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A FIXED POINT THEOREM ON SOME MULTI-VALUED MAPS IN MODULAR SPACES

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Abstract. Fixed point theory has been a flourishing area of mathematical research for decades, because of its many diverse applications. In this paper, we present a fixed point theorem for $s-\rho-$contractive type multi-valued mappings in modular spaces which will generalize some old results.

1. Introduction

The concept of fixed point plays a key role in analysis. Also, fixed point theorems are mainly used in existence theory of random differential equations, numerical methods like Newton-Rapshon method and Picards existence theorem and in other related areas. Fixed point theorems based on the consideration of order have importance in algebra, the theory of automata, mathematical linguistics, linear functional analysis, approximation theory and theory of critical points. Fixed point theorems play a key role in applications of variational inequalities, linear inequalities, optimization techniques and approximation

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theory. Thus the theory of fixed point has been studied by many researchers extensively.

By a contraction on a metric space (X, d), we understand a mapping $T : X \to X$ satisfying for all $x, y \in X$: $d(Tx, Ty) \leq kd(x, y)$, where k is a real in $[0, 1)$.

In 1922, Banach proved the following theorem.

Theorem 1.1. ([3]) *Let (X, d) be a complete metric space. Let $T : X \to X$ be a contraction. Then:*

(i) *T has a unique fixed point $x \in X$.*
(ii) *For every $x_0 \in X$, the sequence $\{x_n\}$, where $x_{n+1} = Tx_n$, converges to x.*
(iii) *We have the following estimate: For every $x \in X$,*

$$d(x_n, x) \leq \frac{k^n}{1-k} d(x_0, x_1), \ \ n \in \mathbb{N}.$$

The previous theorem has been extended and generalized in different directions. Recently, Azennar [1] gave some fixed point results for generalized $\alpha-$contraction.

In 1969, Nadler [12] extended the result of Banach to the case of multi-valued mappings by using the Hausdorff metric on closed and bounded subsets of a given metric space. Fixed point theory in modular function spaces is closely related to the metric fixed point theory, in that it provides modular equivalents of norm and metric concepts. Modular spaces are extensions of the classical Lebesgue and Orlicz spaces, and in many instances conditions cast in this framework are more natural and more easily verified than their metric analogs.

The fixed point results in modular function spaces were obtained by Khamsi et al. [7]. Taleb and Hanebaly [16] obtained a fixed point theorem of Banach type in modular space, where they have used some convenient constants in the Banach contraction assumptions and studied the generalization of Banach fixed point theorem in some classes of modular spaces. As an application they also studied the existence of a solution for an integral equation of Lipschitz type in Musielak-Orlicz spaces.

Kutbi and Latif [10] studied some fixed point results of multi-valued mappings in modular function spaces and proved the existence of fixed points for contractive type and nonexpansive type multi-valued maps in these spaces For some more interesting results on different types of mappings in modular function spaces, we refer the reader to [5, 8, 13, 14].

In this paper, we present a generalization of fixed point theorem of multi-valued mappings in modular spaces, where the modular is s-convex, having the Fatou property and satisfying the Δ_2-condition.

2. Preliminaries

We begin by recalling some basic concepts of modular spaces, for more information, we refer to the books by Musielak [11] and Kozlowski [9].

Definition 2.1. Let X be an arbitrary vector space over $K(=\mathbb{R}$ or $\mathbb{C})$. A functional $\rho : X \to [0, +\infty]$ is called modular if, for any $x, y \in X$, the following hold:

(1) $\rho(x) = 0$ if and only if $x = 0$.
(2) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$.
(3) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ provided that $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$.

The modular ρ is called an s-convex modular if

$$\rho(\alpha x + \beta y) \le \alpha^s \rho(x) + \beta^s \rho(y) \text{ for } \alpha, \beta \ge 0, \alpha^s + \beta^s = 1 \text{ with an } s \in]0, 1[$$

If $s = 1$, ρ is called convex modular.

The vector space X_ρ given by

$$X_\rho = \{x \in X, \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$$

is called a modular space. Generally, the modular ρ is not subadditive and therefore does not behave as a norm or a distance.

Modular space X_ρ can be equipped with the norm called the Luxemburg norm defined by

$$\|x\|_\rho = \inf\left\{\alpha > 0, p\left(\frac{x}{\alpha}\right) \le 1\right\}.$$

As a classical example, we would like to mention the Musielak-Orlicz space denoted by L^ϕ [11] and the modular function space denoted by L^ρ [9].

Definition 2.2. Let X_ρ be a modular space.

(1) A sequence $\{x_n\}$ in X_ρ is said to be :
  (i) ρ-convergent to x if $\rho(x_n - x) \to 0$ as $n \to \infty$.
  (ii) ρ-Cauchy to x if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(2) X_ρ is ρ-complete if any ρ-Cauchy sequence is ρ-convergent.
(3) A subset $B \subset X_\rho$ is said to be ρ-closed if for any sequence $\{x_n\} \subset B$, if $x_n \to x$ then $x \in B$. $\overline{B}^\rho$ denotes the closure of B in the sense of ρ.
(4) A ρ-closed subset $B \subset X_\rho$ is called ρ-compact if any sequence $\{x_n\} \subset B$ has a ρ-convergent subsequence.

(5) A subset $B \subset X_\rho$ is called ρ-bounded if

$$\delta_\rho(B) = \sup_{x,y \in B} \rho(x-y) < \infty,$$

$\delta_\rho(B)$ is called the ρ-diameter of B.

(6) Define the ρ-ball, $B_\rho(x,r)$, centered at $x \in X_\rho$ with radius r as

$$B_\rho(x,r) = \{y \in X_\rho; \rho(x-y) \leq r\}.$$

(7) Let $H_\rho(,)$ be the ρ-Hausdorff distance on X_ρ, that is,

$$H_\rho(A,B) = \max\left\{\sup_{x \in A} \text{dist}_\rho(x,B), \sup_{y \in B} \text{dist}_\rho(y,A)\right\}, \quad A, B \in X_\rho.$$

(8) We say that ρ has the Fatou property if

$$\rho(x-y) \leq \underline{\lim}\, \rho(x_n - y_n),$$

whenever $x_n \xrightarrow{\rho} x$ and $y_n \xrightarrow{\rho} y$ as $n \to \infty$.

(9) ρ is said to satisfy the Δ_2-condition if:

$$\rho(2x_n) \to 0 \text{ as } n \to \infty \quad \text{whenever} \quad \rho(x_n) \to 0 \text{ as } n \to \infty$$

(10) Let X_ρ be a modular space. We say that $T: X_\rho \to X_\rho$ is ρ-continuous when if $\rho(x_n - x) \to 0$, then $\rho(Tx_n - Tx) \to 0$ as $n \to \infty$.

Example 2.3. The Orlicz modular is defined for every measurable real function f by the formula

$$\rho(f) = \int_{\mathbb{R}} \varphi(|f(t)|) dm(t),$$

where m denotes the Lebesgue measure in $\mathbb{R}$ and $\varphi : \mathbb{R} \to [0, \infty[$ is continuous.

We also assume that $\varphi(u) = 0$ iff $u = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$.

The modular space induced by the Orlicz modular ρ_φ is called the Orlicz space L^φ.

Definition 2.4. Let X_ρ be a modular space, an element $x \in X_\rho$ is said to be a fixed point of a multivalued mapping $T : X_\rho \to 2^{X_\rho}$ if $x \in T(x)$.

Theorem 2.5. ([16, Theorem I.2]) *Let X_ρ be a ρ-complete modular space. Let $\{F_n\}$ be a decreasing sequence of nonempty $\rho-$closed subsets of X_ρ with $\delta_\rho(F_n) \to 0$ as $n \to \infty$. Then $\cap_n F_n$ is reduced to one point.*

3. Main results

Definition 3.1. Let X_ρ be a modular space. Let B be a ρ-closed subset of X_ρ, we assume that ρ is a s-convex. A multi-valued mapping $T : B \to 2^B$ is $s-\rho-$contractive type if there exist $c, k \in \mathbb{R}^+$, $c > \max(1, k)$ such that for any $x, y \in B$ and for any $X \in Tx$, there exists $Y \in Ty$ such that

$$\rho(c(X - Y)) \leq k^s \rho(x - y). \tag{3.1}$$

Remark 3.2. By the s-convexity and the monotonicity of ρ, the contraction (1.1) is also true for any constant c_0 such that $1 < c_0 < c$.

Kutbi and Latif [10] have proved that every ρ-contraction, $T : C \to F_\rho(C)$ has a fixed point where ρ is a convex function modular satisfying the so-called Δ_2-condition, C is a nonempty ρ-bounded ρ-closed subset of L_ρ and $F_\rho(C)$ a family of ρ-closed subsets of C.

Taleb and Hanebaly [16] have proved generalization of Banach's fixed point theorem in some classes of modular spaces, where the modular is s-convex, having the Fatou property and satisfying the Δ_2-condition.

We present a generalization of [16] and [10], where the modular ρ is s-convex, having the Fatou property and satisfying the Δ_2-condition.

Theorem 3.3. *Let X_ρ be a modular space. Assume that ρ is an s-convex modular satisfying the Δ_2-condition and has the Fatou property. Let B be a ρ-closed subset of X_ρ and $T : B \to 2^B$ be a multi-valued $s-\rho-$contractive type mapping. Then T has a fixed point.*

Proof. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, and consider the sets defined by

$$M_{\varepsilon_n} = \{x \in B \mid \exists y \in Tx, \rho(L(x - y)) \leq \varepsilon_n\},$$

where $L = \max\{c, 2\alpha\}$ and α is the s-conjugate of c, i.e. $\frac{1}{c^s} + \frac{1}{\alpha^s} = 1$.

First notice that $\{M_\varepsilon\}$ is decreasing, because (ε_n) is decreasing. We will show that M_{ε_n} satisfies the hypotheses of the Theorem 2.5.

Let $x_0 \in B$. Without loss of generality, assume that x_0 is not a fixed point of T. Then there exists $x_1 \in T(x_0)$ such that $x_1 \neq x_0$. Hence $\rho(x_0 - x_1) > 0$. Since T is $s-\rho-$contractive type, there exists $x_2 \in T(x_1)$ such that

$$\rho(c(x_1 - x_2)) \leq k^s \rho(x_0 - x_1).$$

By induction, one can easily construct a sequence $\{x_n\} \subset B$ such that $x_{n+1} \in T(x_n)$ and

$$\rho(c(x_{n+1} - x_n)) \leq k^s \rho(x_n - x_{n-1}).$$

By the monotonicity of $\lambda \to \rho(\lambda x)$ $(\lambda \geq 0)$ and for any $n \geq 1$.

In particular by iteration, we have

$$\rho\left(c(x_{n+1}-x_n)\right) \leq \left(\frac{k}{c}\right)^{s(n-1)} \rho\left(x_1-x_0\right)$$

and since $\frac{k}{c}<1$ we have $\rho\left(c\left(x_{n+1}-x_n\right)\right) \rightarrow 0$ as $n \rightarrow \infty$.
Thus by Δ_2-condition $\rho\left(L\left(x_{n+1}-x_n\right)\right) \rightarrow 0$ as $n \rightarrow \infty$. Hence there exists $q \in \mathbb{N}^*$ such that $\rho\left(L\left(x_{q+1}-x_q\right)\right) \leq \varepsilon_n$. Then for $y=x_{q+1} \in T x_q$ and $x=x_q \in M_{\varepsilon_n}$.

Let's show that M_{ε_n} is ρ-closed.

Let $\{x_n\}$ be a sequence in M_{ε_n}. Then there exist $y_n \in T x_n$ such that

$$\rho(L(x_n-y_n)) \leq \varepsilon_n.$$

Assume that $\{x_n\}$ is ρ -convergent to $x \in X_\rho$, since $\left(x_n\right)_n \subset B$ and B is ρ-closed, it follows that $x \in B$. We have $\rho\left(x_n-x\right) \rightarrow 0$ as $n \rightarrow \infty$, by Δ_2-condition, we have that

$$\rho\left(L\left(x_n-x\right)\right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

So, T is $s-\rho-$contractive type and $y_n \in T x_n$, then there exists $y \in T(x)$ such that

$$\begin{aligned} \rho\left(c(y_n-y)\right) & \leq k^s \rho(x_n-x) \\ & \leq k^s \rho(L(x_n-x)). \end{aligned}$$

Then $\rho\left(c\left(y_n-y\right)\right) \rightarrow 0$ as $n \rightarrow \infty$.

Again by Δ_2-condition $\rho\left(L\left(y_n-y\right)\right) \rightarrow 0$ as $n \rightarrow \infty$. The Fatou property implies that

$$\rho(L(x-y)) \leq \lim \rho\left(L\left(x_n-y_n\right)\right) \leq \varepsilon_n.$$

Finally, there exists $y \in T x$ such that $\rho(L(x-y)) \leq \varepsilon_n$. Therefore $x \in M_{\varepsilon_n}$ and hence M_{ε_n} is ρ-closed.

We still have to show that $\delta_\rho\left(M_{\varepsilon_\mathrm{n}}\right) \rightarrow 0$ as $n \rightarrow \infty$.

Let $x, x' \in M_{\varepsilon_n}$. Then there exist $y \in T x$ and $y' \in T x'$ such that

$$\rho(L(x-y)) \leq \varepsilon_n$$

and

$$\rho(L(x'-y')) \leq \varepsilon_n.$$

Since α is the s-conjugate of c, and T is a s-ρ-contractive type multi-valued mapping, by Δ_2-condition, we show that

$$\begin{aligned}
\rho\left(x-x'\right) &= \rho\left(\frac{\alpha(x-y)}{\alpha}+\frac{c(y-y')}{c}+\frac{\alpha(y'-x')}{\alpha}\right)\\
&\leq \frac{1}{\alpha^s}\rho(\alpha(x-y)+\alpha(y'-x'))+\frac{1}{c^s}\rho(c(y-y'))\\
&\leq \frac{1}{2^s\alpha^s}\rho(2\alpha(x-y))+\frac{1}{2^s\alpha^s}\rho(2\alpha(x'-y'))+\frac{1}{c^s}\rho(c(y-y'))\\
&\leq \frac{1}{2^s\alpha^s}(\rho(L(x-y))+\rho(L(x'-y')))+\frac{1}{c^s}\rho(c(y-y'))\\
&\leq \frac{1}{2^s\alpha^s}2\varepsilon_n+\left(\frac{k}{c}\right)^s\rho(x-x')\\
&\leq \varepsilon_n 2^{1-s}\frac{c^s-1}{c^s-k^s}.
\end{aligned}$$

Then $\delta_\rho\left(M_{\varepsilon_n}\right)=\sup_{x,x'\in M_{\varepsilon_n}}\rho(x-x')\to 0$ as $n\to\infty$. From Theorem 3.3, we have $\cap_n M_{\varepsilon_n}=\{z\}$ and for all $n\in\mathbb{N}, z\in M_{\varepsilon_n}$, therefore there exist $y\in Tz$ such that

$$\rho(L(z-y))\leq\varepsilon_n\to 0,$$

this means that $y=z$. Hence, we have $z\in Tz$. □

Corollary 3.4. ([16, Theorem 1.1]) *Let X_ρ be a modular space. Assume that ρ is an s-convex modular satisfying the Δ_2-condition and has the Fatou property. Let B be a ρ-closed subset of X_ρ and $T: B\to B$ be an $s-\rho-$contractive type mapping. Then T has a fixed point.*

Corollary 3.5. ([10, Theorem 2.1]) *Let X_ρ be a modular space. Assume that ρ is an 1-convex modular satisfying the Δ_2-condition and has the Fatou property. Let B be a ρ-closed subset of X_ρ and $T: B\to 2^B$ be a multi-valued $1-\rho-$contractive type mapping with $c=1$ and $0<k<1$. Then T has a fixed point.*

References

[1] R. Azennar, S. BenTahar and D. Mentagui. *Numerical Approximations For Fractional Equations In $\mathbb{R}$ Via The New Method Of Fixed Point And Application,* J. Interdisciplinary Math., **2021**, DOI:10.1080/09720502.2021.1887606.

[2] R. Azennar and D. Mentagui, *A solution for the minimax problem via fixed point theory in complete ordered locally convex spaces,* Commu. Opti. Theory, **2020** (2020), Article ID 11.

[3] S. Banach, *Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales,* Fund. Math., **3** (1922), 133-181.

[4] M. Edelstein, *An extension of Banach's contraction principle,* Proc. Amer. Math. Soc., **12**(1) (1961), 7-10.

[5] H. Iqbal, M. Abbas and S.H. Khan, *ρ-attractive elements in modular function spaces,* Kragujevac J. Math., **45**(1) (2021), 47-61.

[6] A. Kari, M. Rossafi, E. Marhrani and M. Aamri, *New fixed point theorems for $\theta-\phi-$contraction on rectangular b-metric spaces,* Abst. Appl. Anal., **2020**, Article ID 8833214, 2020.

[7] M.A. Khamsi, W.M. Kozlowski and S. Reich, *Fixed point theory in modular function spaces*, Nonlinear Anal., **14** (1990), 935-935.

[8] S.H. Khan, *Approximating fixed points of (λ, ρ)-firmly nonexpansive mappings in modular function spaces,* Arab. J. Math., **7** (2018), 281-287.

[9] W.M. Kozlowski, *Modular function spaces,* Dekker, New York/Basel, 1988.

[10] M.A. Kutbi and A. Latif, *Fixed Points of Multivalued Maps in Modular Function Spaces,* Fixed Point Theory Appl., **2009**, Article ID 786357, 2009.

[11] J, Musielak, *Orlicz spaces and modular spaces,* Lecture Notes in Mathematics, **1034**, Springer-Verlag, 1983.

[12] S.B. Nadler, *Multivalued contraction mappings,* Pacific J. Math. **30** (1969), 475-488.

[13] G.A. Okeke, S.A. Bishop and S.H. Khan, *Iterative approximation of fixed point of multivalued ρ-quasi-nonexpansive mappings in modular function spaces with applications,* J. Funct. Spaces, **2018**, Article ID 1785702, 2018.

[14] G.A. Okeke and J.K. Kim, *Approximation of common fixed point of three multi-valued ρ-quasi-nonexpansive mappings in modular function spaces,* Nonlinear Funct. Anal. Appl., **24**(4) (2019), 651-664.

[15] F. Ouzine, R. Azennar and D. Mentagui, *Common Coupled Fixed Point Results For Multivalued Mappings In Ordered Banach Spaces,* Dyna. Syst. Appl., **30**(2) (2021), 237-250

[16] A.A. Taleb and E. Hanebaly, *A Fixed Point Theorem And Its Application To Integral Equations In Modular Function Spaces,* Proc. Amer. Math. Soc., **128**(2) (1999), 419-426.

[17] D. Wardowski, *Fixed points of new type of contractive mappings in complete metric space* Fixed Point Theory Appl., **2012**:94 (2012).