. Under certain conditions we prove the existence and uniqueness of common fixed points for these mappings. An example is constructed to show that Theorems 2.1 and 2.2 are different from Theorems 1.1-1.5.

Recall that a pair of self mappings f and g in a metric space (X,d) are said to be weakly compatible if they commute at their coincidence points.

The following lemma plays a key role in this paper.

Lemma 1.7. ([7]) *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = \int_0^a \varphi(t)dt.$$

2. Common fixed point theorems

Our main results are as follows:

Theorem 2.1. *Let (X, d) be a metric space, f and $g : X \to X$ be weakly compatible mappings satisfying*

$$\phi\bigg(\int_0^{d(fx,fy)} \varphi(t)dt\bigg) \le \phi\bigg(\int_0^{M_1(x,y)} \varphi(t)dt\bigg) - \int_0^{\psi(M_1(x,y))} \varphi(t)dt, \quad \forall x, y \in X, \tag{2.1}$$

where $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. If $f(X) \subseteq g(X)$, $g(X)$ is complete and $M_1(x, y)$ is defined by (1.8). Then f and g have a unique common fixed point in X.

Proof. Firstly, we attest that f and g have at most a common fixed point in X. Assume that f and g have two different common fixed points $a, b \in X$. Using (1.8), (2.1) and $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, we infer that

$$\begin{aligned}
M_1(a,b) = \max\bigg\{ & d(ga,gb), d(fa,ga), d(fb,gb), \frac{1}{2}[d(fa,gb)+d(ga,fb)], \\
& \frac{d(fa,gb)d(ga,fb)}{1+d(fa,fb)}, \frac{[1+d(fa,ga)]d(fb,gb)}{1+d(fa,fb)}, \\
& \frac{d(fa,ga)[1+d(fb,gb)]}{1+d(fa,fb)}, \frac{[1+d(fa,gb)]d(ga,fb)}{2+d(fa,fb)}, \\
& \frac{d(fa,gb)[1+d(ga,fb)]}{2+d(fa,fb)}, \frac{[1+d(fa,gb)]d(ga,fb)}{2+d(ga,gb)}, \\
& \frac{d(fa,gb)[1+d(ga,fb)]}{2+d(ga,gb)}\bigg\} \\
= \max\bigg\{ & d(a,b), 0, 0, d(a,b), \frac{d^2(a,b)}{1+d(a,b)}, 0, 0, \\
& \frac{[1+d(a,b)]d(a,b)}{2+d(a,b)}, \frac{d(a,b)[1+d(a,b)]}{2+d(a,b)}, \\
& \frac{[1+d(a,b)]d(a,b)}{2+d(a,b)}, \frac{d(a,b)[1+d(a,b)]}{2+d(a,b)}\bigg\} \\
= d(a,b) &
\end{aligned}$$

and

$$\begin{aligned}
0 &< \phi\bigg(\int_0^{d(a,b)} \varphi(t)dt\bigg) \\
&= \phi\bigg(\int_0^{d(fa,fb)} \varphi(t)dt\bigg) \\
&\le \phi\bigg(\int_0^{M_1(a,b)} \varphi(t)dt\bigg) - \int_0^{\psi(M_1(a,b))} \varphi(t)dt
\end{aligned}$$

$$
\begin{aligned}
&= \phi\bigg(\int_0^{d(a,b)} \varphi(t)dt\bigg) - \int_0^{\psi(d(a,b))} \varphi(t)dt \\
&< \phi\bigg(\int_0^{d(a,b)} \varphi(t)dt\bigg),
\end{aligned}
$$

which is impossible. Accordingly, f and g have at most a common fixed point.

Secondly, we claim that f and g have a common fixed point in X. Let x_0 be an arbitrary point in X. Since $f(X) \subseteq g(X)$, it follows that there exists a sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in X satisfying

$$fx_n = gx_{n+1}, \quad \forall n \in \mathbb{N}_0. \tag{2.2}$$

Put $d_n = d(fx_n, fx_{n+1})$ for all $n \in \mathbb{N}_0$. Assume that $d_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$. It follows that

$$fx_{n_0} = fx_{n_0+1} = gx_{n_0+1} \tag{2.3}$$

and

$$f^2x_{n_0+1} = fgx_{n_0+1} = gfx_{n_0+1} = g^2x_{n_0+1}. \tag{2.4}$$

Now we assert that $fx_{n_0+1} = f^2x_{n_0+1}$. Otherwise, in view of (1.8), (2.1)-(2.4) and $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, we deduce that

$$
\begin{aligned}
&M_1(fx_{n_0+1}, x_{n_0+1}) \\
&= \max\bigg\{ d(gfx_{n_0+1}, gx_{n_0+1}), d(f^2x_{n_0+1}, gfx_{n_0+1}), d(fx_{n_0+1}, gx_{n_0+1}), \\
&\qquad \frac{1}{2}[d(f^2x_{n_0+1}, gx_{n_0+1}) + d(gfx_{n_0+1}, fx_{n_0+1})], \\
&\qquad \frac{d(f^2x_{n_0+1}, gx_{n_0+1})d(gfx_{n_0+1}, fx_{n_0+1})}{1 + d(f^2x_{n_0+1}, fx_{n_0+1})}, \\
&\qquad \frac{[1 + d(f^2x_{n_0+1}, gfx_{n_0+1})]d(fx_{n_0+1}, gx_{n_0+1})}{1 + d(f^2x_{n_0+1}, fx_{n_0+1})}, \\
&\qquad \frac{d(f^2x_{n_0+1}, gfx_{n_0+1})[1 + d(fx_{n_0+1}, gx_{n_0+1})]}{1 + d(f^2x_{n_0+1}, fx_{n_0+1})}, \\
&\qquad \frac{[1 + d(f^2x_{n_0+1}, gx_{n_0+1})]d(gfx_{n_0+1}, fx_{n_0+1})}{2 + d(f^2x_{n_0+1}, fx_{n_0+1})}, \\
&\qquad \frac{d(f^2x_{n_0+1}, gx_{n_0+1})[1 + d(gfx_{n_0+1}, fx_{n_0+1})]}{2 + d(f^2x_{n_0+1}, fx_{n_0+1})}, \\
&\qquad \frac{[1 + d(f^2x_{n_0+1}, gx_{n_0+1})]d(gfx_{n_0+1}, fx_{n_0+1})}{2 + d(gfx_{n_0+1}, gx_{n_0+1})}, \\
&\qquad \frac{d(f^2x_{n_0+1}, gx_{n_0+1})[1 + d(gfx_{n_0+1}, fx_{n_0+1})]}{2 + d(gfx_{n_0+1}, gx_{n_0+1})} \bigg\}
\end{aligned}
$$

$$\begin{aligned}
&= \max\Big\{ d(f^2x_{n_0+1}, fx_{n_0+1}), 0, 0, d(f^2x_{n_0+1}, fx_{n_0+1}), \\
&\quad \frac{d^2(f^2x_{n_0+1}, fx_{n_0+1})}{1 + d(f^2x_{n_0+1}, fx_{n_0+1})}, 0, 0, \\
&\quad \frac{[1 + d(f^2x_{n_0+1}, fx_{n_0+1})]d(f^2x_{n_0+1}, fx_{n_0+1})}{2 + d(f^2x_{n_0+1}, fx_{n_0+1})}, \\
&\quad \frac{d(f^2x_{n_0+1}, fx_{n_0+1})[1 + d(f^2x_{n_0+1}, fx_{n_0+1})]}{2 + d(f^2x_{n_0+1}, fx_{n_0+1})}, \\
&\quad \frac{[1 + d(f^2x_{n_0+1}, fx_{n_0+1})]d(f^2x_{n_0+1}, fx_{n_0+1})}{2 + d(f^2x_{n_0+1}, fx_{n_0+1})}, \\
&\quad \frac{d(f^2x_{n_0+1}, fx_{n_0+1})[1 + d(f^2x_{n_0+1}, fx_{n_0+1})]}{2 + d(f^2x_{n_0+1}, fx_{n_0+1})} \Big\} \\
&= d(f^2x_{n_0+1}, fx_{n_0+1})
\end{aligned}$$

and

$$\begin{aligned}
0 &< \phi\Big(\int_0^{d(f^2x_{n_0+1}, fx_{n_0+1})} \varphi(t)dt \Big) \\
&\le \phi\Big(\int_0^{M_1(fx_{n_0+1}, x_{n_0+1})} \varphi(t)dt \Big) - \int_0^{\psi(M_1(fx_{n_0+1}, x_{n_0+1}))} \varphi(t)dt \\
&= \phi\Big(\int_0^{d(f^2x_{n_0+1}, fx_{n_0+1})} \varphi(t)dt \Big) - \int_0^{\psi(d(f^2x_{n_0+1}, fx_{n_0+1}))} \varphi(t)dt \\
&< \phi\Big(\int_0^{d(f^2x_{n_0+1}, fx_{n_0+1})} \varphi(t)dt \Big),
\end{aligned}$$

which is a contradiction. Therefore,

$$fx_{n_0+1} = f^2x_{n_0+1},$$

which together with (2.4) means that fx_{n_0+1} is a common fixed point of f and g in X.

Assume that $d_n \neq 0$ for all $n \in \mathbb{N}_0$. We show that $d_n \le d_{n-1}$ for all $n \in \mathbb{N}$. Or else, $d_n > d_{n-1}$ for some $n \in \mathbb{N}$. Making use of (1.8), (2.1), (2.2) and $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, we conclude that

$$\begin{aligned}
M_1(x_n, x_{n+1}) = \max\bigg\{ & d(gx_n, gx_{n+1}), d(fx_n, gx_n), d(fx_{n+1}, gx_{n+1}), \\
& \frac{1}{2}[d(fx_n, gx_{n+1}) + d(gx_n, fx_{n+1})], \\
& \frac{d(fx_n, gx_{n+1})d(gx_n, fx_{n+1})}{1 + d(fx_n, fx_{n+1})}, \\
& \frac{[1 + d(fx_n, gx_n)]d(fx_{n+1}, gx_{n+1})}{1 + d(fx_n, fx_{n+1})}, \\
& \frac{d(fx_n, gx_n)[1 + d(fx_{n+1}, gx_{n+1})]}{1 + d(fx_n, fx_{n+1})}, \\
& \frac{[1 + d(fx_n, gx_{n+1})]d(gx_n, fx_{n+1})}{2 + d(fx_n, fx_{n+1})}, \\
& \frac{d(fx_n, gx_{n+1})[1 + d(gx_n, fx_{n+1})]}{2 + d(fx_n, fx_{n+1})}, \\
& \frac{[1 + d(fx_n, gx_{n+1})]d(gx_n, fx_{n+1})}{2 + d(gx_n, gx_{n+1})}, \\
& \frac{d(fx_n, gx_{n+1})[1 + d(gx_n, fx_{n+1})]}{2 + d(gx_n, gx_{n+1})}\bigg\} \\
= \max\bigg\{ & d_{n-1}, d_{n-1}, d_n, \frac{1}{2}d(fx_{n-1}, fx_{n+1}), 0, \frac{(1 + d_{n-1})d_n}{1 + d_n}, \\
& \frac{d_{n-1}(1 + d_n)}{1 + d_n}, \frac{d(fx_{n-1}, fx_{n+1})}{2 + d_n}, 0, \frac{d(fx_{n-1}, fx_{n+1})}{2 + d_{n-1}}, 0\bigg\} \\
= \max & \{d_{n-1}, d_n\} \\
= d_n &
\end{aligned}$$

and

$$\begin{aligned}
0 < \phi\bigg(\int_0^{d_n} \varphi(t)dt\bigg) &= \phi\bigg(\int_0^{d(fx_n, fx_{n+1})} \varphi(t)dt\bigg) \\
&\le \phi\bigg(\int_0^{M_1(x_n, x_{n+1})} \varphi(t)dt\bigg) - \int_0^{\psi(M_1(x_n, x_{n+1}))} \varphi(t)dt \\
&= \phi\bigg(\int_0^{d_n} \varphi(t)dt\bigg) - \int_0^{\psi(d_n)} \varphi(t)dt \\
&< \phi\bigg(\int_0^{d_n} \varphi(t)dt\bigg),
\end{aligned}$$

which is illogical. Hence, $d_n \le d_{n-1}$ for all $n \in \mathbb{N}$ and

$$M_1(x_n, x_{n+1}) = d_{n-1}, \quad \forall n \in \mathbb{N}. \tag{2.5}$$

It is apparent that the sequence $\{d_n\}_{n\in\mathbb{N}_0}$ is nonincreasing and bounded, which implies that there exists r with

$$\lim_{n\to\infty} d_n = r \geq 0. \tag{2.6}$$

Now, we certify that $r = 0$. Otherwise, $r > 0$. Put

$$\liminf_{n\to\infty} \psi(d_n) = \alpha. \tag{2.7}$$

It follows that there exists a subsequence $\{d_{n(k)-1}\}_{k\in\mathbb{N}}$ of $\{d_n\}_{n\in\mathbb{N}_0}$ satisfying

$$\lim_{k\to\infty} \psi(d_{n(k)-1}) = \alpha. \tag{2.8}$$

Note that $\psi \in \Phi_3$ and (2.6)-(2.8) yield that

$$\alpha \geq \psi(r) > 0. \tag{2.9}$$

In terms of (2.1), (2.5)-(2.9), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.1, we get that

$$\begin{aligned}
0 &< \phi\bigg(\int_0^r \varphi(t)dt\bigg) = \limsup_{k\to\infty} \phi\bigg(\int_0^{d_{n(k)}} \varphi(t)dt\bigg) \\
&= \limsup_{k\to\infty} \phi\bigg(\int_0^{d(fx_{n(k)}, fx_{n(k)+1})} \varphi(t)dt\bigg) \\
&\leq \limsup_{k\to\infty} \bigg[\phi\bigg(\int_0^{M_1(x_{n(k)}, x_{n(k)+1})} \varphi(t)dt\bigg) - \int_0^{\psi(M_1(x_{n(k)}, x_{n(k)+1}))} \varphi(t)dt\bigg] \\
&\leq \limsup_{k\to\infty} \phi\bigg(\int_0^{M_1(x_{n(k)}, x_{n(k)+1})} \varphi(t)dt\bigg) - \liminf_{k\to\infty} \int_0^{\psi(M_1(x_{n(k)}, x_{n(k)+1}))} \varphi(t)dt \\
&= \limsup_{k\to\infty} \phi\bigg(\int_0^{d_{n(k)-1}} \varphi(t)dt\bigg) - \liminf_{k\to\infty} \int_0^{\psi(d_{n(k)-1})} \varphi(t)dt \\
&= \phi\bigg(\int_0^r \varphi(t)dt\bigg) - \int_0^{\alpha} \varphi(t)dt \\
&\leq \phi\bigg(\int_0^r \varphi(t)dt\bigg) - \int_0^{\psi(r)} \varphi(t)dt \\
&< \phi\bigg(\int_0^r \varphi(t)dt\bigg),
\end{aligned}$$

which is absurd. Thus $r = 0$, that is,

$$\lim_{n\to\infty} d_n = 0. \tag{2.10}$$

Next, we verify that $\{fx_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. If not, there exist a constant $\varepsilon > 0$ and two sequences $\{m(k)\}_{n\in\mathbb{N}_0}$ and $\{n(k)\}_{n\in\mathbb{N}_0}$ in $\mathbb{N}$ such that

$k < m(k) < n(k) < m(k+1)$ and

$$d(fx_{m(k)}, fx_{n(k)}) \geq \varepsilon \text{ and } d(fx_{m(k)}, fx_{n(k)-1}) < \varepsilon, \quad \forall k \in \mathbb{N}. \tag{2.11}$$

Notice that

$$\begin{aligned} &d(fx_{m(k)}, fx_{n(k)}) \leq d(fx_{m(k)}, fx_{n(k)-1}) + d_{n(k)-1}, \quad \forall k \in \mathbb{N}; \\ &|d(fx_{m(k)}, fx_{n(k)}) - d(fx_{m(k)}, fx_{n(k)-1})| \leq d_{n(k)-1}, \quad \forall k \in \mathbb{N}; \\ &|d(fx_{m(k)}, fx_{n(k)}) - d(fx_{m(k)-1}, fx_{n(k)})| \leq d_{m(k)-1}, \quad \forall k \in \mathbb{N}; \\ &|d(fx_{m(k)-1}, fx_{n(k)-1}) - d(fx_{m(k)-1}, fx_{n(k)})| \leq d_{n(k)-1}, \quad \forall k \in \mathbb{N}. \end{aligned} \tag{2.12}$$

By means of (2.10)-(2.12), we obtain that

$$\begin{aligned} \varepsilon &= \lim_{k\to\infty} d(fx_{m(k)}, fx_{n(k)}) = \lim_{k\to\infty} d(fx_{m(k)}, fx_{n(k)-1}) \\ &= \lim_{k\to\infty} d(fx_{m(k)-1}, fx_{n(k)}) = \lim_{k\to\infty} d(fx_{m(k)-1}, fx_{n(k)-1}). \end{aligned} \tag{2.13}$$

On account of (1.8), (2.2), (2.10) and (2.13), we receive that

$$\begin{aligned} M_1(x_{m(k)}, x_{n(k)}) = \max\Big\{ &d(gx_{m(k)}, gx_{n(k)}), d(fx_{m(k)}, gx_{m(k)}), d(fx_{n(k)}, gx_{n(k)}), \\ &\frac{1}{2}[d(fx_{m(k)}, gx_{n(k)}) + d(gx_{m(k)}, fx_{n(k)})], \\ &\frac{d(fx_{m(k)}, gx_{n(k)}) d(gx_{m(k)}, fx_{n(k)})}{1 + d(fx_{m(k)}, fx_{n(k)})}, \\ &\frac{[1 + d(fx_{m(k)}, gx_{m(k)})] d(fx_{n(k)}, gx_{n(k)})}{1 + d(fx_{m(k)}, fx_{n(k)})}, \\ &\frac{d(fx_{m(k)}, gx_{m(k)})[1 + d(fx_{n(k)}, gx_{n(k)})]}{1 + d(fx_{m(k)}, fx_{n(k)})}, \\ &\frac{[1 + d(fx_{m(k)}, gx_{n(k)})] d(gx_{m(k)}, fx_{n(k)})}{2 + d(fx_{m(k)}, fx_{n(k)})}, \\ &\frac{d(fx_{m(k)}, gx_{n(k)})[1 + d(gx_{m(k)}, fx_{n(k)})]}{2 + d(fx_{m(k)}, fx_{n(k)})}, \\ &\frac{[1 + d(fx_{m(k)}, gx_{n(k)})] d(gx_{m(k)}, fx_{n(k)})}{2 + d(gx_{m(k)}, gx_{n(k)})}, \\ &\frac{d(fx_{m(k)}, gx_{n(k)})[1 + d(gx_{m(k)}, fx_{n(k)})]}{2 + d(gx_{m(k)}, gx_{n(k)})} \Big\} \\ &\to \max\left\{ \varepsilon, 0, 0, \varepsilon, \frac{\varepsilon^2}{1+\varepsilon}, 0, 0, \frac{1+\varepsilon}{2+\varepsilon}\varepsilon, \frac{1+\varepsilon}{2+\varepsilon}\varepsilon, \frac{1+\varepsilon}{2+\varepsilon}\varepsilon, \frac{1+\varepsilon}{2+\varepsilon}\varepsilon \right\} \\ &= \varepsilon \quad \text{as } k \to \infty. \end{aligned}$$

Set

$$\liminf_{k\to\infty} \psi(M_1(x_{m(k)}, x_{n(k)})) = \beta. \tag{2.14}$$

Then, there exist two subsequences $\{x_{m(k_j)}\}_{j\in\mathbb{N}} \subseteq \{x_{m(k)}\}_{k\in\mathbb{N}}$ and $\{x_{n(k_j)}\}_{j\in\mathbb{N}} \subseteq \{x_{n(k)}\}_{k\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} \psi(M_1(x_{m(k_j)}, x_{n(k_j)})) = \beta. \tag{2.15}$$

Since ψ is lower semi-continuous, it follows from (2.13)-(2.15) that $\beta \geq \psi(\varepsilon) > 0$. On the basis of (2.1), (2.13)-(2.15), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 1.1, we deduce that

$$\begin{aligned}
0 &< \phi\bigg(\int_0^{\varepsilon} \varphi(t)dt\bigg) \\
&= \limsup_{j\to\infty} \phi\bigg(\int_0^{d(fx_{m(k_j)}, fx_{n(k_j)})} \varphi(t)dt\bigg) \\
&\leq \limsup_{j\to\infty} \bigg[\phi\bigg(\int_0^{M_1(x_{m(k_j)}, x_{n(k_j)})} \varphi(t)dt\bigg) - \int_0^{\psi(M_1(x_{m(k_j)}, x_{n(k_j)}))} \varphi(t)dt\bigg] \\
&\leq \limsup_{j\to\infty} \phi\bigg(\int_0^{M_1(x_{m(k_j)}, x_{n(k_j)})} \varphi(t)dt\bigg) - \liminf_{j\to\infty} \int_0^{\psi(M_1(x_{m(k_j)}, x_{n(k_j)}))} \varphi(t)dt \\
&= \phi\bigg(\int_0^{\varepsilon} \varphi(t)dt\bigg) - \int_0^{\beta} \varphi(t)dt \\
&\leq \phi\bigg(\int_0^{\varepsilon} \varphi(t)dt\bigg) - \int_0^{\psi(\varepsilon)} \varphi(t)dt \\
&< \phi\bigg(\int_0^{\varepsilon} \varphi(t)dt\bigg),
\end{aligned}$$

which is a contradiction. Hence $\{fx_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Since $g(X)$ is complete, it follows that there exist $u, v \in X$ with

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u = gv. \tag{2.16}$$

Suppose that $fv \neq u$. Note that

$$\begin{aligned}
M_1(v,x_n) = \max\Big\{& d(gv,gx_n), d(fv,gv), d(fx_n,gx_n), \\
& \frac{1}{2}[d(fv,gx_n)+d(gv,fx_n)], \frac{d(fv,gx_n)d(gv,fx_n)}{1+d(fv,fx_n)}, \\
& \frac{[1+d(fv,gv)]d(fx_n,gx_n)}{1+d(fv,fx_n)}, \frac{d(fv,gv)[1+d(fx_n,gx_n)]}{1+d(fv,fx_n)}, \\
& \frac{[1+d(fv,gx_n)]d(gv,fx_n)}{2+d(fv,fx_n)}, \frac{d(fv,gx_n)[1+d(gv,fx_n)]}{2+d(fv,fx_n)}, \\
& \frac{[1+d(fv,gx_n)]d(gv,fx_n)}{2+d(gv,gx_n)}, \frac{d(fv,gx_n)[1+d(gv,fx_n)]}{2+d(gv,gx_n)}\Big\} \\
\to \max\Big\{& d(u,u), d(fv,u), d(u,u), \frac{1}{2}[d(fv,u)+d(u,u)], \\
& \frac{d(fv,u)d(u,u)}{1+d(fv,u)}, \frac{[1+d(fv,u)]d(u,u)}{1+d(fv,u)}, \frac{d(fv,u)[1+d(u,u)]}{1+d(fv,u)}, \\
& \frac{[1+d(fv,u)]d(u,u)}{2+d(fv,u)}, \frac{d(fv,u)[1+d(u,u)]}{2+d(fv,u)}, \\
& \frac{[1+d(fv,u)]d(u,u)}{2+d(u,u)}, \frac{d(fv,u)[1+d(u,u)]}{2+d(u,u)}\Big\} \\
= \max\Big\{& 0, d(fv,u), 0, \frac{1}{2}d(fv,u), 0, 0, \frac{d(fv,u)}{1+d(fv,u)}, \\
& 0, \frac{d(fv,u)}{2+d(fv,u)}, 0, \frac{1}{2}d(fv,u)\Big\} \\
= d(fv,u) & \quad \text{as } n\to\infty.
\end{aligned}$$

Put

$$\liminf_{n\to\infty} \psi(M_1(v,x_n)) = \gamma.$$

Obviously, there exists a subsequence $\{x_{n(k)}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}_0}$ such that

$$\lim_{k\to\infty} \psi(M_1(v,x_{n(k)})) = \gamma \geq \psi(d(fv,u)) > 0.$$

In accordance with (2.1), (2.16), $(\varphi,\phi,\psi)\in\Phi_1\times\Phi_2\times\Phi_3$ and Lemma 1.1, we arrive at

$$
\begin{aligned}
0 &< \phi\bigg(\int_0^{d(fv,u)} \varphi(t)dt\bigg) \\
&= \limsup_{k\to\infty} \phi\bigg(\int_0^{d(fv,fx_{n(k)})} \varphi(t)dt\bigg) \\
&\le \limsup_{k\to\infty} \bigg[\phi\bigg(\int_0^{M_1(v,x_{n(k)})} \varphi(t)dt\bigg) - \int_0^{\psi(M_1(v,x_{n(k)}))} \varphi(t)dt\bigg] \\
&\le \limsup_{k\to\infty} \phi\bigg(\int_0^{M_1(v,x_{n(k)})} \varphi(t)dt\bigg) - \liminf_{k\to\infty} \int_0^{\psi(M_1(v,x_{n(k)}))} \varphi(t)dt \\
&= \phi\bigg(\int_0^{d(fv,u)} \varphi(t)dt\bigg) - \int_0^{\gamma} \varphi(t)dt \\
&\le \phi\bigg(\int_0^{d(fv,u)} \varphi(t)dt\bigg) - \int_0^{\psi(d(fv,u))} \varphi(t)dt \\
&< \phi\bigg(\int_0^{d(fv,u)} \varphi(t)dt\bigg),
\end{aligned}
$$

which is impossible. Consequently, $u = fv = gv$.

Note that f and g are weakly compatible. It follows that

$$
fu = f^2v = fgv = gfv = g^2v = gu. \tag{2.17}
$$

Suppose that $u \neq fu$. In view of (1.8), (2.1), (2.17) and $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, we give that

$$
\begin{aligned}
M_1(v, gv) = \max \bigg\{ & d(gv, g^2v), d(fv, gv), d(fgv, g^2v), \\
& \frac{1}{2}[d(fv, g^2v) + d(gv, fgv)], \frac{d(fv, g^2v)d(gv, fgv)}{1 + d(fv, fgv)}, \\
& \frac{[1 + d(fv, gv)]d(fgv, g^2v)}{1 + d(fv, fgv)}, \frac{d(fv, gv)[1 + d(fgv, g^2v)]}{1 + d(fv, fgv)}, \\
& \frac{[1 + d(fv, g^2v)]d(gv, fgv)}{2 + d(fv, fgv)}, \frac{d(fv, g^2v)[1 + d(gv, fgv)]}{2 + d(fv, fgv)}, \\
& \frac{[1 + d(fv, g^2v)]d(gv, fgv)}{2 + d(gv, g^2v)}, \frac{d(fv, g^2v)[1 + d(gv, fgv)]}{2 + d(gv, g^2v)} \bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \max\Big\{d(u,fu), d(u,u), d(fu,fu), \frac{1}{2}[d(u,fu)+d(u,fu)],\\
&\qquad \frac{d^2(u,fu)}{1+d(u,fu)}, \frac{[1+d(u,u)]d(fu,fu)}{1+d(u,fu)}, \frac{d(u,u)[1+d(fu,fu)]}{1+d(u,fu)},\\
&\qquad \frac{[1+d(u,fu)]d(u,fu)}{2+d(u,fu)}, \frac{d(u,fu)[1+d(u,fu)]}{2+d(u,fu)},\\
&\qquad \frac{[1+d(u,fu)]d(u,fu)}{2+d(u,fu)}, \frac{d(u,fu)[1+d(u,fu)]}{2+d(u,fu)}\Big\}\\
&= d(u,fu)
\end{aligned}
$$

and

$$
\begin{aligned}
0 < \phi\Big(\int_0^{d(u,fu)} \varphi(t)dt\Big) &= \phi\Big(\int_0^{d(fv,fgv)} \varphi(t)dt\Big)\\
&\le \phi\Big(\int_0^{M_1(v,gv)} \varphi(t)dt\Big) - \int_0^{\psi(M_1(v,gv))} \varphi(t)dt\\
&= \phi\Big(\int_0^{d(u,fu)} \varphi(t)dt\Big) - \int_0^{\psi(d(u,fu))} \varphi(t)dt\\
&< \phi\Big(\int_0^{d(u,fu)} \varphi(t)dt\Big),
\end{aligned}
$$

which is a contradiction. That is, $u = fu = gu$. Consequently, f and g have a common fixed point $u \in X$. This completes the proof. □

Similar to the proof of Theorem 2.1, we have the following results and omit their proofs.

Theorem 2.2. *Let (X,d) be a metric space, f and $g : X \to X$ be weakly compatible mappings satisfying*

$$
\phi\Big(\int_0^{d(fx,fy)} \varphi(t)dt\Big) \le \phi\Big(\int_0^{M_2(x,y)} \varphi(t)dt\Big) - \int_0^{\psi(M_2(x,y))} \varphi(t)dt, \quad \forall x,y \in X, \tag{2.18}
$$

where $(\varphi,\phi,\psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. If $f(X) \subseteq g(X)$, $g(X)$ is complete and $M_2(x,y)$ is defined by (1.9). *Then f and g have a unique common fixed point in X.*

Theorem 2.3. *Let (X,d) be a metric space, f and $g : X \to X$ be weakly compatible mappings satisfying*

$$
\phi\Big(\int_0^{d(fx,fy)} \varphi(t)dt\Big) \le \phi\Big(\int_0^{M_3(x,y)} \varphi(t)dt\Big) - \int_0^{\psi(M_3(x,y))} \varphi(t)dt, \quad \forall x,y \in X, \tag{2.19}
$$

where $(\varphi,\phi,\psi)\in\Phi_1\times\Phi_2\times\Phi_3$. *If* $f(X)\subseteq g(X)$, $g(X)$ *is complete and* $M_3(x,y)$ *is defined by* (1.10). *Then* f *and* g *have a unique common fixed point in* X.

Theorem 2.4. *Let* (X,d) *be a metric space,* f *and* $g:X\to X$ *be weakly compatible mappings satisfying*

$$\phi\bigg(\int_0^{d(fx,fy)}\varphi(t)dt\bigg)\leq\phi\bigg(\int_0^{M_4(x,y)}\varphi(t)dt\bigg)-\int_0^{\psi(M_4(x,y))}\varphi(t)dt,\quad\forall x,y\in X,\tag{2.20}$$

where $(\varphi,\phi,\psi)\in\Phi_1\times\Phi_2\times\Phi_3$. *If* $f(X)\subseteq g(X)$, $g(X)$ *is complete and* $M_4(x,y)$ *is defined by* (1.11). *Then* f *and* g *have a unique common fixed point in* X.

Remark 2.5. It is clear that Theorem 2.3 extends Theorem 1.5 and Theorem 2.4 generalizes Theorems 1.3 and 1.4. Example 2.1 below shows that Theorems 2.1 and 2.2 differ from Theorems 1.1-1.5, respectively.

Example 2.6. Let $X=\mathbb{R}$ be endowed with the Euclidean metric $d(x,y)=|x-y|$ for all $x,y\in X$. Let $f,g:X\to X$ be defined by

$$fx=\begin{cases}4, & \forall x\in X\backslash\{3\},\\ \frac{7}{2}, & x=3,\end{cases}\qquad gx=\frac{1}{4}x^2,\quad\forall x\in X.$$

Clearly, $f(X)=\{\frac{7}{2},4\}\subset\mathbb{R}^+=g(X)$, $g(X)$ is complete and f and g are weakly compatible.

Firstly, we prove that Theorems 1.1 and 1.3 cannot be applied to verify the existence of fixed points of the mapping f in X. Suppose that there exist $c\in[0,1)$ and $\varphi\in\Phi_1$ satisfying the conditions of Theorem 1.1. In virtue of (1.1), $c\in[0,1)$ and $\varphi\in\Phi_1$, we get that

$$\begin{aligned}0<\int_0^{\frac{1}{2}}\varphi(t)dt&=\int_0^{d(f3,f\frac{7}{2})}\varphi(t)dt\\&\leq c\int_0^{d(3,\frac{7}{2})}\varphi(t)dt=c\int_0^{\frac{1}{2}}\varphi(t)dt<\int_0^{\frac{1}{2}}\varphi(t)dt,\end{aligned}$$

which is absurd.

Suppose that there exists $\varphi\in\Phi_5$ satisfying the conditions of Theorem 1.3. In light of (1.3), we deduce that

$$\frac{1}{2}=d\bigg(f3,f\frac{7}{2}\bigg)\leq d\bigg(3,\frac{7}{2}\bigg)-\varphi\bigg(d\bigg(3,\frac{7}{2}\bigg)\bigg)=\frac{1}{2}-\varphi\bigg(\frac{1}{2}\bigg)<\frac{1}{2},$$

which is a contradiction.

Now we claim that Theorem 1.2 cannot be used to prove the existence of common fixed points of the mappings f and g in X. Suppose that there exist $c \in [0,1)$ and $\varphi \in \Phi_1$ satisfying the conditions of Theorem 1.2. By means of (1.2), $c \in [0,1)$ and $\varphi \in \Phi_1$, we infer that

$$\begin{aligned} 0 &< \int_0^{\frac{1}{2}} \varphi(t)dt = \int_0^{d(f3, f\sqrt{7})} \varphi(t)dt \\ &\leq c \int_0^{d(g3, g\sqrt{7})} \varphi(t)dt = c \int_0^{\frac{1}{2}} \varphi(t)dt < \int_0^{\frac{1}{2}} \varphi(t)dt, \end{aligned}$$

which is impossible.

Next, we certify that Theorems 1.4 and 1.5 cannot be used to prove the existence of fixed points of the mapping f in X. Suppose that there exists $(\varphi, \psi) \in \Phi_1 \times \Phi_4$ satisfying the conditions of Theorems 1.4 and 1.5. On the basis of (1.4), we deduce that

$$\begin{aligned} 0 &< \int_0^{\frac{1}{2}} \varphi(t)dt = \int_0^{d(f3, f\frac{7}{2})} \varphi(t)dt \\ &\leq \int_0^{d(3, \frac{7}{2})} \varphi(t)dt - \int_0^{\psi(d(3, \frac{7}{2}))} \varphi(t)dt \\ &= \int_0^{\frac{1}{2}} \varphi(t)dt - \int_0^{\psi(\frac{1}{2})} \varphi(t)dt < \int_0^{\frac{1}{2}} \varphi(t)dt, \end{aligned}$$

which is impossible.

Using (1.5), we have

$$\begin{aligned} M\left(3, \frac{7}{2}\right) &= \max\left\{ d\left(3, \frac{7}{2}\right), d(3, f3), d\left(\frac{7}{2}, f\frac{7}{2}\right), \frac{1}{2}\left[d\left(3, f\frac{7}{2}\right) + d\left(\frac{7}{2}, f3\right)\right]\right\} \\ &= \max\left\{ d\left(3, \frac{7}{2}\right), d\left(3, \frac{7}{2}\right), d\left(\frac{7}{2}, 4\right), \frac{1}{2}\left[d(3,4) + d\left(\frac{7}{2}, \frac{7}{2}\right)\right]\right\} \\ &= \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} 0 &< \int_0^{\frac{1}{2}} \varphi(t)dt = \int_0^{d(f3, f\frac{7}{2})} \varphi(t)dt \\ &\leq \int_0^{M(3, \frac{7}{2})} \varphi(t)dt - \int_0^{\psi(M(3, \frac{7}{2}))} \varphi(t)dt \\ &= \int_0^{\frac{1}{2}} \varphi(t)dt - \int_0^{\psi(\frac{1}{2})} \varphi(t)dt < \int_0^{\frac{1}{2}} \varphi(t)dt, \end{aligned}$$

which is absurd.

Finally, we prove the existence of common fixed points of the mappings f and g in X by employing Theorems 2.1 and 2.2, respectively. Define $\varphi, \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\varphi(t) = 2t, \quad \phi(t) = t, \quad \forall t \in \mathbb{R}^+ \quad \text{and} \quad \psi(t) = \begin{cases} t, & \forall t \in [0, \frac{1}{3}), \\ \frac{1}{3}, & \forall t \in [\frac{1}{3}, +\infty). \end{cases}$$

It is obvious that $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, $\psi(t) \leq t$ for each $t \in \mathbb{R}^+$. Let $x, y \in X$. In order to collate (2.1) and (2.18), we consider the following three possible cases.

Case 1. $x, y \in X \backslash \{3\}$ or $x = y = 3$. It follows that for $i \in \{1, 2\}$

$$\begin{aligned} \phi\bigg(\int_0^{d(fx,fy)} \varphi(t)dt\bigg) &= 0 \leq (M_i(x,y))^2 - (\psi(M_i(x,y)))^2 \\ &= \phi\bigg(\int_0^{M_i(x,y)} \varphi(t)dt\bigg) - \int_0^{\psi(M_i(x,y))} \varphi(t)dt; \end{aligned}$$

Case 2. $x = 3$ and $y \in X \backslash \{3\}$. It follows that

$$M_i(3,y) \geq d(f3, g3) = d\bigg(\frac{7}{2}, \frac{9}{4}\bigg) = \frac{5}{4}, \quad i \in \{1, 2\}$$

and

$$\begin{aligned} \phi\bigg(\int_0^{d(f3,fy)} \varphi(t)dt\bigg) &= \int_0^{d(\frac{7}{2},4)} 2tdt = \frac{1}{4} < \frac{25}{16} - \frac{1}{9} = \frac{209}{144} \\ &= \phi\bigg(\int_0^{d(\frac{7}{2},\frac{9}{4})} 2tdt\bigg) - \int_0^{\psi(d(\frac{7}{2},\frac{9}{4}))} 2tdt \\ &\leq \phi\bigg(\int_0^{M_i(3,y)} \varphi(t)dt\bigg) - \int_0^{\psi(M_i(3,y))} \varphi(t)dt, \quad i \in \{1, 2\}; \end{aligned}$$

Case 3. $y = 3$ and $x \in X \backslash \{3\}$. It follows that

$$M_i(x,3) \geq d(f3, g3) = d\bigg(\frac{7}{2}, \frac{9}{4}\bigg) = \frac{5}{4}, \quad i \in \{1, 2\}$$

and

$$\begin{aligned} \phi\bigg(\int_0^{d(fx,f3)} \varphi(t)dt\bigg) &= \int_0^{d(4,\frac{7}{2})} 2tdt = \frac{1}{4} < \frac{25}{16} - \frac{1}{9} = \frac{209}{144} \\ &= \phi\bigg(\int_0^{d(\frac{7}{2},\frac{9}{4})} 2tdt\bigg) - \int_0^{\psi(d(\frac{7}{2},\frac{9}{4}))} 2tdt \\ &\leq \phi\bigg(\int_0^{M_i(x,3)} \varphi(t)dt\bigg) - \int_0^{\psi(M_i(x,3))} \varphi(t)dt, \quad i \in \{1, 2\}. \end{aligned}$$

Consequently, (2.1) and (2.18) hold. That is, the conditions of Theorems 2.1 and 2.2 are satisfied. Hence, each of Theorems 2.1 and 2.2 guarantees that f and g have a unique common fixed point $4 \in X$.

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