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STUDY OF A CRITICAL Φ-KIRCHHOFF TYPE EQUATIONS IN ORLICZ-SOBOLEV SPACES

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Abstract. This paper is concerned with the existence of solutions for a class of Φ-Kirchhoff type equations with critical exponent in Orlicz-Sobolev spaces. Our technical approach is based on variational methods.

1. Introduction

In this article, we consider the existence and multiplicity of solutions for the following nonlocal problems:

$$\begin{cases} -M\Big(\int_\Omega \Phi(|\nabla u|)\Big)\,\mathrm{div}\,\big(\phi(|\nabla u|)\nabla u\big) = \lambda f(x,u) + \phi_*(|u|)u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary, λ is a positive parameter, and the functions M, f, ϕ and Φ will be specified later.

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Equation (1.1) has been widely studied in the case when $\phi(t) = t^{p-2}$, and corresponds to the well-known nonlocal problem involving the classical p-Laplacian operator $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$:

$$\begin{cases} -M\Big(\displaystyle\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = \lambda f(x,u) + |u|^{p^*-2}u, & \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.2}$$

The problem (1.1) becomes a nonlinear and non-homogeneous problem, which has been received considerable attention in recent years and studied by some authors in Orlicz-Sobolev spaces, We refer to the overview papers [3, 8, 14, 15, 16, 17, 18, 21, 22] for the advances and references of this area.

In the present work, motivated by results found in [11, 12], we generalize the results of the paper [23, 25], in which the authors study a p-Kirchhoff type equation involving the critical Sobolev exponent, to nonlocal problems in Orlicz-Sobolev spaces with critical growth. So, we will show the existence of infinitely many solutions for the nonlocal problem (1.1). The difficulty in this case, is due to the lack of compactness of the imbedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\Phi_*(x)}(\Omega)$ and the Palais-Smale condition for the corresponding energy functional could not be checked directly. To deal with this difficulty, we use a version of the concentration compactness lemma due to Lions for Orlicz-Sobolev space found in Fukagai et al. [10].

Throughout the sequel, we make that the function $\phi : (0, +\infty) \to \mathbb{R}$ is a continuous satisfying

(ϕ_1)

$$(\phi(t)t)' > 0, \quad \forall t > 0. \tag{1.3}$$

(ϕ_2) There exist $l, m \in (1, N)$ such that

$$l \le \frac{\phi(|t|)t^2}{\Phi(t)} \le m, \quad \forall t \neq 0, \tag{1.4}$$

where

$$\Phi(t) = \int_0^{|t|} \phi(s)s\, ds, \quad l \le m < l^*, \quad l^* = \frac{lN}{N-l}, \quad m^* = \frac{mN}{N-m}.$$

Moreover, ϕ_* is such that the Sobolev conjugate function Φ_* of Φ is its primitive; that is, $\Phi_*(t) = \int_0^{|t|} \phi_*(s)sds$.

Related to functions $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $M : \mathbb{R}^+ \to \mathbb{R}^+$, we assume that

(M_1) There exists $m_0 > 0$ such that $M(t) \ge m_0$ for all $t \ge 0$.

(M_2) There exists $\sigma > m/l^*$ such that

$$\widehat{M}(t) \ge \sigma M(t)t, \;\; \forall\, t \ge 0,$$

where $\widehat{M}(t) = \int_0^t M(s)ds$.

(f_1) $f(x,t) = o\big(\phi(|t|)|t|\big)$ as $|t| \to 0$ uniformly in $x \in \Omega$;

(f_2) $f(x,t) = o\big(\phi_*(|t|)|t|\big)$ as $|t| \to +\infty$ uniformly in $x \in \Omega$;

(f_3) There exists $\theta \in (m/\sigma, l^*)$ such that

$$0 < \theta F(x,t) \leq t f(x,t), \quad \forall\, t \in \mathbb{R},\ a.e.\ x \in \Omega,$$

where $F(x,t) = \int_0^t f(x,s)ds$, and σ is given in assumption (M_2).

The difficulty to prove main theorem is related to the nonlinearity of f and her critical growth. In this case, it is not clear that functional energy associated with (1.1) satisfies the well-known (PS) condition, once that the embedding $W^{1,\Phi}(\Omega) \hookrightarrow L_{\Phi_*}(\Omega)$ is not compact. To overcome this difficulty, we use a version of the concentration compactness lemma due to Lions for Orlicz-Sobolev space found in Fukagai, Ito and Narukawa [10].

2. Preliminaries on Orlicz-Sobolev spaces

Here, we state some interesting properties of the theory of Orlicz- Sobolev spaces that will be useful to discuss problem (1.1). To be more precise, let ϕ be a real-valued function defined in $[0,\infty)$ having the following properties: ϕ is nondecreasing, $\phi(0) = 0$, $\phi(t) > 0$ if $t > 0$ and $\lim_{t\to\infty} \phi(t) = \infty$, ϕ is right continuous, that is, $\lim_{s\to t^+} \phi(s) = \phi(t)$. Then, the real-valued function Φ defined on $\mathbb{R}$ by

$$\Phi(u) = \int_0^{|t|} \phi(s)s\, ds$$

is called an N-function. We say that a N-function Φ satisfies the Δ_2-condition if, there exists $t_0 \geq 0$ and $k > 0$ such that

$$\Phi(2t) \leq k\Phi(t), \quad \forall t \geq t_0.$$

For an N-function Φ satisfying Δ_2-condition and an open set $\Omega \subset \mathbb{R}^N$, the space $L_\Phi(\Omega)$ is the vectorial space of the measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega \Phi(|u|) < \infty.$$

The space $L_\Phi(\Omega)$ endowed with Luxemburg norm,

$$\|u\|_\Phi = \inf\Big\{\tau > 0 : \int_\Omega \Phi\big(\frac{|u|}{\tau}\big) \leq 1\Big\}$$

is a Banach space. The complement function of Φ, denoted by $\widetilde{\Phi}(s)$, is given by the Legendre transformation,

$$\widetilde{\Phi}(s) = \max_{t\geq 0}\{st - \Phi(t)\} \quad \text{for } s \geq 0.$$

The functions Φ and $\widetilde{\Phi}$ are complementary each other. Moreover, we have the Young's inequality

$$st \leq \Phi(t) + \widetilde{\Phi}(s), \quad \forall t, s \geq 0. \tag{2.1}$$

Using this inequality, it is possible to prove the Hölder type inequality

$$\Big|\int_\Omega uv\,dx\Big| \leq 2\|u\|_\Phi \|v\|_{\widetilde{\Phi}}, \quad \forall u \in L_\Phi(\Omega) \text{ and } v \in L_{\widetilde{\Phi}}(\Omega). \tag{2.2}$$

We denote by $W_0^{1,\Phi}(\Omega)$ the corresponding Orlicz-Sobolev space for problem (1.1), defined by the completion of $C_0^\infty(\Omega)$ with norm

$$\|u\|_{1,\Phi} = \|u\|_\Phi + \||\nabla u|\|_\Phi$$

for all $u \in W_0^{1,\Phi}(\Omega)$, there is $c > 0$ such that

$$\|u\|_\Phi < c\||\nabla u|\|_\Phi, \text{ and } \int_\Omega \Phi(u)dx < c\int_\Omega \Phi(|\nabla u|)dx.$$

In this case, we see that $\|u\| := \||\nabla u|\|_\Phi$ and $\|.\|_{1,\Phi}$ are equivalent norms in $W_0^{1,\Phi}(\Omega)$. In the following, we will use $\|.\|$ instead of $\|.\|_{1,\Phi}$ on $X := W_0^{1,\Phi}(\Omega)$.

Another important function related to function Φ is the Sobolev's conjugate function Φ_* of Φ, defined by

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}}ds \quad \text{for } t > 0.$$

An important property is that: If Φ and $\widetilde{\Phi}$ satisfy the Δ_2-condition, then the spaces $L_\Phi(\Omega)$ and $W^{1,\Phi}(\Omega)$ are reflexive and separable. Moreover, the Δ_2-condition also implies that

$$u_n \to u \text{ in } L_\Phi(\Omega) \iff \int_\Omega \Phi(|u_n - u|) \to 0 \tag{2.3}$$

and

$$u_n \to u \text{ in } W^{1,\Phi}(\Omega) \iff \int_\Omega \Phi(|u_n - u|) \to 0 \text{ and } \int_\Omega \Phi(|\nabla u_n - \nabla u|) \to 0. \tag{2.4}$$

Another important inequality was proved by Donaldson and Trudinger [7], which establishes that for all open $\Omega \subset \mathbb{R}^N$ and there is a constant $S = S(N) > 0$ such that

$$\|u\|_{\Phi_*} \leq S\|u\|, \quad u \in X. \tag{2.5}$$

This inequality shows the following embedding is continuous

$$W_0^{1,\Phi}(\Omega) \hookrightarrow L_{\Phi_*}(\Omega).$$

If Ω is a bounded domain and the two limits hold

$$\limsup_{t\to 0}\frac{\Psi(t)}{\Phi(t)}<+\infty, \quad \limsup_{|t|\to+\infty}\frac{\Psi(t)}{\Phi_*(t)}=0, \tag{2.6}$$

then the embedding

$$W_0^{1,\Phi}(\Omega)\hookrightarrow L_\Psi(\Omega) \tag{2.7}$$

is compact.

The following four lemmas involving the functions $\Phi, \widetilde{\Phi}$ and Φ_*, are useful in the proof of our results, and their proofs can be found in [10]. Hereafter, Φ is the N-function given in the introduction and $\widetilde{\Phi}, \Phi_*$ are the complement and conjugate functions of Φ, respectively.

Lemma 2.1. *Assume* (1.3) *and* (1.4). *Then*

$$\Phi(t)=\int_0^{|t|}s\phi(s)ds,$$

is a N-function with $\Phi, \widetilde{\Phi}\in\Delta_2$. Hence, $L_\Phi(\Omega), W^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ are reflexive and separable spaces.

Lemma 2.2. *The functions Φ, Φ_*, $\widetilde{\Phi}$ and $\widetilde{\Phi}_*$ satisfy the inequalities*

$$\widetilde{\Phi}(\phi(|t|)t)\le\Phi(2t), \quad \widetilde{\Phi}_*(\phi_*(|t|)t)\le\Phi_*(2t), \quad \forall t\ge 0. \tag{2.8}$$

Lemma 2.3. *Assume that* (1.3) *and* (1.4) *hold and let $\xi_0(t)=\min\{t^l,t^m\}$, $\xi_1(t)=\max\{t^l,t^m\}$ for all $t\ge 0$. Then*

$$\xi_0(\rho)\Phi(t)\le\Phi(\rho t)\le\xi_1(\rho)\Phi(t) \quad \textit{for } \rho,t\ge 0,$$

$$\xi_0(\|u\|_\Phi)\le\int_\Omega\Phi(u)\le\xi_1(\|u\|_\Phi) \quad \textit{for } u\in L_\Phi(\Omega).$$

Lemma 2.4. *The function Φ_* satisfies the inequality*

$$l^*\le\frac{\Phi_*'(t)t}{\Phi_*(t)}\le m^* \quad \textit{for } t>0.$$

As an immediate consequence of the Lemma 2.4, we have the following result

Lemma 2.5. *Assume that* (1.3) *and* (1.4) *hold and let $\xi_2(t)=\min\{t^{l^*},t^{m^*}\}$, $\xi_3(t)=\max\{t^{l^*},t^{m^*}\}$ for all $t\ge 0$. Then*

$$\xi_2(\rho)\Phi_*(t)\le\Phi_*(\rho t)\le\xi_3(\rho)\Phi_*(t) \quad \textit{for } \rho,t\ge 0,$$

$$\xi_2(\|u\|_{\Phi_*})\le\int_\Omega\Phi_*(u)dx\le\xi_3(\|u\|_{\Phi_*}) \quad \textit{for } u\in L_{\Phi_*}(\Omega).$$

Lemma 2.6. *Let $\widetilde{\Phi}$ be the complement of Φ and put*

$$\xi_4(s) = \min\{s^{\frac{l}{l-1}}, s^{\frac{m}{m-1}}\}, \quad \xi_5(s) = \max\{s^{\frac{l}{l-1}}, s^{\frac{m}{m-1}}\}, \quad s \geq 0.$$

Then the following inequalities hold

$$\xi_4(r)\widetilde{\Phi}(s) \leq \widetilde{\Phi}(rs) \leq \xi_5(r)\widetilde{\Phi}(s),\ r, s \geq 0;$$

$$\xi_4(\|u\|_{\widetilde{\Phi}}) \leq \int_\Omega \widetilde{\Phi}(u)dx \leq \xi_5(\|u\|_{\widetilde{\Phi}}),\ u \in L_{\widetilde{\Phi}}(\Omega).$$

3. Proof of main result

In the sequel, we derive some results related to the mountain pass Theorem and the Palais-Smale compactness condition. We define the energy functional corresponding to problem (1.1) as $I_\lambda : X \mapsto \mathbb{R}$,

$$I_\lambda(u) = \widehat{M}\Big(\int_\Omega \Phi(|\nabla u|)dx\Big) - \lambda \int_\Omega F(x,u)dx - \int_\Omega \Phi_*(u)dx.$$

Clearly, $I_\lambda \in C^1(X, \mathbb{R})$ with the derivatives given by

$$\begin{aligned}\langle I'_\lambda(u), v\rangle = M\Big(\int_\Omega \Phi(|\nabla u|)dx\Big) \int_\Omega \phi(|\nabla u|)\nabla u \nabla v \\ - \lambda \int_\Omega f(x,u)vdx - \int_\Omega \phi_*(|u|)uvdx,\end{aligned}$$

for any u, $v \in X$, and the critical points of it are weak solutions of problem (1.1). For simplicity, we use c_i $(i = 1, 2, ..)$, to denote the general nonnegative or positive constant. The exact value may change from line to line.

Lemma 3.1. *Under the conditions (M_1), (f_1) and (f_2), there exist $\beta, \varrho > 0$ such that $I_\lambda(u) \geq \beta$ for any $u \in X$, with $||u|| = \varrho$.*

Proof. By (f_1) and (f_2), it follows that for any $\varepsilon > 0$, there exists $C_\varepsilon = C(\varepsilon) > 0$ depending on ε such that

$$|F(x,t)| \leq \varepsilon\Phi(t) + C_\varepsilon \Phi_*(t) \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{3.1}$$

Together with (M_1), we have

$$I_\lambda(u) \geq m_0 \int_\Omega \Phi(|\nabla u|)dx - \lambda\varepsilon \int_\Omega \Phi(u)dx - (1 + \lambda C_\varepsilon) \int_\Omega \Phi_*(u)dx.$$

Hence for ε sufficiently small, we get

$$I_\lambda(u) \geq m_0 \int_\Omega \Phi(|\nabla u|)dx - (1 + \lambda C_\varepsilon) \int_\Omega \Phi_*(u)dx.$$

Due to (2.5) for the continuous embedding $X \hookrightarrow L_{\Phi_*}(\Omega)$, and by Lemma 2.3 and Lemma 2.5, for $||u|| = \varrho$ with $0 < \varrho < \min\{1, 1/S\}$ and taking into account the last inequalities, we deduce that

$$I_\lambda(u) \geq m_0||u||^m - c_1||u||^{l^*}.$$

Since $m < l^*$, there exists $\beta > 0$ such that $I_\lambda(u) \geq \beta$ for $||u|| = \varrho$, where ϱ is chosen sufficiently small. The proof of Lemma 3.1 is complete. □

Lemma 3.2. *Assume that conditions* (f_2), (M_2) *hold. Then for all* $\lambda > 0$, *there exists a nonnegative function* $e \in X$ *with* $\|e\| > \varrho$ *(where* ϱ *is given in Lemma* 3.1*) such that* $I_\lambda(e) < 0$.

Proof. Choose a nonnegative function $v_0 \in C_0^\infty(\Omega)$ with $\|v_0\| = 1$. By integrating (M_2), we obtain

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_0)}{t_0^{1/\sigma}} t^{\frac{1}{\sigma}} = c_3 t^{1/\sigma} \quad \text{for all } t \geq t_0 > 0. \tag{3.2}$$

From (f_2), for $\varepsilon > 0$ there is a constant $M_\varepsilon > 0$ such that

$$F(x,t) > -M_\varepsilon - \varepsilon \Phi_*(t) \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{3.3}$$

Thus

$$I_\lambda(u) \leq c_3 \Big(\int_\Omega \Phi(|\nabla u|) dx \Big)^{1/\sigma} + (\lambda\varepsilon - 1) \int_\Omega \Phi_*(u) dx + \lambda M_\varepsilon |\Omega|.$$

By choosing $\varepsilon = \frac{1}{2\lambda}$, for all $t \geq 1$, according to Lemma 2.5 we obtain

$$I_\lambda(tv_0) \leq c_3 t^{m/\sigma} \Big(\int_\Omega \Phi(|\nabla v_0|) dx \Big)^{1/\sigma} - \frac{1}{2} t^{l^*} \int_\Omega \Phi_*(v_0) dx + \lambda M_\varepsilon |\Omega|. \tag{3.4}$$

Since $\sigma > m/l^*$, $I_\lambda(tv_0) \to -\infty$ as $n \to \infty$, and the result follows by considering $e = t_* v_0$ for some $t_* > 0$ large enough. □

From Lemmas 3.1 and 3.2, using a version of the mountain pass theorem due to Ambrosetti and Rabinowitz [2], without (PS) condition (see [24]), there exists a sequence $\{u_n\} \subset H$ such that

$$I_\lambda(u_n) \to c_\lambda \text{ and } I'_\lambda(u_n) \to 0,$$

where

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0 \tag{3.5}$$

with

$$\Gamma = \{\gamma \in C([0,1], H) : \gamma(0) = 0, \, I_\lambda(\gamma(1)) < 0\}.$$

Lemma 3.3. *If the conditions* $(\mathrm{M}_1), (\mathrm{M}_2)$ *and* $(f_1) - (f_3)$ *hold, then*

$$\lim_{\lambda\to\infty} c_\lambda = 0.$$

Proof. For $e \in X$ given by Lemma 3.2, we have $\lim_{t\to+\infty} I_\lambda(te) = -\infty$, then there exists $t_\lambda > 0$ such that $I_\lambda(t_\lambda e) = \max_{t\geq 0} I_\lambda(te)$. Therefore,

$$\begin{aligned} &M\Big(\int_\Omega \Phi(t_\lambda|\nabla e|)dx\Big)\int_\Omega \phi(|\nabla t_\lambda e|)(t_\lambda \nabla e)^2 dx \\ &= \lambda \int_\Omega f(x, t_\lambda e)t_\lambda e\, dx + \int_\Omega \phi_*(|t_\lambda e|)(t_\lambda e)^2 dx. \end{aligned}$$

By (M_2), (f_3), Lammas 2.4 and 2.5, it follows that, for $t_\lambda > t_0$,

$$\begin{aligned} l^* t_\lambda^{l^*} \int_\Omega \Phi_*(e)dx &\leq l^* \int_\Omega \Phi_*(t_\lambda e)dx \\ &\leq \lambda \int_\Omega f(x, t_\lambda e)t_\lambda e\, dx + \int_\Omega \phi_*(|t_\lambda e|)(t_\lambda e)^2 dx \\ &= M\Big(\int_\Omega \Phi(t_\lambda|\nabla e|)dx\Big)\int_\Omega \phi(|\nabla t_\lambda e|)(t_\lambda \nabla e)^2 dx \\ &\leq mM\Big(\int_\Omega \Phi(t_\lambda|\nabla e|)dx\Big)\int_\Omega \Phi(t_\lambda|\nabla e|)dx \\ &\leq \frac{m}{\sigma}\widehat{M}\Big(\int_\Omega \Phi(t_\lambda|\nabla e|)dx\Big) \\ &\leq \frac{mc_3}{\sigma} t_\lambda^{m/\sigma}\Big(\int_\Omega \Phi(|\nabla e|)\Big)^{1/\sigma}. \end{aligned} \tag{3.6}$$

Since $l^* > m/\sigma$, $\{t_\lambda\}$ is bounded, there exists a sequence $\lambda_n \to +\infty$ and $t_* \geq 0$ such that $t_{\lambda_n} \to t_*$ as $n \to \infty$. Hence, there exists $c_5 > 0$ such that

$$M\Big(\int_\Omega \Phi(t_{\lambda_n}|\nabla e|)dx\Big)\int_\Omega \phi(|\nabla t_\lambda e|)(t_{\lambda_n}\nabla e)^2 \leq c_5 \quad \text{for all } n,$$

which yields,

$$\lambda_n \int_\Omega f(x, t_{\lambda_n} e)t_{\lambda_n} e\, dx + \int_\Omega \phi_*(|t_{\lambda_n} e|)(t_{\lambda_n} e)^2 dx \leq c_5 \quad \text{for all } n.$$

We claim that $t_* = 0$, otherwise, if $t_* > 0$ then the last inequality becomes

$$\lambda_n \int_\Omega f(x, t_{\lambda_n} e)t_{\lambda_n} e\, dx + \int_\Omega \phi_*(|t_{\lambda_n} e|)(t_{\lambda_n} e)^2 dx \to +\infty, \quad \text{as } n \to \infty,$$

which is impossible, so $t_* = 0$.

Now, let us consider the path $\gamma_*(t) = te$ for $t \in [0, 1]$, which belongs to Γ, thus

$$0 < c_{\lambda_n} \leq \max_{t_{\lambda_n} \geq 0} I(\gamma_*(t_{\lambda_n})) = I(t_{\lambda_n} e) \leq \widehat{M}\Big(\int_\Omega \Phi(t_{\lambda_n}|\nabla e|)dx\Big).$$

Since $t_{\lambda_n} \to 0$ from (3.6) we obtain

$$\lim_{\lambda\to\infty} \widehat{M}\Big(\int_\Omega \Phi(t_{\lambda_n}|\nabla e|)dx\Big) = 0,$$

which leads to $\lim_{\lambda\to\infty} c_\lambda = 0$. □

As consequence of the above Lemma 3.3, there exists $\lambda_* > 0$ such that

$$c_\lambda < \left(\frac{1}{\theta} - \frac{1}{l^*}\right) S^N m_0^{N/m} \text{ for every } \lambda \geq \lambda_*. \tag{3.7}$$

Now we are in a position to prove the main theorem.

Theorem 3.4. *Suppose that* $(M_1) - (M_2)$ *and* $(f_1) - (f_3)$ *hold. Then, there exists* $\lambda_* > 0$*, such that problem* (1.1) *has at least one nontrivial solution for all* $\lambda \geq \lambda_*$.

Proof. From Lemmas 3.1 and 3.2, there exists a bounded sequence $\{u_n\} \subset X$ such that

$$I_\lambda(u_n) \to c_\lambda \text{ and } I'_\lambda(u_n) \to 0.$$

First, let us show that the sequence $\{u_n\}$ is bounded in X. Indeed, from (f_3), for n large enough, it follows from (M_1) and (M_2) that

$$\begin{aligned}
&c + 1 + ||u_n|| \\
&\geq I_\lambda(u_n) - \frac{1}{\theta}\langle I'_\lambda(u_n), u_n\rangle \\
&= \widehat{M}\Big(\int_\Omega \Phi(|\nabla u_n|)dx\Big) - \frac{1}{\theta} M\Big(\int_\Omega \Phi(|\nabla u_n|)dx\Big)\int_\Omega \phi(|\nabla u_n|)(\nabla u_n)^2 dx \\
&\quad - \int_\Omega \Phi_*(u_n)dx + \frac{1}{\theta}\int_\Omega \phi_*(|u_n|)u_n^2 dx + \lambda \int_\Omega \left(\frac{1}{\theta} f(x,u_n)u_n - F(x,u_n)\right)dx \\
&\geq \Big(\sigma - \frac{m}{\theta}\Big) M\Big(\int_\Omega \Phi(|\nabla u_n|)dx\Big)\int_\Omega \phi(|\nabla u_n|)(\nabla u_n)^2 dx + \Big(\frac{l^*}{\theta} - 1\Big)\int_\Omega \Phi_*(u_n)dx \\
&\geq m_0\Big(\sigma - \frac{m}{\theta}\Big)\xi_0\Big(||u_n||_\Phi\Big) + \Big(\frac{l^*}{\theta} - 1\Big)\int_\Omega \Phi_*(u_n)dxdx.
\end{aligned}$$

Since $\theta \in (m/\sigma, l^*)$, $\{u_n\}$ is bounded. So, up to subsequence, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u \ \text{ in } W_0^{1,\Phi}(\Omega);\\ &u_n \rightharpoonup u \ \text{ in } L_{\Phi_*}(\Omega);\\ &u_n \to u \ \text{ in } L_{\Phi}(\Omega);\\ &u_n(x) \to u(x) \ \text{ a.e. in } \Omega. \end{aligned}$$

From the concentration compactness lemma of Lions in Orlicz-Sobolev space found in [10], there exist two nonnegative measures $\mu, \nu \in \mathcal{M}(\mathbb{R}^N)$, a countable set $\mathcal{J}$, points $\{x_j\}_{j\in\mathcal{J}}$ in $\overline{\Omega}$ and sequences $\{\mu_j\}_{j\in\mathcal{J}}, \{\nu_j\}_{j\in\mathcal{J}} \subset [0, +\infty)$, such that

$$\Phi(|\nabla u_n|) \to \mu \geq \Phi(|\nabla u|) + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j} \quad \text{in } \mathcal{M}(\mathbb{R}^N), \tag{3.8}$$

$$\Phi_*(u_n) \to \nu = \Phi_*(u) + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j} \quad \text{in } \mathcal{M}(\mathbb{R}^N), \tag{3.9}$$

$$\nu_j \leq \max\{S_N^{l^*} \mu_j^{\frac{l^*}{l}}, S_N^{m^*} \mu_j^{\frac{m^*}{l}}, S_N^{l^*} \mu_j^{\frac{l^*}{m}}, S_N^{m^*} \mu_j^{\frac{m^*}{m}}\}, \tag{3.10}$$

where S_N satisfies (2.5).

Now, we claim that $u_n \to u$ in $L_{\Phi_*}(\Omega)$. To prove this, let $\phi \in C_0^\infty\left(\mathbb{R}^N\right)$ such that

$$0 \leq \psi \leq 1, \quad \psi \equiv 1 \text{ in } B_{1/2}(0), \quad \psi = 0 \text{ in } \mathbb{R}^N \backslash B_1(0).$$

For $\varepsilon > 0$ and $j \in \mathcal{J}$ denote

$$\psi_\varepsilon^j(x) = \psi\left(\frac{x - x_j}{\varepsilon}\right) \ \text{ for all } \ x \in \mathbb{R}^N.$$

Because $\{u_n \psi_\varepsilon^j\}$ is bounded for each $j \in \mathcal{J}$, $\langle I'_\lambda(u_n), u_n \psi_\varepsilon^j\rangle = o_n(1)$, that is,

$$\begin{aligned} &M\Big(\int_\Omega \Phi(|\nabla u_n|)dx\Big) \int_\Omega \phi(|\nabla u_n|)(\nabla u_n)^2 \psi_\varepsilon^j \\ &= -M\Big(\int_\Omega \Phi(|\nabla u_n|)dx\Big) \int_\Omega \phi(|\nabla u_n|) u_n \nabla u_n \nabla \psi_\varepsilon^j \\ &\quad + \int_\Omega \phi_*(|u_n|) u_n^2 \psi_\varepsilon^j dx + \lambda \int_\Omega f(x, u_n) u_n \psi_\varepsilon^j dx + o_n(1). \end{aligned} \tag{3.11}$$

By Hölder inequality and the boundedness of $\{u_n\}$ and Lemma 2.3, 2.5, 2.6, we have

$$\begin{aligned} 0 &\leq \lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big|\Big(\int_\Omega \Phi(|\nabla u_n|)dx\Big)\int_\Omega \phi(|\nabla u_n|)u_n\nabla u_n\nabla\psi_\varepsilon^j\Big| \\ &\leq 2\xi_1(\|\nabla u\|_\Phi)\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big(\|u_n\nabla\psi_\varepsilon^j\|_\Phi\|\phi(|\nabla u_n|)\nabla u_n\|_{\widetilde{\Phi}}\Big) \\ &\leq 2c_6 2\xi_1(\|\nabla u\|_\Phi)\xi_1(\|u\|_\Phi)\xi_5(\|\nabla u\|_{\widetilde{\Phi}})\lim_{\varepsilon\to 0}\Big(\|\nabla\psi_\varepsilon^j)\|_{\widetilde{\Phi}}\Big) = 0. \end{aligned} \tag{3.12}$$

By the compactness lemma of Strauss [5], we obtain

$$\lim_{n\to+\infty}\int_\Omega f(x,u_n)u_n\phi_\varepsilon^j dx = \int_\Omega f(x,u)u\phi_\varepsilon^j dx. \tag{3.13}$$

Since ϕ_ε^j has compact support, $n\to\infty$ in (3.11), from (3.12) and (3.13), we obtain

$$\begin{aligned} m_0\int_\Omega \phi_\varepsilon^j d\mu \leq & c_6\Big(\int_{B_\varepsilon(x_j)}\Phi(|\nabla u|)dx\Big)^{1/\sigma}\Big(\int_{B_\varepsilon(x_j)}|\nabla\phi_\varepsilon^j|^N dx\Big)^{1/N} \\ &+\int_\Omega \phi_\varepsilon^j d\nu + \lambda\int_{B_\varepsilon(x_j)} f(x,u)u\phi_\varepsilon^j dx. \end{aligned}$$

Letting $\varepsilon\to 0$, we derive $m_0\mu_j \leq \nu_j$. Therefore,

$$\begin{aligned} \nu_j &\leq \max\{S_N^{l^*}\mu_j^{\frac{l^*}{l}}, S_N^{m^*}\mu_j^{\frac{m^*}{l}}, S_N^{l^*}\mu_j^{\frac{l^*}{m}}, S_N^{m^*}\mu_j^{\frac{m^*}{m}}\} \\ &\leq \frac{\nu_j}{m_0}\max\{S_N^{l^*}(\frac{\nu_j}{m_0})^{\frac{l^*}{l}-1}, S_N^{m^*}(\frac{\nu_j}{m_0})^{\frac{m^*}{l}-1}, S_N^{l^*}(\frac{\nu_j}{m_0})^{\frac{l^*}{m}-1}, S_N^{m^*}(\frac{\nu_j}{m_0})^{\frac{m^*}{m}-1}\}. \end{aligned}$$

Then either $\nu_j = 0$ or $\nu_j \geq S^N m_0^{N/m}$. We assert that $\nu_j = 0$ for each j. If not, assume that $\nu_j \geq S^N m_0^{N/m}$ for some j, then because $\{u_n\}$ is a Palais-Smale sequence, we have

$$\begin{aligned} c_\lambda &= I_\lambda(u_n) - \frac{1}{\theta}\langle I'_\lambda(u_n), u_n\rangle + o_n(1) \\ &= \widehat{M}\Big(\int_\Omega \Phi(|\nabla u_n|)dx\Big) - \frac{1}{\theta}M\Big(\int_\Omega \Phi(|\nabla u_n|)dx\Big)\int_\Omega \phi(|\nabla u_n|)(\nabla u_n)^2 dx \\ &\quad - \int_\Omega \Phi_*(u_n)dx + \frac{1}{\theta}\int_\Omega \phi_*(|u_n|)u_n^2 dx \\ &\quad + \lambda\int_\Omega\left(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right)dx + o_n(1) \\ &\geq m_0\Big(\frac{\sigma}{m} - \frac{1}{\theta}\Big)\int_\Omega \Phi(|\nabla u_n|)dx + \int_\Omega\left(\frac{1}{\theta} - \frac{1}{l^*}\right)\Phi_*(u_n)dx \\ &\quad + \lambda\int_\Omega\left(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right)dx + o_n(1). \end{aligned}$$

From (f_3) and the fact that $\theta \in (m/\sigma, l^*)$ we obtain

$$c_\lambda \geq \int_\Omega \left(\frac{1}{\theta} - \frac{1}{l^*}\right) \Phi_*(u_n)dx + o_n(1).$$

Using the concentration compactness lemma of Lions in Orlicz-Sobolev, we have

$$\begin{aligned} c_\lambda &\geq \lim_{n\to+\infty} \int_\Omega \left(\frac{1}{\theta} - \frac{1}{l^*}\right) \Phi_*(u_n)dx \\ &\geq \left(\frac{1}{\theta} - \frac{1}{l^*}\right) \left(\int_\Omega \Phi_*(u_n)dx + \sum_{j\in\mathcal{J}} \nu_j\right) \\ &\geq \left(\frac{1}{\theta} - \frac{1}{l^*}\right) \nu_j \\ &\geq \left(\frac{1}{\theta} - \frac{1}{l^*}\right) S^N m_0^{N/m}. \end{aligned}$$

Hence for $m_0 > 1$

$$c_\lambda \geq \left(\frac{1}{\theta} - \frac{1}{l^*}\right) S^N m_0^{N/m},$$

which contradicts to (3.7), and so $\nu_j = 0$ for all $j \in \mathcal{J}$. Hence

$$\lim_{n\to+\infty} \int_\Omega \Phi_*(u_n)dx = \int_\Omega \Phi_*(u)dx.$$

This implies $\lim\limits_{n\to+\infty} \int_\Omega \Phi_*(|u_n - u|)dx = 0$, we deduce

$$u_n \to u \text{ in } L_{\Phi_*}(\Omega).$$

Now, using again the Hölder type inequality, it follows that

$$\lim_{n\to+\infty} \int_\Omega \phi(|u_n|)u_n(u_n - u)dx = 0 \text{ and } \lim_{n\to+\infty} \int_\Omega f(x, u_n)(u_n - u)dx = 0. \tag{3.14}$$

On the other hand, we have

$$\begin{aligned} \langle I'(u_n), u_n - u\rangle &= M\Big(\int_\Omega \Phi(|\nabla u|)dx\Big) \int_\Omega \phi(|\nabla u|)\nabla u\nabla(u_n - u) \\ &\quad - \lambda \int_\Omega f(x, u)(u_n - u)dx - \int_\Omega \phi_*(|u|)u(u_n - u)dx \\ &= o_n(1). \end{aligned}$$

Moreover, from the continuity of M and the boundedness of $\{u_n\}$ in X, we may find $c_8 \geq 0$ such that

$$M\Big(\int_\Omega \Phi(|\nabla u_n|)dx\Big) \to M(c_8) > 0.$$

Combining the last limit with (3.14) we give

$$\int_{\Omega} \phi(|\nabla u|)\nabla u \nabla(u_n - u)dx = 0,$$

finally we conclude that

$$u_n \to u \text{ in } X. \tag{3.15}$$

This completes the proof. □

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