Nonlinear Functional Analysis and Applications
Vol. 27, No. 3 (2022), pp. 621-640
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2022.27.03.10
http://nfaa.kyungnam.ac.kr/journal-nfaa

MEAN CONVERGENCE THEOREMS FOR DOUBLE ARRAY OF FUZZY RANDOM VARIABLES IN METRIC SPACES

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Abstract. The aim of this study is to establish some mean convergence theorems for double array of fuzzy random variables in metric space endowed with a convex combination operation under various assumptions.

1. Introduction

Mean convergence theorems for sequences of random variables have been studied and extended to the array case by many researchers. For example, Cabrera and Volodin [3] derived mean convergence theorems and weak laws of large numbers for weighted sums of dependence random variables under the condition of integrability and appropriate conditions on the array of weights, Thanh [16] proved L^p-convergence for double arrays of independent random variables under the conditions that the series of p-th order moments is convergent or the random variables are dominated in distribution. In addition, various mean convergence theorems for arrays of random variables or arrays of random elements in Banach spaces were also established in [1, 6, 8, 9, 13, 20, 21].

Recently, Thuan and Quang [18] have proved the mean convergence theorem for sequences of pairwise independent random elements in convex combination spaces under the condition that the sequence of random elements is compactly

[0]Received October 13, 2021. Revised February 22, 2022. Accepted April 10, 2022.
[0]2020 Mathematics Subject Classification: 28A20, 60B05, 60F99.
[0]Keywords: Convex combination space, fuzzy random variable, mean convergence theorem, compactly uniformly r-th order integrable in Cesàro sense (Cesàro r-th CUI).

uniformly r-th order integrable in Cesàro sense ($r \geq 1$), this result extends the corresponding result of Chen and Wang in [4] from Banach spaces to convex combinatorial spaces. The convex combination space is a metric space endowed with a convex combination operation and introduced by Terán and Molchanov [14]. The class of these metric spaces is not only larger than the class of Banach spaces but also larger than the class of hyperspace of compact subsets, as well as the class of upper semicontinuous functions (fuzzy sets) with compact support in Banach space. Since Puri and Ralescu [10] introduced the concept of fuzzy random variables as a natural generalization of random sets, several authors have studied limit theorems for fuzzy random variables. Continuing in this direction, in this study we establish some results on mean convergence for double arrays of fuzzy random variables in a convex combination space and with or without compactly uniformly integrable condition.

This paper is organized as follows: In Section 2, we state and summarize basic results in a convex combination space and some related concepts. Main results, some results on mean convergence theorems for double arrays of fuzzy random variables are established in Section 3. First, we give mean convergence theorem for double arrays of levelwise pairwise independent and (α, α^+)-levelwise Cesàro r-th CUI ($r \geq 1$) fuzzy random variables. Then, we establish necessary and sufficient conditions for mean convergence of double arrays of fuzzy random variables under the restrictive assumptions.

2. Preliminaries

Throughout this paper, $(\Omega, \mathcal{A}, P)$ is a complete probability space. For notational convenience, for $a, b \in \mathbb{R}$, $\max(a, b)$ and $\min(a, b)$ are denoted by $a \vee b$ and $a \wedge b$ respectively. For $A \in \mathcal{A}$, the notation $I\{A\}$ (or I_A) is the indicator function of A. At first, we present a short introduction to the approach given by Terán and Molchanov [14]. Let $(\mathfrak{X}, d)$ be a metric space. Based on $\mathfrak{X}$, introduce a *convex combination operation* which for all $n \geq 2$, numbers $\lambda_1, ..., \lambda_n > 0$ satisfying $\sum_{i=1}^n \lambda_i = 1$, and all $u_1, ..., u_n \in \mathfrak{X}$, this operation produces an element of $\mathfrak{X}$, which is denoted by $[\lambda_i, u_i]_{i=1}^n$ or $[\lambda_1, u_1; ...; \lambda_n, u_n]$. Assume that $[1, u] = u$ for every $u \in \mathfrak{X}$ and the following axioms are satisfied:

(CC.i) **(Commutativity)** $[\lambda_i, u_i]_{i=1}^n = [\lambda_{\sigma(i)}, u_{\sigma(i)}]_{i=1}^n$ for every permutation σ of $\{1, ..., n\}$;

(CC.ii) **(Associativity)**
$[\lambda_i, u_i]_{i=1}^{n+2} = [\lambda_1, u_1; ...; \lambda_n, u_n; \lambda_{n+1} + \lambda_{n+2}, [\frac{\lambda_{n+j}}{\lambda_{n+1}+\lambda_{n+2}}, u_{n+j}]_{j=1}^2]$;

(CC.iii) **(Continuity)** If $u, v \in \mathfrak{X}$ and $\lambda^{(k)} \to \lambda \in (0, 1)$ as $k \to \infty$, then $[\lambda^{(k)}, u; 1 - \lambda^{(k)}, v] \to [\lambda, u; 1 - \lambda, v]$;

(CC.iv) **(Negative curvature)** If $u_1, u_2, v_1, v_2 \in \mathfrak{X}$ and $\lambda \in (0, 1)$, then $d([\lambda, u_1; 1 - \lambda, u_2], [\lambda, v_1; 1 - \lambda, v_2]) \leq \lambda d(u_1, v_1) + (1 - \lambda) d(u_2, v_2)$;

(CC.v) **(Convexification)** For each $u \in \mathfrak{X}$, there exists $\lim_{n\to\infty} [n^{-1}, u]_{i=1}^n$, which will be denoted by $K_{\mathfrak{X}}u$ (or Ku so no confusion can arise), and K is called the *convexification operator*.

The metric space $\mathfrak{X}$ endowed with a convex combination operation is referred to as the *convex combination space* (CC space for short).

Note that, based on the inductive method and (CC.ii), we can be extended (CC.iv) to convex combinations of n elements, as follows: if $u_i, v_i \in \mathfrak{X}, \lambda_i \in (0;1), \sum_{i=1}^n \lambda_i = 1$, then $d([\lambda_i, u_i]_{i=1}^n, [\lambda_i, v_i]_{i=1}^n) \leq \sum_{i=1}^n \lambda_i d(u_i, v_i)$. From axiom (CC.v) we see that if $\mathfrak{X}$ is linear space, then K is the identity operator. In a general $\mathfrak{X}$ it is well possible that $[n^{-1}, u]_{i=1}^n \neq u$, so Ku and u may be not identical. If $[\lambda_i, u]_{i=1}^n = u$ for all $n \geq 2$ and $\lambda_1, ..., \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$, then u will be called *convexly decomposable element* (or *convex element*) of $\mathfrak{X}$ and $K(\mathfrak{X})$ coincides with the family of convexly decomposable elements of $\mathfrak{X}$. Moreover, if $(\mathfrak{X}, d)$ is a separable and complete CC space, then so is $(K(\mathfrak{X}), d)$ (see Proposition 2.1 [17]). The axioms (CC.i)–(CC.v) imply the following properties:

(1) For every $u_{11}, ..., u_{mn} \in \mathfrak{X}$ and $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n > 0$ with $\sum_{i=1}^m \alpha_i = \sum_{j=1}^n \beta_j = 1$, we have

$$[\alpha_i, [\beta_j, u_{ij}]_{j=1}^n]_{i=1}^m = [\alpha_i\beta_j, u_{ij}]_{i=1,j=1}^{i=m,j=n}.$$

(2) The convex combination operation is jointly continuous in its $2n$ arguments.

(3) The convexification operator K is linear, that is $K([\lambda_j, u_j]_{j=1}^n) = [\lambda_j, Ku_j]_{j=1}^n$.

(4) If $u \in \mathfrak{X}$ and $\lambda_1, ..., \lambda_n > 0$ with $\sum_{j=1}^n \lambda_j = 1$, then $K([\lambda_j, u]_{j=1}^n) = Ku = [\lambda_j, Ku]_{j=1}^n$.

(5) For every $\lambda_1, \lambda_2, \lambda_3 > 0$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $u, v \in \mathfrak{X}$,

$$[\lambda_1, u; \lambda_2, Kv; \lambda_3, Kv] = [\lambda_1, u; (\lambda_2 + \lambda_3), Kv].$$

(6) The mapping K is nonexpansive with respect to metric d, which means that $d(Ku, Kv) \leq d(u, v)$, for all $u, v \in \mathfrak{X}$.

A mapping $X : \Omega \to \mathfrak{X}$ is called an $\mathfrak{X}$-valued random element (or $\mathcal{A}$-measurable) if $X^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}(\mathfrak{X})$, where $\mathcal{B}(\mathfrak{X})$ is the Borel σ-algebra on $\mathfrak{X}$. When an $\mathfrak{X}$-valued random element X takes finite values, it is called a simple random element.

The collection of $\mathfrak{X}$-valued random elements $\{X_i : i \in I\}$ is said to be independent (pairwise independent, respectively) if the collection of σ-algebras $\{\sigma(X_i) : i \in I\}$ is independent (pairwise independent, respectively), where $\sigma(X) = \{X^{-1}(B) : B \in \mathcal{B}(\mathfrak{X})\}$.

Next,we assume that $(\mathfrak{X}, d)$ is a separable and complete CC space. If X is a simple random element that takes a distinct value $x_i \in \mathfrak{X}$ for each non-null set Ω_i, $i = 1, ..., n$, the *expectation* of X is defined by $EX = [P(\Omega_i), Kx_i]_{i=1}^n$.

We fix $u_0 \in K(\mathfrak{X})$ (by (CC.v), $K(\mathfrak{X}) \neq \emptyset$) and u_0 will be considered as the special element of $\mathfrak{X}$. An $\mathfrak{X}$-valued random element X is said to be *integrable* if $Ed(u_0, X) < \infty$. Note that this definition does not depend on the selection of the element u_0. The space of all integrable $\mathfrak{X}$-valued random elements will be denoted by $\mathcal{L}^1_{\mathfrak{X}}$, and the metric on $\mathcal{L}^1_{\mathfrak{X}}$ is defined by $\Delta(X, Y) = Ed(X, Y)$. By continuity of all Borel functions $X \in \mathcal{L}^1_{\mathfrak{X}}$, then for $X \in \mathcal{L}^1_{\mathfrak{X}}$, the *expectation* of X is defined as the limit of the expectations sequence of simple random elements. Note that, if $X, Y \in \mathcal{L}^1_{\mathfrak{X}}$ then $d(EX, EY) \leq Ed(X, Y)$, and $EX \in K(\mathfrak{X})$ if $X \in \mathcal{L}^1_{\mathfrak{X}}$. Moreover, Corollary 4.2 [17] shows that, if $X_1, X_2 \in \mathcal{L}^1_{\mathfrak{X}}$ and $\lambda_1, \lambda_2 \in (0; 1), \lambda_1 + \lambda_2 = 1$, then $E[\lambda_1, X_1; \lambda_2, X_2] = [\lambda_1, EX_1; \lambda_2, EX_2]$. By axiom (CC.ii), we also have $E[\lambda_i, X_i]_{i=1}^n = [\lambda_i, EX_i]_{i=1}^n$, for all $X_i \in \mathcal{L}^1_{\mathfrak{X}}$ and $\lambda_i \in (0; 1), \sum_{i=1}^n \lambda_i = 1$.

Let $k(\mathfrak{X})$ be the set of nonempty compact subsets of $\mathfrak{X}$ and denote by d_H the Hausdorff metric on $k(\mathfrak{X})$, that is

$$d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(b, a)\}$$

for $A, B \in k(\mathfrak{X})$. It follows from Theorem 6.2 [14] that if $\mathfrak{X}$ is a separable complete CC space then the space $k(\mathfrak{X})$ with the convex combination

$$[\lambda_i, A_i]_{i=1}^n = \{[\lambda_i, u_i]_{i=1}^n : u_i \in A_i, \text{ for all } i\}$$

and Hausdorff metric d_H is a separable complete CC space, where the convexification operator $K_{k(\mathfrak{X})}$ is given by

$$K_{k(\mathfrak{X})} A = \overline{co} K_{\mathfrak{X}} A = \overline{co}\{K_{\mathfrak{X}} u : u \in A\},$$

where $\overline{co} K_{\mathfrak{X}} A$ denotes the closed convex hull of $K_{\mathfrak{X}} A$. Based on this property, if a result holds for elements in CC space $\mathfrak{X}$ then it can be uplifted to the space of nonempty compact subsets $k(\mathfrak{X})$. We denote the expectation of an integrable random element X in $(k(\mathfrak{X}), d_H)$ by $E_{k(\mathfrak{X})} X$.

The notion of compactly uniformly integrable in Cesàro sense for a sequence of random elements taking values in Banach space was discussed by many authors (for example, see [2, 4]). To this end the summary, we introduce this notion for double array of random elements in metric space, which is also naturally extended from Banach space to metric space. Let $r > 0$. Then a double array $\{X_{mn} : m \geq 1, n \geq 1\}$ of $\mathfrak{X}$-valued ($k(\mathfrak{X})$-valued, respectively) random elements is said to be *compactly uniformly r-th order integrable in Cesàro sense* (Cesàro r-th CUI for short) if for every $\varepsilon > 0$, there exists a compact subset $\mathcal{K}_\varepsilon$ of $\mathfrak{X}$ ($k(\mathfrak{X})$, respectively) such that

$$\sup_{m,n\geq 1}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}E\big[d^r(u_0,X_{ij})I\{X_{ij}\notin\mathcal{K}_\varepsilon\}\big]\leq\varepsilon$$

$$\Big(\sup_{m,n\geq 1}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}E\big[d_H^r(\{u_0\},X_{ij})I\{X_{ij}\notin\mathcal{K}_\varepsilon\}\big]\leq\varepsilon,\quad\text{respectively}\Big).$$

3. Mean convergence theorems for double array of fuzzy random variables

In this section, we will establish mean convergence theorems for double arrays of fuzzy random variables in convex combination space. For u_0 is a fixed element of $K(\mathfrak{X})$, which is mentioned in Section 2. We denote $\|x\|_{u_0} := d(x,u_0)$, $\|A\|_{\{u_0\}} := d_H(A,\{u_0\})$ for all $x\in\mathfrak{X}, A\in k(\mathfrak{X})$. First, we introduce some related concepts.

Let $\mathcal{F}(\mathfrak{X})$ denote the family of all fuzzy set $v:\mathfrak{X}\to[0,1]$ that satisfy the following properties:

(i) v is a upper semicontinuous function,
(ii) v is normal, that is, there exists $x\in\mathfrak{X}$ such that $v(x)=1$,
(iii) $\operatorname{supp} v=\operatorname{cl}\{x\in\mathfrak{X}: v(x)>0\}$ is compact in $\mathfrak{X}$, where $\operatorname{cl}(A)$ denotes the closure of A in $\mathfrak{X}$.

For $v\in\mathcal{F}(\mathfrak{X})$, its α-*level set* is denoted by $L_\alpha v=\{x\in\mathfrak{X}: v(x)\geq\alpha\}\in k(\mathfrak{X})$ with $\alpha\in(0,1]$. We also denote $L_\alpha^+ v=\operatorname{cl}\{x\in\mathfrak{X}: v(x)>\alpha\}$ for $\alpha\in[0,1)$, and especially $L_0^+ v=\operatorname{supp} v$.

Theorem 3 [15] shows that if $\mathfrak{X}$ is a convex combination space then the space $\mathcal{F}(\mathfrak{X})$ with the convex combination operator given by

$$L_\alpha([\lambda_i,v_i]_{i=1}^n)=[\lambda_i,L_\alpha v_i]_{i=1}^n,\ \alpha\in(0,1]$$

and the metric

$$d_\infty(v_1,v_2)=\sup_{\alpha\in(0,1]}d_H(L_\alpha v_1,L_\alpha v_2)$$

is a convex combination space, where the convexification operator $K_{\mathcal{F}(\mathfrak{X})}$ is given by

$$L_\alpha(K_{\mathcal{F}(\mathfrak{X})}v)=K_{k(\mathfrak{X})}L_\alpha v=\overline{co}K_{\mathfrak{X}}(L_\alpha v),\ \alpha\in(0,1].$$

Furthermore, Theorem 4 [15] also shows that the space $\mathcal{F}(\mathfrak{X})$ with the same convex combination operation described above and the metric

$$d_p(v_1,v_2)=\Big(\int_0^1 d_H^p(L_\alpha v_1,L_\alpha v_2)d\alpha\Big)^{1/p}$$

is a convex combination space, where $K_{\mathcal{F}(\mathfrak{X})}$ is given by

$$L_\alpha(K_{\mathcal{F}(\mathfrak{X})}v) = K_{k(\mathfrak{X})}L_\alpha v,\ \alpha \in (0,1].$$

A mapping $X : \Omega \to \mathcal{F}(\mathfrak{X})$ is called a *fuzzy random variable* (also called *fuzzy random set*) if X is $(\mathcal{F}(\mathfrak{X}), d_p)$-valued random variable, for any $p \geq 1$. Note that this condition is equivalent to the condition: $L_\alpha X$ is $k(\mathfrak{X})$-valued random variable for all $\alpha \in (0,1]$ (see Theorem 5 [15]).

A fuzzy random variable X is called *integrably bounded* if $\|L_0^+ X\|_{\{u_0\}} \in \mathcal{L}^1_{\mathbb{R}}$, and we denote $X \in \mathcal{L}^1(\mathcal{F}(\mathfrak{X}))$. Also in [15], Terán and Molchanov define the expectation of $X \in \mathcal{L}^1(\mathcal{F}(\mathfrak{X}))$ as follows. For $X \in \mathcal{L}^1(\mathcal{F}(\mathfrak{X}))$, the *expectation* of X is a fuzzy set, denoted by $E_{\mathcal{F}(\mathfrak{X})}X$, such that for each $\alpha \in (0,1]$

$$L_\alpha(E_{\mathcal{F}(\mathfrak{X})}X) = E_{k(\mathfrak{X})}(L_\alpha X).$$

Proposition 3.1. ([19]) *For* $\alpha \in [0,1)$, *we have*

(2) $L_\alpha^+([\lambda_i, v_i]_{i=1}^n) = [\lambda_i, L_\alpha^+ v_i]_{i=1}^n$, *for* $v_i \in \mathcal{F}(\mathfrak{X})$*;*
(2) $L_\alpha^+(E_{\mathcal{F}(\mathfrak{X})}X) = E_{k(\mathfrak{X})}(L_\alpha^+ X)$, *for* $X \in \mathcal{L}^1(\mathcal{F}(\mathfrak{X}))$.

A collection $\{X_i : i \in \mathcal{I}\}$ of fuzzy random variables is said to be *levelwise independent* (*resp. levelwise pairwise independent*) if $\{L_\alpha X_i : i \in \mathcal{I}\}$ is a collection of independent (resp. pairwise independent) $k(\mathfrak{X})$-valued random elements for each $\alpha \in (0,1]$. Note that, if $\{X_i : i \in \mathcal{I}\}$ is a collection of levelwise independent (resp. levelwise pairwise independent) fuzzy random variables, then $\{L_\alpha^+ X_i : i \in \mathcal{I}\}$ is also collection of independent (resp. pairwise independent) $k(\mathfrak{X})$-valued random elements for each $\alpha \in (0,1]$ (see Lemma 4.3 [19]).

A collection $\{X_i : i \in \mathcal{I}\}$ of fuzzy random variables is said to be (α, α^+)-*levelwise Cesàro r-th CUI* if $\{L_\alpha X_i : i \in \mathcal{I}\}$ is $k(\mathfrak{X})$-valued Cesàro r-th CUI for each $\alpha \in (0,1]$ and $\{L_\alpha^+ X_i : i \in \mathcal{I}\}$ is $k(\mathfrak{X})$-valued Cesàro r-th CUI for each $\alpha \in [0,1)$. Note that, the concept of (α, α^+)-levelwise Cesàro r-th CUI extends really the concept of CUI which has been introduced in [5, 7].

We now present some lemmas which will be used later.

Lemma 3.2. ([11]) *Let* $v \in \mathcal{F}(\mathfrak{X})$. *Then for each* $\varepsilon > 0$, *there exists a partition* $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_p = 1$ *of* $[0,1]$ *such that*

$$\max_{1 \leq k \leq p} d_H(L^+_{\alpha_{k-1}} v, L_{\alpha_k} v) < \varepsilon.$$

The next lemma is extended from Theorem 3.4(b) [18] to the case of double arrays.

Lemma 3.3. *Let* $\{X_{ij} : i \geq 1, j \geq 1\}$ *be an array of pairwise independent and Cesàro r-th CUI* $(r \geq 1)$ $\mathfrak{X}$*-valued random elements. Then*

$$Ed^r([m^{-1}, [n^{-1}, X_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1}, [n^{-1}, EX_{ij}]_{j=1}^n]_{i=1}^m) \to 0 \text{ as } m \vee n \to \infty.$$

Proof. For $\varepsilon > 0$ arbitrarily small, by Cesàro r-th CUI hypothesis, there exist a compact subset $\mathcal{K}_\varepsilon$ of $\mathfrak{X}$ such that for all m, n

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E\big(\|X_{ij}\|_{u_0}^r I\{X_{ij} \notin \mathcal{K}_\varepsilon\}\big) \leq \varepsilon.$$

By the compactness of $\mathcal{K}_\varepsilon$, there exists $\{c_1, c_2, ..., c_p\} \subset \mathcal{K}_\varepsilon$ such that

$$\mathcal{K}_\varepsilon \subset \bigcup_{t=1}^{p} B(c_t, \varepsilon), \text{ where } B(c_t, \varepsilon) = \{x \in \mathfrak{X} : d(x, c_t) < \varepsilon\}.$$

For each $i \geq 1, j \geq 1$, we define the $\mathfrak{X}$-valued random elements as follows:

$$Y_{ij}(\omega) = \begin{cases} c_0 := u_0, & \text{if } X_{ij}(\omega) \notin \mathcal{K}_\varepsilon \\ c_1, & \text{if } X_{ij}(\omega) \in B(c_1, \varepsilon) \cap \mathcal{K}_\varepsilon \\ c_t, & \text{if } X_{ij}(\omega) \in B(c_t, \varepsilon) \cap \{\cup_{k=1}^{t-1} B(c_k, \varepsilon)\}^c \cap \mathcal{K}_\varepsilon, t = 2, ..., p \end{cases}$$

and

$$Z_{ij}(\omega) = \begin{cases} X_{ij}(\omega), & \text{if } X_{ij}(\omega) \in \mathcal{K}_\varepsilon \\ u_0, & \text{if } X_{ij}(\omega) \notin \mathcal{K}_\varepsilon. \end{cases}$$

By triangular inequality, we have

$$\begin{aligned} &d([m^{-1}, [n^{-1}, X_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1}, [n^{-1}, EX_{ij}]_{j=1}^n]_{i=1}^m) \\ &\leq d([m^{-1}, [n^{-1}, X_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1}, [n^{-1}, Z_{ij}]_{j=1}^n]_{i=1}^m) \\ &\quad + d([m^{-1}, [n^{-1}, Z_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1}, [n^{-1}, Y_{ij}]_{j=1}^n]_{i=1}^m) \\ &\quad + d([m^{-1}, [n^{-1}, Y_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1}, [n^{-1}, KY_{ij}]_{j=1}^n]_{i=1}^m) \\ &\quad + d([m^{-1}, [n^{-1}, KY_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1}, [n^{-1}, EY_{ij}]_{j=1}^n]_{i=1}^m) \\ &\quad + d([m^{-1}, [n^{-1}, EY_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1}, [n^{-1}, EZ_{ij}]_{j=1}^n]_{i=1}^m) \\ &\quad + d([m^{-1}, [n^{-1}, EZ_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1}, [n^{-1}, EX_{ij}]_{j=1}^n]_{i=1}^m) \\ &:= (I_1) + (I_2) + (I_3) + (I_4) + (I_5) + (I_6). \end{aligned}$$

Let us estimate the parts above as follows:
For (I_1), we have

$$(I_1) \leq \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} d(X_{ij}, Z_{ij}) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} \|X_{ij}\|_{u_0} I\{X_{ij} \notin \mathcal{K}_\varepsilon\}.$$

By Cesàro r-th CUI hypothesis and Jensen's inequality,

$$E(I_1)^r \leq E\Big(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\|X_{ij}\|_{u_0} I\{X_{ij}\notin \mathcal{K}_\varepsilon\}\Big)^r \leq \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E\big(\|X_{ij}\|_{u_0}^r I\{X_{ij}\notin \mathcal{K}_\varepsilon\}\big) \leq \varepsilon.$$

For (I_2), by the definition of Y_{ij} and Z_{ij},

$$(I_2) \leq \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} d(Z_{ij}, Y_{ij}) \leq \varepsilon.$$

For (I_3), we will prove that $(I_3) < \varepsilon$ for all $\omega \in \Omega$ when $m \vee n$ is sufficiently large. Indeed, for $t = 0, 1, ..., p$, we put

$$Q_{mn}^t = \text{card}\{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n, Y_{ij} = c_t\} = \sum_{i=1}^{m}\sum_{j=1}^{n} I\{Y_{ij} = c_t\},$$

$$\mathcal{T}_{mn} = \{t : 0 \leq t \leq p, Q_{mn}^t > 0\}\ ;\quad m, n \geq 1.$$

By properties (2.1) and (2.4),

$$[m^{-1}, [n^{-1}, Y_{ij}]_{j=1}^n]_{i=1}^m = [(mn)^{-1} Q_{mn}^t, [(Q_{mn}^t)^{-1}, c_t]_{i=1}^{Q_{mn}^t}]_{t\in\mathcal{T}_{mn}}$$

and

$$[m^{-1}, [n^{-1}, KY_{ij}]_{j=1}^n]_{i=1}^m = [(mn)^{-1} Q_{mn}^t, [(Q_{mn}^t)^{-1}, Kc_t]_{i=1}^{Q_{mn}^t}]_{t\in\mathcal{T}_{mn}}.$$

Therefore,

$$(I_3) = d([m^{-1}, [n^{-1}, Y_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1}, [n^{-1}, KY_{ij}]_{j=1}^n]_{i=1}^m) \leq \sum_{t\in\mathcal{T}_{mn}} \frac{Q_{mn}^t}{mn} d([(Q_{mn}^t)^{-1}, c_t]_{i=1}^{Q_{mn}^t}, Kc_t).$$

By the definition of K, we have

$$\lim_{n\to\infty} d([n^{-1}, c_t]_{i=1}^n, Kc_t) = 0.$$

Thus, there exists $n_1(\varepsilon) \in \mathbb{N}$ such that

$$d([n^{-1}, c_t]_{i=1}^n, Kc_t) < \frac{\varepsilon}{p+1} \quad \text{for all} \quad n \geq n_1(\varepsilon) \quad \text{and for all} \quad t = 0, 1, ..., p.$$

We put

$$M_t(\varepsilon) = \max_{1\leq k<n_1(\varepsilon)} d([k^{-1}, c_t]_{i=1}^k, Kc_t), \quad M(\varepsilon) = \max_{0\leq t\leq p} M_t(\varepsilon)$$

and choose the smallest integer number $n(\varepsilon)$ such that

$$n(\varepsilon) \geq \varepsilon^{-1}(p+1)M(\varepsilon)n_1(\varepsilon).$$

Now, for all $m \vee n \geq n(\varepsilon)$:
If $Q^t_{mn} \geq n_1(\varepsilon)$, then

$$\frac{Q^t_{mn}}{mn} d([(Q^t_{mn})^{-1}, c_t]_{i=1}^{Q^t_{mn}}, Kc_t) < \frac{\varepsilon}{p+1} \quad (\text{since } \ m^{-1}n^{-1}Q^t_{mn} \leq 1).$$

If $0 < Q^t_{mn} < n_1(\varepsilon)$, then

$$\frac{Q^t_{mn}}{mn} d([(Q^t_{mn})^{-1}, c_t]_{i=1}^{Q^t_{mn}(\omega)}, Kc_t) < \frac{n_1(\varepsilon)}{n(\varepsilon)} M(\varepsilon) \leq \frac{\varepsilon}{p+1}.$$

Hence, for $m \vee n \geq n(\varepsilon)$ and for all $\omega \in \Omega$

$$\frac{Q^t_{mn}}{mn} d([(Q^t_{mn})^{-1}, c_t]_{i=1}^{Q^t_{mn}}, Kc_t) < \frac{\varepsilon}{p+1}.$$

This implies that

$$(I_3) \leq \sum_{t \in \mathcal{T}_{mn}} \frac{Q^t_{mn}}{mn} d([(Q^t_{mn})^{-1}, c_t]_{i=1}^{Q^t_{mn}}, Kc_t) < \sum_{t=0}^{p} \frac{\varepsilon}{p+1} = \varepsilon$$

for values of $m \vee n$ that are sufficiently large.
For (I_4), by property (2.1) and Lemma 3.3 [12],

$$\begin{aligned}
(I_4) &= d([m^{-1}, [n^{-1}, KY_{ij}]_{j=1}^{n}]_{i=1}^{m}, [m^{-1}, [n^{-1}, EY_{ij}]_{j=1}^{n}]_{i=1}^{m}) \\
&= d([m^{-1}, [n^{-1}, [I\{Y_{ij} = c_t\}, Kc_t]_{t=0}^{t=p}]_{j=1}^{n}]_{i=1}^{m}, \\
&\qquad [m^{-1}, [n^{-1}, [P\{Y_{ij} = c_t\}, Kc_t]_{t=0}^{t=p}]_{j=1}^{n}]_{i=1}^{m}) \\
&\leq \sum_{t=0}^{p} \Big| \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \big(I\{Y_{ij} = c_t\} - P\{Y_{ij} = c_t\} \big) \Big| \|c_t\|_{u_0} \\
&\leq C \sum_{t=1}^{p} \Big| \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \big(I\{Y_{ij} = c_t\} - P\{Y_{ij} = c_t\} \big) \Big|,
\end{aligned}$$

where $C = \max_{1 \leq t \leq p} \|c_t\|_{u_0}$. Hence, Jensen's inequality yields

$$\begin{aligned}
E(I_4)^r &\leq C^r p^{r-1} \sum_{t=1}^{p} E \Big| \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \big(I\{Y_{ij} = c_t\} - P\{Y_{ij} = c_t\} \big) \Big|^r \\
&\leq C^r p^{r-1} \sum_{t=1}^{p} \frac{1}{mn} E \Big| \sum_{i=1}^{m} \sum_{j=1}^{n} \big(I\{Y_{ij} = c_t\} - P\{Y_{ij} = c_t\} \big) \Big| \\
&\leq C^r p^{r-1} \sum_{t=1}^{p} \frac{1}{mn} \Big(E \Big| \sum_{i=1}^{m} \sum_{j=1}^{n} \big(I\{Y_{ij} = c_t\} - P\{Y_{ij} = c_t\} \big) \Big|^2 \Big)^{1/2} \\
&\leq C^r p^r (mn)^{-1/2}.
\end{aligned}$$

For (I_5), by the definition of Y_{ij} and Z_{ij},

$$\begin{aligned}(I_5) &\le \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} d(EY_{ij}, EZ_{ij})\\ &\le \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} Ed(Y_{ij}, Z_{ij}) \le \varepsilon.\end{aligned}$$

For (I_6),

$$\begin{aligned}(I_6) &\le \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} d(EZ_{ij}, EX_{ij})\\ &\le \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} Ed(Z_{ij}, X_{ij})\\ &= \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E\big(\|X_{ij}\|_{u_0} I\{X_{ij} \notin \mathcal{K}_\varepsilon\}\big).\end{aligned}$$

By Cesàro r-th CUI hypothesis and Jensen's inequality again

$$\begin{aligned}(I_6)^r &\le \Big[\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E\big(\|X_{ij}\|_{u_0} I\{X_{ij} \notin \mathcal{K}_\varepsilon\}\big)\Big]^r\\ &\le \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E\big(\|X_{ij}\|_{u_0}^r I\{X_{ij} \notin \mathcal{K}_\varepsilon\}\big) \le \varepsilon.\end{aligned}$$

Combining the parts above and for $m \vee n$ large enough, by Jensen's inequality, we obtain

$$\begin{aligned}&Ed^r([m^{-1}, [n^{-1}, X_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1}, [n^{-1}, EX_{ij}]_{j=1}^n]_{i=1}^m)\\ &\quad\le 6^{r-1}\big[E(I_1)^r + E(I_1)^r + E(I_2)^r + E(I_3)^r + E(I_4)^r + E(I_5)^r + E(I_6)^r\big]\\ &\quad\le 6^{r-1}\big[2\varepsilon + 3\varepsilon^r + C^r p^r (mn)^{-1/2}\big].\end{aligned}$$

Letting $m \vee n \to \infty$ and by the arbitrariness of ε, we derive the conclusion. □

Remark 3.4. Lemma 3.3 extends Theorem 1.2 [4] for sequence of pairwise independent and Cesàro r-th CUI ($r \ge 1$) random elements in Banach space to double array of pairwise independent and Cesàro r-th CUI random elements in CC space.

In the first theorem, we establish mean convergence conditions for double array of levelwise pairwise independent and (α, α^+)-levelwise Cesàro r-th CUI ($r \ge 1$) fuzzy random variables.

Theorem 3.5. *Let $\{X_{mn} : m \geq 1, n \geq 1\}$ be an array of levelwise pairwise independent and (α, α^+)-levelwise Cesàro r-th CUI $(r \geq 1)$ fuzzy random variables, and for each $\varepsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_p = 1$ of $[0,1]$ such that for all m, n*

$$\max_{1\leq k\leq p} d_H(L^+_{\alpha_{k-1}}[m^{-1}, [n^{-1}, E_{\mathcal{F}(\mathfrak{X})}X_{ij}]^n_{j=1}]^m_{i=1}, L_{\alpha_k}[m^{-1}, [n^{-1}, E_{\mathcal{F}(\mathfrak{X})}X_{ij}]^n_{j=1}]^m_{i=1}) < \varepsilon. \tag{3.1}$$

Then

$$Ed^r_\infty([m^{-1}, [n^{-1}X_{ij}]^n_{j=1}]^m_{i=1}, [m^{-1}, [n^{-1}, E_{\mathcal{F}(\mathfrak{X})}X_{ij}]^n_{j=1}]^m_{i=1}) \to 0$$

as $m \vee n \to \infty$.

Proof. By condition (3.1), for $\varepsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_p = 1$ of $[0,1]$ such that for all m, n,

$$\max_{1\leq k\leq p} d_H(L^+_{\alpha_{k-1}}[m^{-1}, [n^{-1}, E_{\mathcal{F}(\mathfrak{X})}X_{ij}]^n_{j=1}]^m_{i=1}, L_{\alpha_k}[m^{-1}, [n^{-1}, E_{\mathcal{F}(\mathfrak{X})}X_{ij}]^n_{j=1}]^m_{i=1}) < \varepsilon.$$

Note that if A, A_1, A_2, B, B_1, B_2 are compact sets such that $A_1 \subset A \subset A_2$ and $B_1 \subset B \subset B_2$, then

$$\begin{aligned} d_H(A,B) &\leq \max\{d_H(A_1,B_2), d_H(A_2,B_1)\} \\ &\leq d_H(A_1,B_2) + d_H(A_2,B_1). \end{aligned}$$

We have

$$\begin{aligned} &\sup_{\alpha_{k-1}<\alpha\leq\alpha_k} d_H(L_\alpha[m^{-1}, [n^{-1}X_{ij}]^n_{j=1}]^m_{i=1}, L_\alpha[m^{-1}, [n^{-1}, E_{\mathcal{F}(\mathfrak{X})}X_{ij}]^n_{j=1}]^m_{i=1}) \\ &\leq d_H(L_{\alpha_k}[m^{-1}, [n^{-1}, X_{ij}]^n_{j=1}]^m_{i=1}, L^+_{\alpha_{k-1}}[m^{-1}, [n^{-1}, E_{\mathcal{F}(\mathfrak{X})}X_{ij}]^n_{j=1}]^m_{i=1}) \\ &\quad + d_H(L^+_{\alpha_{k-1}}[m^{-1}, [n^{-1}, X_{ij}]^n_{j=1}]^m_{i=1}, L_{\alpha_k}[m^{-1}, [n^{-1}, E_{\mathcal{F}(\mathfrak{X})}X_{ij}]^n_{j=1}]^m_{i=1}) \\ &= d_H(L_{\alpha_k}[m^{-1}, [n^{-1}, X_{ij}]^n_{j=1}]^m_{i=1}, [m^{-1}, [n^{-1}, E_{k(\mathfrak{X})}L^+_{\alpha_{k-1}}X_{ij}]^n_{j=1}]^m_{i=1}) \\ &\quad + d_H([m^{-1}, [n^{-1}, L^+_{\alpha_{k-1}}X_{ij}]^n_{j=1}]^m_{i=1}, [m^{-1}, [n^{-1}, E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]^n_{j=1}]^m_{i=1}) \\ &\leq d_H([m^{-1}, [n^{-1}, L_{\alpha_k}X_{ij}]^n_{j=1}]^m_{i=1}, [m^{-1}, [n^{-1}, E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]^n_{j=1}]^m_{i=1}) \\ &\quad + d_H([m^{-1}, [n^{-1}, L^+_{\alpha_{k-1}}X_{ij}]^n_{j=1}]^m_{i=1}, [m^{-1}, [n^{-1}, E_{k(\mathfrak{X})}L^+_{\alpha_{k-1}}X_{ij}]^n_{j=1}]^m_{i=1}) \\ &\quad + 2d_H([m^{-1}, [n^{-1}, E_{k(\mathfrak{X})}L^+_{\alpha_{k-1}}X_{ij}]^n_{j=1}]^m_{i=1}, [m^{-1}, [n^{-1}, E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]^n_{j=1}]^m_{i=1}) \\ &\leq d_H([m^{-1}, [n^{-1}, L_{\alpha_k}X_{ij}]^n_{j=1}]^m_{i=1}, [m^{-1}, [n^{-1}, E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]^n_{j=1}]^m_{i=1}) \\ &\quad + d_H([m^{-1}, [n^{-1}, L^+_{\alpha_{k-1}}X_{ij}]^n_{j=1}]^m_{i=1}, [m^{-1}, [n^{-1}, E_{k(\mathfrak{X})}L^+_{\alpha_{k-1}}X_{ij}]^n_{j=1}]^m_{i=1}) + 2\varepsilon. \end{aligned}$$

Therefore

$$\begin{aligned}
&d_\infty([m^{-1},[n^{-1}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)\\
&=\max_{1\le k\le p}\sup_{\alpha_{k-1}<\alpha\le\alpha_k} d_H(L_\alpha[m^{-1},[n^{-1}X_{ij}]_{j=1}^n]_{i=1}^m,\\
&L_\alpha[m^{-1},[n^{-1}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)\\
&\le \max_{1\le k\le p} d_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m)\\
&\quad+\max_{1\le k\le p} d_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m,\\
&\qquad\qquad [m^{-1},[n^{-1},E_{k(\mathfrak{X})}L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m)+2\varepsilon.
\end{aligned}$$

Using the above estimation and Jensen's inequality, we have

$$\begin{aligned}
&Ed^r_\infty([m^{-1},[n^{-1}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)\\
&\le E\Big(\max_{1\le k\le p} d_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m)\\
&\qquad+\max_{1\le k\le p} d_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m,\\
&\qquad\qquad [m^{-1},[n^{-1},E_{k(\mathfrak{X})}L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m)+2\varepsilon\Big)^r\\
&\le E\Big(\sum_{k=1}^p d_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m)\\
&\qquad+\sum_{k=1}^p d_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m,\\
&\qquad\qquad [m^{-1},[n^{-1},E_{k(\mathfrak{X})}L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m)+2\varepsilon\Big)^r\\
&\le 3^{r-1}\Big[E\Big(\sum_{k=1}^p d_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m,\\
&\qquad\qquad [m^{-1},[n^{-1},E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m)\Big)^r\\
&\qquad+E\Big(\sum_{k=1}^p d_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m,\\
&\qquad\qquad [m^{-1},[n^{-1},E_{k(\mathfrak{X})}L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m)\Big)^r+(2\varepsilon)^r\Big]
\end{aligned}$$

$$\le 3^{r-1}\Big[p^{r-1}\sum_{k=1}^{p} Ed_H^r([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m,$$
$$[m^{-1},[n^{-1},E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m)$$
$$+\,p^{r-1}\sum_{k=1}^{p} Ed_H^r([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m,$$
$$[m^{-1},[n^{-1},E_{k(\mathfrak{X})}L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m)+(2\varepsilon)^r\Big].$$

Note that $\{L_{\alpha_k}X_{mn} : m \ge 1, n \ge 1\}$ and $\{L^+_{\alpha_{k-1}}X_{mn} : m \ge 1, n \ge 1\}$ are arrays of pairwise independent and Cesàro r-th CUI ($r \ge 1$) $k(\mathfrak{X})$-valued random elements, for all $k = 1, 2, ..., p$. Thus, applying Lemma 3.3 to these arrays, we get

$$Ed_H^r([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m)\to 0$$

as $m \vee n \to \infty$, and

$$Ed_H^r([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},E_{k(\mathfrak{X})}L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m)\to 0$$

as $m \vee n \to \infty$. Hence

$$\limsup_{m\vee n\to\infty} Ed_\infty^r([m^{-1},[n^{-1}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)$$
$$\le 3^{r-1}(2\varepsilon)^r.$$

By the arbitrariness of ε, the proof is completed. □

In the next theorem, we will establish necessary and sufficient conditions for mean convergence of double arrays of fuzzy random variables under the restrictive assumptions without Cesàro r-th CUI hypothesis.

Theorem 3.6. *Let $\{X_{mn} : m \ge 1, n \ge 1\}$ be an array of fuzzy random variables. Suppose that for each $\varepsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_p = 1$ of $[0, 1]$ such that for all m, n*

$$E\Big[\max_{1\le k\le p} d_H(L^+_{\alpha_{k-1}}[m^{-1},[n^{-1}X_{ij}]_{j=1}^n]_{i=1}^m, L_{\alpha_k}[m^{-1},[n^{-1}X_{ij}]_{j=1}^n]_{i=1}^m)\Big] < \varepsilon. \tag{3.2}$$

Then

$$Ed_\infty([m^{-1},[n^{-1}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)\to 0$$

as $m \vee n \to \infty$, if and only if for each $\alpha \in [0; 1]$

$$Ed_H([m^{-1},[n^{-1}L_\alpha X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},E_{k(\mathfrak{X})}L_\alpha X_{ij}]_{j=1}^n]_{i=1}^m)\to 0$$

as $m \vee n \to \infty$.

Proof. The necessity is obvious. To prove the sufficiency, for $\varepsilon > 0$ arbitrarily small, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_p = 1$ of $[0,1]$ such that condition (3.2) is satisfied. First we have following estimations

$$\begin{aligned}
&\sup_{\alpha_{k-1}<\alpha\le\alpha_k} d_H(L_\alpha[m^{-1},[n^{-1},X_{ij}]_{j=1}^n]_{i=1}^m, L_\alpha[m^{-1},[n^{-1},E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m) \\
&\quad\le \sup_{\alpha_{k-1}<\alpha\le\alpha_k} d_H([m^{-1},[n^{-1},L_\alpha X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m) \\
&\qquad + d_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m) \\
&\qquad + \sup_{\alpha_{k-1}<\alpha\le\alpha_k} d_H([m^{-1},[n^{-1},E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m, \\
&\qquad\qquad [m^{-1},[n^{-1},E_{k(\mathfrak{X})}L_\alpha X_{ij}]_{j=1}^n]_{i=1}^m) \\
&\quad= d_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m) \\
&\qquad + d_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m) \\
&\qquad + d_H([m^{-1},[n^{-1},E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m, \\
&\qquad\qquad [m^{-1},[n^{-1},E_{k(\mathfrak{X})}L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m).
\end{aligned}$$

Therefore

$$\begin{aligned}
&Ed_\infty([m^{-1},[n^{-1}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m) \\
&= E\Big[\max_{1\le k\le p}\sup_{\alpha_{k-1}<\alpha\le\alpha_k} d_H(L_\alpha[m^{-1},[n^{-1}X_{ij}]_{j=1}^n]_{i=1}^m, \\
&\qquad\qquad L_\alpha[m^{-1},[n^{-1}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)\Big] \\
&\le E\Big[\max_{1\le k\le p} d_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m)\Big] \\
&\quad + E\Big[\max_{1\le k\le p} d_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m, \\
&\qquad\qquad [m^{-1},[n^{-1},E_{k(\mathfrak{X})}L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m)\Big] \\
&\quad + \max_{1\le k\le p} d_H([m^{-1},[n^{-1},EL_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m, \\
&\qquad\qquad [m^{-1},[n^{-1},E_{k(\mathfrak{X})}L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m) \\
&:= (II_1)+(II_2)+(II_3).
\end{aligned}$$

For (II_1), by (3.2) we have $(II_1) < \varepsilon$.
For (II_2), by assumption we have $(II_2) \to 0$ as $m \vee n \to \infty$.

For (II_3), by (3.2) we have

$$\begin{aligned}(II_3) &= \max_{1\le k\le p} d_H(E_{k(\mathfrak{X})}[m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m,\\ &\qquad E_{k(\mathfrak{X})}[m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m)\\ &\le \max_{1\le k\le p} Ed_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m)\\ &\le E\Big[\max_{1\le k\le p} d_H(L^+_{\alpha_{k-1}}[m^{-1},[n^{-1},X_{ij}]_{j=1}^n]_{i=1}^m,\\ &\qquad L_{\alpha_k}[m^{-1},[n^{-1},X_{ij}]_{j=1}^n]_{i=1}^m)\Big] < \varepsilon.\end{aligned}$$

Combining the parts above, we obtain

$$\limsup_{m\vee n\to\infty} Ed_\infty([m^{-1},[n^{-1}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m) \le 2\varepsilon.$$

This completes the proof. □

Applying Theorem 3.6, we obtain the following corollary.

Corollary 3.7. *Let $\{X_{mn} : m \ge 1, n \ge 1\}$ be an array of fuzzy random variables. Suppose that for each $\varepsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_p = 1$ of $[0,1]$ such that for all m, n*

$$E\Big[\max_{1\le k\le p} d_H(L^+_{\alpha_{k-1}}X_{mn}, L_{\alpha_k}X_{mn})\Big] < \varepsilon. \tag{3.3}$$

Then

$$Ed_\infty([m^{-1},[n^{-1},X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m) \to 0$$

as $m\vee n\to\infty$, if and only if for each $\alpha\in[0;1]$

$$Ed_H([m^{-1},[n^{-1},L_\alpha X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},E_{k(\mathfrak{X})}L_\alpha X_{ij}]_{j=1}^n]_{i=1}^m) \to 0$$

as $m\vee n\to\infty$.

Proof. We have

$$\begin{aligned}&E\Big[\max_{1\le k\le p} d_H(L^+_{\alpha_{k-1}}[m^{-1},[n^{-1},X_{ij}]_{j=1}^n]_{i=1}^m, L_{\alpha_k}[m^{-1},[n^{-1},X_{ij}]_{j=1}^n]_{i=1}^m)\Big]\\ &\le E\Big[\max_{1\le k\le p}\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n d_H(L^+_{\alpha_{k-1}}X_{ij}, L_{\alpha_k}X_{ij})\Big]\\ &\le \frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n E\Big[\max_{1\le k\le p} d_H(L^+_{\alpha_{k-1}}X_{ij}, L_{\alpha_k}X_{ij})\Big].\end{aligned}$$

Thus, if (3.3) holds then so does (3.2). The proof is completed by applying Theorem 3.6. □

In the case, the expectations of double arrays of fuzzy random variables are convergent, we obtain the following result.

Theorem 3.8. *Let* $\{X_{mn} : m \geq 1, n \geq 1\}$ *be an array of fuzzy random variables. If there exists* $v \in \mathcal{F}(\mathfrak{X})$ *such that*

$$d_\infty([m^{-1}, [n^{-1}, E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m, v) \to 0 \text{ as } m \vee n \to \infty, \tag{3.4}$$

then

$$Ed_\infty([m^{-1}, [n^{-1}, X_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1}, [n^{-1}, E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m) \to 0$$

as $m \vee n \to \infty$*, if and only if for each* $\alpha \in [0; 1)$

$$Ed_H([m^{-1}, [n^{-1}, L_\alpha X_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1}, [n^{-1}, E_{k(\mathfrak{X})}L_\alpha X_{ij}]_{j=1}^n]_{i=1}^m) \to 0$$

as $m \vee n \to \infty$*, and for each* $\alpha \in (0; 1]$

$$Ed_H([m^{-1}, [n^{-1}L_\alpha^+ X_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1}, [n^{-1}, E_{k(\mathfrak{X})}L_\alpha^+ X_{ij}]_{j=1}^n]_{i=1}^m) \to 0$$

as $m \vee n \to \infty$*.*

Proof. The necessity is obvious. To prove the sufficiency, it suffices to prove that

$$Ed_\infty([m^{-1}, [n^{-1}X_{ij}]_{j=1}^n]_{i=1}^m, v) \to 0 \text{ as } m \vee n \to \infty.$$

For $\varepsilon > 0$, by Lemma 3.2, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_p = 1$ of $[0; 1]$ such that

$$\max_{1 \leq k \leq p} d_H(L_{\alpha_{k-1}}^+ v, L_{\alpha_k} v) < \varepsilon. \tag{3.5}$$

By condition (3.4), there exists $N \in \mathbb{N}$ such that for all $\alpha \in [0; 1]$ and $m \vee n > N$,

$$d_H([m^{-1}, [n^{-1}, L_\alpha E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m, L_\alpha v) < \varepsilon \tag{3.6}$$

and

$$d_H([m^{-1}, [n^{-1}, L_\alpha^+ E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m, L_\alpha^+ v) < \varepsilon. \tag{3.7}$$

Now, by (3.5), (3.6) and (3.7), for $m \vee n > N$, we have

$$\begin{aligned}
&\sup_{\alpha_{k-1}<\alpha\le\alpha_k} d_H(L_\alpha[m^{-1},[n^{-1},X_{ij}]_{j=1}^n]_{i=1}^m, L_\alpha v)\\
&\le d_H(L_{\alpha_k}[m^{-1},[n^{-1},X_{ij}]_{j=1}^n]_{i=1}^m, L^+_{\alpha_{k-1}}v)\\
&\quad + d_H(L^+_{\alpha_{k-1}}[m^{-1},[n^{-1},X_{ij}]_{j=1}^n]_{i=1}^m, L_{\alpha_k}v)\\
&= d_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m, L^+_{\alpha_{k-1}}v)\\
&\quad + d_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m, L_{\alpha_k}v)\\
&\le d_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m, L_{\alpha_k}v)\\
&\quad + d_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m, L^+_{\alpha_{k-1}}v) + 2d_H(L^+_{\alpha_{k-1}}v, L_{\alpha_k}v)\\
&\le d_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m, L_{\alpha_k}v)\\
&\quad + d_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m, L^+_{\alpha_{k-1}}v) + 2\varepsilon\\
&\le d_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1},[n^{-1},L_{\alpha_k}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)\\
&\quad + d_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m,\\
&\qquad\quad [m^{-1},[n^{-1},L^+_{\alpha_{k-1}}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)\\
&\quad + d_H([m^{-1},[n^{-1},L_{\alpha_k}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m, L_{\alpha_k}v)\\
&\quad + d_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m, L^+_{\alpha_{k-1}}v) + 2\varepsilon\\
&\le d_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m, [m^{-1},[n^{-1},L_{\alpha_k}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)\\
&\quad + d_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m,\\
&\qquad\quad [m^{-1},[n^{-1},L^+_{\alpha_{k-1}}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)\\
&\quad + 4\varepsilon.
\end{aligned}$$

Thus for $m \vee n > N$,

$$\begin{aligned}
&d_\infty([m^{-1},[n^{-1}X_{ij}]_{j=1}^n]_{i=1}^m, v)\\
&= \max_{1\le k\le p}\sup_{\alpha_{k-1}<\alpha\le\alpha_k} d_H(L_\alpha[m^{-1},[n^{-1},X_{ij}]_{j=1}^n]_{i=1}^m, L_\alpha v)\\
&\le \max_{1\le k\le p} d_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m,\\
&\qquad\qquad [m^{-1},[n^{-1},L_{\alpha_k}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)\\
&\quad + \max_{1\le k\le p} d_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m,\\
&\qquad\qquad [m^{-1},[n^{-1},L^+_{\alpha_{k-1}}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m) + 4\varepsilon.
\end{aligned}$$

Therefore, by the assumption we obtain

$$\begin{aligned}
&Ed_\infty([m^{-1},[n^{-1}X_{ij}]_{j=1}^n]_{i=1}^m, v)\\
&\le E\Big[\max_{1\le k\le p} d_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m,\\
&\qquad\qquad [m^{-1},[n^{-1},L_{\alpha_k}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)\Big]\\
&\quad + E\Big[\max_{1\le k\le p} d_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m,\\
&\qquad\qquad [m^{-1},[n^{-1},L^+_{\alpha_{k-1}}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)\Big] + 4\varepsilon\\
&\le \sum_{k=1}^p Ed_H([m^{-1},[n^{-1},L_{\alpha_k}X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},L_{\alpha_k}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)\\
&\quad + \sum_{k=1}^p Ed_H([m^{-1},[n^{-1},L^+_{\alpha_{k-1}}X_{ij}]_{j=1}^n]_{i=1}^m,\\
&\qquad\qquad [m^{-1},[n^{-1},L^+_{\alpha_{k-1}}E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m) + 4\varepsilon\\
&= o(1) + 4\varepsilon \quad\text{as}\quad m\vee n\to\infty.
\end{aligned}$$

The proof is completed. □

Applying Theorem 3.8, we obtain the following corollary.

Corollary 3.9. *Let $\{X_{mn} : m \ge 1, n \ge 1\}$ be an array of identically distributed fuzzy random variables with $X_{11} \in \mathcal{L}^1(\mathcal{F}(\mathfrak{X}))$. Then*

$$Ed_\infty([m^{-1},[n^{-1},X_{ij}]_{j=1}^n]_{i=1}^m, E_{\mathcal{F}(\mathfrak{X})}X_{11}) \to 0 \text{ as } m\vee n\to\infty$$

if and only if for each $\alpha \in [0;1)$

$$Ed_H([m^{-1},[n^{-1},L_\alpha X_{ij}]_{j=1}^n]_{i=1}^m, E_{k(\mathfrak{X})}L_\alpha X_{11}) \to 0 \text{ as } m\vee n\to\infty$$

and for each $\alpha \in (0;1]$

$$Ed_H([m^{-1},[n^{-1}L^+_\alpha X_{ij}]_{j=1}^n]_{i=1}^m, E_{k(\mathfrak{X})}L^+_\alpha X_{11}) \to 0 \text{ as } m\vee n\to\infty.$$

Proof. The necessity is obvious. To prove the sufficiency, we note that

$$L_\alpha(E_{\mathcal{F}(\mathfrak{X})}X_{11}) = E_{k(\mathfrak{X})}(L_\alpha X_{11}) \in K_{k(\mathfrak{X})}(k(\mathfrak{X})), \text{ for all } \alpha\in(0;1].$$

Therefore, for all m, n,

$$\begin{aligned}
&d_\infty([m^{-1},[n^{-1},E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m, E_{\mathcal{F}(\mathfrak{X})}X_{11})\\
&= \sup_{\alpha\in(0;1]} d_H(L_\alpha[m^{-1},[n^{-1},E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m, L_\alpha E_{\mathcal{F}(\mathfrak{X})}X_{11})\\
&= \sup_{\alpha\in(0;1]} d_H([m^{-1},[n^{-1},L_\alpha E_{\mathcal{F}(\mathfrak{X})}X_{11}]_{j=1}^n]_{i=1}^m, L_\alpha E_{\mathcal{F}(\mathfrak{X})}X_{11}) = 0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&d_\infty([m^{-1},[n^{-1},X_{ij}]_{j=1}^n]_{i=1}^m,E_{\mathcal{F}(\mathfrak{X})}X_{11})\\&\le d_\infty([m^{-1},[n^{-1},X_{ij}]_{j=1}^n]_{i=1}^m,[m^{-1},[n^{-1},E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m)\\&\quad+d_\infty([m^{-1},[n^{-1},E_{\mathcal{F}(\mathfrak{X})}X_{ij}]_{j=1}^n]_{i=1}^m,E_{\mathcal{F}(\mathfrak{X})}X_{11}).\end{aligned}$$

The proof is completed by applying Theorem 3.8. □

Acknowledgments: The author would like to thank the referees for carefully reading the manuscript and for offering some very perceptive comments that helped me to improve this study.

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