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# SOME FIXED POINT THEOREMS FOR GENERALIZED KANNAN TYPE MAPPINGS IN RECTANGULAR $b ext{-METRIC SPACES}$

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**Abstract.** This present paper extends some fixed point theorems in rectangular *b*-metric spaces using subadditive altering distance and establishing the existence and uniqueness of fixed point for Kannan type mappings. Non-trivial examples are further provided to support the hypotheses of our results.

#### 1. Introduction

In 1968, Kannan proved that a contractive mapping with a fixed point need not be necessarily continuous and presented the following fixed point result.

**Theorem 1.1.** ([13]) Let (X,d) be a complete metric space and  $T: X \to X$  be a mapping such that there exists  $0 < k < \frac{1}{2}$  satisfying

$$d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)], \ \, \forall x,y \in X.$$

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Then, T has a unique fixed point  $u \in X$ , and for any  $x \in X$  the sequence of iterates  $\{T^n x\}$  converges to u and

$$d(T^{n+1}x, u) \le k(\frac{k}{1-k})^n d(x, Tx), \quad n = 0, 1, 2, \dots$$

The concept of metric space, as an ambient space in fixed point theory, has been generalized in several directions. In particular, b-metric spaces were introduced by Bakhtin [1] and Czerwik [2], in such a way that triangle inequality is replaced by the b-triangle inequality:

$$d(x,y) \le b \left( d(x,z) + d(z,y) \right)$$

for all pairwise distinct points x, y, z and  $b \ge 1$ . Various fixed point results were established on such spaces, see [3, 4, 5, 10, 11, 14, 15, 17, 18, 19, 20].

In this paper, we provide some fixed point results for generalized Kannan type mapping in rectangular b-metric spaces. Moreover, an illustrative examples is presented to support the obtained results.

#### 2. Preliminaries

Combining conditions used to define b-metric and rectangular metric spaces, George et al. [9] announced the notions of b-rectangular metric space as follow:

**Definition 2.1.** ([9]) Let X be a nonempty set,  $b \ge 1$  be a given real number, and let  $d: X \times X \to [0, +\infty[$  be a mapping such that for all  $x, y \in X$  and all distinct points  $u, v \in X$ , each distinct from x and y:

- (1) d(x,y) = 0, if only if x = y;
- (2) d(x,y) = d(y,x);
- (3)  $d(x,y) \le b [d(x,u) + d(u,v) + d(v,y)]$  (b-rectangular inequality).

Then (X, d) is called a b-rectangular metric space.

**Example 2.2.** Let  $X = \mathbb{R}$ . Define d(x,y) = |x-y| where  $x,y \in \mathbb{R}$ . It is easy to verify that d is a rectangular b-metric and  $(X,\mathbb{R},d)$  is a complete rectangular b-metric space.

We try to extend the result of Kannan using the following class of subadditive altering distance functions.

**Definition 2.3.** ([12]) A function  $\varphi : [0, \infty) \to [0, \infty)$  is said to be a subadditive altering distance function if

- (1)  $\varphi$  is an altering distance function (that is,  $\varphi$  is continuous, strictly increasing and  $\varphi(t) = 0$  if and only if t = 0),
- (2)  $\varphi(x+y) \leq \varphi(x) + \varphi(y), \forall x, y \in [0, \infty).$

**Example 2.4.** The functions  $\varphi_1(x) = \sqrt{x}$ ,  $\varphi_2(x) = 3x$  and  $\varphi_3(x) = \log(1+x)$  are subadditive altering distance functions.

We note that, if  $\varphi$  is subadditive, then for any non negative real number k < 1,  $\varphi(d(x,y)) \le k\varphi(d(a,b))$  implies  $d(x,y) \le k'd(a,b)$  for some k' < 1.

### 3. Main result

Consider  $\varphi$  as a subadditive altering distance function and the *b*-metric *d* is assumed to be continuous in the topology generated by it, we give some new fixed point results.

**Theorem 3.1.** Let (X,d) be a complete rectangular b-metric space with coefficient  $b \ge 1$  and  $T: X \to X$  be a mapping such that there exists  $p < \frac{1}{2b+1}$  satisfying:

$$\varphi(d(Tx,Ty)) \le p[\varphi(d(x,y)) + \varphi(d(x,Tx)) + \varphi(d(y,Ty))], \ \forall x,y \in X. \quad (3.1)$$

Then, T has a unique fixed point  $u \in X$ , the sequence  $\{T^n x\}$  converges to u and for  $q = \frac{2p}{1-p} < 1$  we have

$$d(T^{n+1}x, T^nx) \le q^n d(x, Tx), \ n = 0, 1, 2, 3, \cdots$$

*Proof.* Let z = Tx for an arbitrary element  $x \in X$ . Then

$$\varphi(d(z,Tz)) = \varphi(d(Tx,Tz))$$

$$\leq p[\varphi(d(x,z)) + \varphi(d(x,Tx)) + \varphi(d(z,Tz)).$$

Hence we have

$$\varphi(d(z,Tz)) \le q\varphi(d(x,Tx)),$$

where  $q = \frac{2p}{1-p} < 1$ , it implies that

$$d(z, Tz) \le q'd(x, Tx) \tag{3.2}$$

for q' < 1.

Without loss of generality, we assume q = q'. Let  $x_0 \in X$ , consider the sequence  $\{x_n\} \subset X$  such that  $x_{n+1} = Tx_n$  for all  $n \in \mathbb{N}$ . If there exists  $n \in \mathbb{N}$  such that  $x_n = Tx_n$ . Then  $x_n$  is a fixed point of T and the proof is finished. Hence, we assume that  $x_n \neq Tx_n$  for all  $n \in \mathbb{N}$ . Then for  $m \geq 1$  and  $r \geq 1$ , it

follows that

$$\begin{split} &d(x_{m+r},x_m)\\ &\leq b[d(x_{m+r},x_{m+r-1})+d(x_{m+r-1},x_{m+r-2})+d(x_{m+r-2},x_m)]\\ &\leq bd(x_{m+r},x_{m+r-1})+bd(x_{m+r-1},x_{m+r-2})\\ &+b[b[d(x_{m+r-2},x_{m+r-3})+d(x_{m+r-3},x_{m+r-4})+d(x_{m+r-4},x_m)]]\\ &=bd(x_{m+r},x_{m+r-1})+bd(x_{m+r-1},x_{m+r-2})+b^2d(x_{m+r-2},x_{m+r-3})\\ &+b^2d(x_{m+r-3},x_{m+r-4})+b^2d(x_{m+r-4},x_m)\\ &\leq bd(x_{m+r},x_{m+r-1})+bd(x_{m+r-1},x_{m+r-2})+b^2d(x_{m+r-2},x_{m+r-3})\\ &+b^2d(x_{m+r-3},x_{m+r-4})+\cdots+b^{\frac{r-1}{2}}d(x_{m+3},x_{m+2})\\ &+b^{\frac{r-1}{2}}d(x_{m+2},x_{m+1})+b^{\frac{r-1}{2}}d(x_{m+1},x_m)\\ &\leq d(x_1,x_0)(bq^{m+r-1}+b^2q^{m+r-3}+\cdots+b^{\frac{r-1}{2}}q^{m+2}+bq^{m+r-2}\\ &+b^2q^{m+r-4}+\cdots+b^{\frac{r-1}{2}}q^{m+1}+b^{\frac{r-1}{2}}q^m)\\ &=\sum_{k=1}^{\frac{r-1}{2}}b^kq^{m+r-(2k-1)}d(x_1,x_0)+\sum_{k=1}^{\frac{r-1}{2}}b^kq^{m+r-2k}d(x_1,x_0)+b^{\frac{r-1}{2}}q^md(x_1,x_0)\\ &\to 0 \text{ as } m\to\infty. \end{split}$$

Therefore,  $\{x_n\}$  is a Cauchy sequence in X. By completeness of X, there exists an  $x \in X$  such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = x.$$

Since

$$d(Tx, x) \leq b[d(Tx, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, x)],$$
  

$$\varphi(d(Tx, x)) \leq bp[\varphi(d(x, x_n)) + \varphi(d(x, Tx)) + \varphi(x_n, x_{n+1}) + \varphi(d(x_n, x_{n+1})) + \varphi(d(x_n, x_{n+1})) + \varphi(d(x_{n+1}, x_{n+2}))] + b\varphi(d(Tx_{n+1}, x)).$$

Then

$$(1 - bp)\varphi(d(Tx, x)) \le bp[\varphi(d(x, x_n)) + \varphi(x_n, x_{n+1}) + \varphi(d(x_n, x_{n+1}) + \varphi(d(x_n, x_{n+1})) + \varphi(d(x_{n+1}, x_{n+2}))] + b\varphi(d(Tx_{n+1}, x))$$

$$\to 0 \quad \text{as} \quad n \to \infty.$$

This implies that Tx = x, it means that that x is a fixed point of T.

Now if  $y(\neq x)$  is an another fixed point of T, then

$$\varphi(d(x,y)) \leq p[\varphi(d(x,y)) + \varphi(d(x,Tx)) + \varphi(d(y,Ty))],$$

it implies that

$$\varphi(d(x,y)) \le p\varphi(d(x,y)).$$

Since  $\varphi$  is strictly increasing and  $p < \frac{1}{2b+1}$ , d(x,y) = 0, therefore the fixed point of T is unique. From (3.2) we have

$$d(T^{n+1}x, T^n x) \le q d(T^{n-1}x, T^n x),$$

where  $q = \frac{2p}{1-p} < 1$ , that is,

$$d(T^{n+1}x, T^n x) \le q^n d(x, Tx)$$

for all  $n = 0, 1, 2, \cdots$ . This completes the proof.

**Example 3.2.** Let  $X = \mathbb{R}$  and (X, d) the complete rectangular *b*-metric space as given in Example 2.2.

Define  $T: X \to X$ , by  $Tx = \frac{x}{3}$  for all  $x \in X$  and  $\varphi(t) = 2t$ , we have

$$\varphi(d(Tx,Ty)) < \frac{1}{6}(\varphi(d(x,y)) + \varphi(d(x,Tx)) + \varphi(d(y,Ty))), \ \forall x,y \in X.$$

Then T is a continuous map satisfying (3.1) and 0 is a unique fixed point of T and the sequence  $\{T^nx\} = \{\frac{x}{3^n}\}$  for any point  $x \in X$  converges to 0.

**Corollary 3.3.** Let (X,d) be a complete rectangular b-metric space and let  $T: X \to X$  be a mapping such that

$$d(Tx,Ty) \leq p[d(x,y) + d(x,Tx) + d(y,Ty))], \ \forall x,y \in X,$$

where  $p < \frac{1}{2b+1}$ . Then, T has a fixed point in X.

*Proof.* From Theorem 3.1 if we take  $\varphi(x) = x$ , we obtain the result.

**Theorem 3.4.** Let (X,d) be a complete rectangular b-metric space with coefficient  $b \ge 1$  and  $T: X \to X$  be a mapping such that there exists  $p_1, p_2, p_3$  with  $p_1 + p_2 + p_3 < 1$  and  $bp_2 < 1$  satisfying

$$\varphi(d(Tx,Ty)) \le p_1 \varphi(d(x,y)) + p_2 \varphi(d(x,Tx)) + p_3 \varphi(d(y,Ty)), \ \forall x,y \in X. \ (3.3)$$

Then T has a unique fixed point  $u \in X$ , and for any  $x \in X$  the sequence of iterates  $\{T^n x\}$  converges to u and for  $q = \frac{p_1 + p_2}{1 - p_3}$ ,

$$d(T^{n+1}x, T^n x) \le q^n d(x, Tx), \ n = 0, 1, 2, \dots$$

*Proof.* Similarly to the proof of Theorem 3.1 if we consider a metric space (X,d) and  $\varphi(x)=x$ .

**Example 3.5.** Let X = [0,1] and  $d: X \times X \to [0,\infty[$  defined as  $d(x,y) = |x-y|^2$  is a rectangular b-metric and  $T: X \to X$  defined by  $Tx = \frac{x}{2}$ ; if  $x \in [0,1[$  and  $T1 = \frac{1}{3}$ . If we put  $p_1 = \frac{1}{2}$ ,  $p_2 = \frac{1}{3}$  and  $p_3 = \frac{1}{9}$  and  $\varphi(x) = x$ , we obtain that T satisfies (3.3) then T has a unique fixed point.

We can easily prove the following two theorems.

**Theorem 3.6.** Let (X, d) be a rectangular b-metric space with coefficient  $b \ge 1$ , if every mapping  $T: X \to X$  satisfying

$$\varphi(d(Tx,Ty)) \leq p[\varphi(d(x,y)) + \varphi(d(x,Tx)) + \varphi(d(y,Ty))], \ \, \forall x,y \in X,$$
 for some  $0 \leq p < \frac{1}{2b+1}$ , then  $X$  is complete.

**Theorem 3.7.** Let (X,d) be a complete rectangular b-metric space with coefficient  $b \ge 1$ , and  $T: X \to X$  be a mapping such that there exists  $0 \le p < \frac{1}{2b+1}$  satisfying

$$\varphi(d(Tx,Ty)) \leq p(\varphi(d(x,Tx)) + \varphi(d(y,Ty))), \ \, \forall x,y \in X.$$

Then T has a unique fixed point  $u \in X$  and the sequence  $\{T^n x\}$  converges to u.

By the proof of Theorem 3.1, we get the following result which is the Kannan theorem as a consequence.

**Theorem 3.8.** Let (X,d) be a complete rectangular b-metric space with coefficient  $b \ge 1$ , and  $T: X \to X$  be a mapping such that there exists  $p < \frac{1}{2b}$  satisfying

$$\varphi(d(Tx,Ty)) \le p(\varphi(d(x,Tx)) + \varphi(d(y,Ty))), \quad \forall x,y \in X.$$
 (3.4)

Then T has a unique fixed point  $u \in X$ , and for all  $x \in X$  the sequence  $\{T^n x\}$  converges to u and for  $q = \frac{p}{1-p} < 1$ ,

$$d(T^{n+1}x, u) \le q^n d(x, Tx), \ n = 0, 1, 2, \dots$$

*Proof.* Let  $x_0$  be an arbitrary point of X. Consider the iterative sequence  $\{x_n\}$ , where  $x_n = Tx_{n-1}$  for  $n \in \mathbb{N}$ . Then we have

$$\begin{split} \varphi(d(x_n,x_{n+1})) &= \varphi(d(Tx_{n-1},Tx_n)) \\ &\leq p[\varphi(d(x_{n-1},Tx_{n-1})) + \varphi(d(x_n,Tx_n))] \\ &\leq p[\varphi(d(x_{n-1},x_n)) + \varphi(d(x_n,x_{n+1}))]. \end{split}$$

Hence, we get

$$(1-p)\varphi(d(x_n, x_{n+1})) \le p\varphi(d(x_{n-1}, x_n)),$$

that is,

$$\varphi(d(x_n, x_{n+1})) \le \frac{p}{1-p} \varphi(d(x_{n-1}, x_n)).$$

From (3.2), we get

$$d(x_n, x_{n+1}) \le \frac{p}{1-p} d(x_{n-1}, x_n) = q d(x_{n-1}, x_n)$$
$$\le q^n d(x_0, x_1)$$
$$\to 0 \quad \text{as } n \to \infty.$$

For  $m \geq 1$  and  $r \geq 1$ , it follows that

$$\begin{split} &d(x_{m+r},x_m)\\ &\leq b[d(x_{m+r},x_{m+r-1})+d(x_{m+r-1},x_{m+r-2})+d(x_{m+r-2},x_m)]\\ &\leq bd(x_{m+r},x_{m+r-1})+bd(x_{m+r-1},x_{m+r-2})\\ &+b[b[d(x_{m+r-2},x_{m+r-3})+d(x_{m+r-3},x_{m+r-4})+d(x_{m+r-4},x_m)]]\\ &=bd(x_{m+r},x_{m+r-1})+bd(x_{m+r-1},x_{m+r-2})+b^2d(x_{m+r-2},x_{m+r-3})\\ &+b^2d(x_{m+r-3},x_{m+r-4})+b^2d(x_{m+r-4},x_m)\\ &\leq bd(x_{m+r},x_{m+r-1})+bd(x_{m+r-1},x_{m+r-2})+b^2d(x_{m+r-2},x_{m+r-3})\\ &+b^2d(x_{m+r-3},x_{m+r-4})+\ldots+b^{\frac{r-1}{2}}d(x_{m+3},x_{m+2})\\ &+b^{\frac{r-1}{2}}d(x_{m+2},x_{m+1})+b^{\frac{r-1}{2}}d(x_{m+1},x_m)\\ &\leq d(x_1,x_0)(bq^{m+r-1}+b^2q^{m+r-3}+\ldots+b^{\frac{r-1}{2}}q^{m+2}\\ &+bq^{m+r-2}+b^2q^{m+r-4}+\ldots+b^{\frac{r-1}{2}}q^{m+1}+b^{\frac{r-1}{2}}q^m\\ &=\sum_{k=1}^{\frac{r-1}{2}}b^kq^{m+r-(2k-1)}d(x_1,x_0)+\sum_{k=1}^{\frac{r-1}{2}}b^kq^{m+r-2k}d(x_1,x_0)+b^{\frac{r-1}{2}}q^md(x_1,x_0)\\ &\to 0\quad\text{as }m\to\infty. \end{split}$$

Therefore  $\{x_n\}$  is a Cauchy sequence in X. By completeness of X, there exists an  $x \in X$  such that

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} Tx_{n-1} = x.$$

From

$$d(Tx,x) \le b[d(Tx,Tx_n) + d(Tx_n,x_n) + d(x_n,x)],$$

we have

$$\varphi(d(Tx,x)) \le bp[\varphi(d(Tx,Tx_n)) + \varphi(Tx_n,x_n) + \varphi(d(x_n,x)).$$

$$\le bp[\varphi(d(x,Tx)) + \varphi(d(x_n,Tx_n))]$$

$$+ b\varphi(d(Tx_n,x_n)) + b\varphi(d(x_n,x)).$$

Hence, we have

$$(1 - bp)\varphi(d(Tx, x)) \le b(p + 1)\varphi(d(Tx_n, x_n)) + b\varphi(d(x_n, x))$$
  
 $\to 0 \text{ as } n \to \infty.$ 

This implies that Tx = x, it means that x is a fixed point of T.

Now, if  $y(\neq x)$  is an another fixed point of T, then

$$\varphi(d(x,y)) \le p[\varphi(d(x,Tx)) + \varphi(d(y,Ty))].$$

Hence,

$$\varphi(d(x,y)) \le p(\varphi(d(x,x)) + \varphi(d(y,y))) = 0,$$

then d(x,y) = 0. Therefore, the fixed point of T is unique. From (3.2), we have

$$d(T^{n+1}x, T^n x) \le q d(T^{n-1}x, T^n x),$$

where  $q = \frac{p}{1-p} < 1$ , that is,

$$d(T^{n+1}x, T^n x) \le q^n d(x, Tx)$$

for all  $n = 0, 1, 2, \cdots$ .

**Example 3.9.** Consider the complete rectangular b-metric space (X, d), where  $X = \mathbb{R}$  and d(x, y) = |x - y| for all  $x, y \in X$ . Define the mapping  $T : X \to X$  by

$$T(x) = \begin{cases} 0, & \text{if } x \le 1, \\ -\frac{1}{3}, & \text{if } x > 1. \end{cases}$$

Then T is not continuous at 1. For  $\varphi(x) = 3x$ , we have

$$3d(Tx, Ty) \le 3p(d(x, Tx) + d(y, Ty)).$$

For  $x \leq 1$  and  $y \leq 1$ ,

$$d(Tx, Ty) = 0 \le p[d(x, Tx) + d(y, Ty)]$$
  
=  $p[|x| + |y|]$ 

and

$$\varphi(d(Tx,Ty)) \le p[\varphi(|x|) + \varphi(|y|)].$$

For x > 1 and y > 1,

$$d(Tx, Ty) = 0 \le p[d(x, Tx) + d(y, Ty)]$$
$$= p\left[\left|x + \frac{1}{3}\right| + \left|y + \frac{1}{3}\right|\right],$$

$$0 \le p\left(x + y + \frac{2}{3}\right)$$

and

$$\varphi(d(Tx,Ty)) \le 3p\left(x+y+\frac{2}{3}\right).$$

Thus, T satisfies (3.4). Therefore, T has a unique fixed point x = 0.

**Theorem 3.10.** Let (X, d) be a rectangular b-metric space with coefficient  $b \ge 1$ , if every mapping  $T: X \to X$  satisfying

$$\varphi(d(Tx,Ty)) \leq p(\varphi(d(x,Tx)) + \varphi(d(y,Ty))), \ \, \forall x,y \in X$$

for some  $p < \frac{1}{2b}$ , has a unique fixed point, then X is complete.

In 1975, Subrahmanyam [21] proved that a metric space (X, d) is complete if and only if every Kannan mapping has a unique fixed point in X. Later on, Fisher [7] and Khan [16] proved two important fixed point results related to contractive type mappings on compact metric spaces. They proved that a continuous mapping on a compact metric space (X, d) has a unique fixed point if T satisfies

$$d(Tx,Ty)<\frac{1}{2}(d(x,Ty)+d(y,Tx))$$

or

$$d(Tx, Ty) < (d(x, Tx)d(y, Ty))^{\frac{1}{2}}$$

for all  $x, y \in X$  with  $x \neq y$  respectively.

Since sequentially compact rectangular b-metric spaces are complete, the completeness condition in Theorem 3.8 may be replaced by sequential compactness.

A bounded compact metric space [6] is a metric space X in which every bounded sequence in X has a convergent subsequence. The same notion may be defined in the case of rectangular b-metric spaces. The class of bounded compact rectangular b-metric spaces is larger than that of sequentially compact spaces as the rectangular b-metric space  $\mathbb{R}$  of real numbers with the usual metric is not sequentially compact but bounded compact. In the next result, p is independent of the coefficient b of the rectangular b-metric space.

**Theorem 3.11.** Let (X,d) be a bounded compact rectangular b-metric space and  $T: X \to X$  be a continuous mapping satisfying (3.4) for some  $0 \le p < \frac{1}{2}$ . Then T has a unique fixed point  $u \in X$  and for every  $x_0 \in X$ , the sequence  $\{T^n x_0\}$  converges to u.

*Proof.* Let  $x_0 \in X$  be an arbitrary point. Consider a sequence  $\{x_n\}$ , where  $x_n = T^n x_0$  for all  $n \in \mathbb{N}$ . Then by (3.4) we have

$$\varphi(d(x_n, x_{n+1})) = \varphi(d(T^n x_0, T^{n+1} x_0)) 
= \varphi(d(T(T^{n-1} x_0), T(T^n x_0))) 
\leq p(\varphi(d(T^{n-1} x_0, T^n x_0)) + \varphi(d(T^n x_0, T^{n+1} x_0))) 
= p(\varphi(d(x_{n-1}, x_n)) + \varphi(d(x_n, x_{n+1}))).$$

It implies that

$$(1-p)\varphi(d(x_{n-1},x_n)) < p\varphi(d(x_n,x_{n+1})), \ \forall n \in \mathbb{N}.$$

Since  $1 - p \ge p$ ,

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \forall n \in \mathbb{N}.$$

This means that the sequence  $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}}$  is strictly decreasing and hence convergent, so there exists  $t\geq 0$  such that  $\lim_{n\to\infty} d(x_n, x_{n+1}) = t$ .

For  $m, n \in \mathbb{N}$  with n < m, we have

$$\varphi(d(x_m, x_n)) \le \varphi(d(x_{m-1}, x_m) + \varphi(d(x_{n-1}, x_n))),$$

and hence  $\varphi(d(x_m,x_n)) \leq \varphi(t)$  as  $m,n \to \infty$ . This implies that  $d(x_m,x_n) \leq t$  as  $m,n \to \infty$ , therefore,  $\{x_n\}$  is a bounded sequence. Hence,  $\{x_n\}$  has a subsequence which converges to u, that is,  $\lim_{k\to\infty} x_{n_k} = u$ . By the continuity of T we have  $Tu = T(\lim_{k\to\infty} T^{n_k}x_0) = \lim Tx_{n_k+1}x_0 = u$ , thus, u is a fixed point of T.

Next, we show the uniqueness of the fixed point of T. Let  $z(\neq u)$  be an another fixed point of T. Then

$$\varphi(d(Tz,Tu)) \le p(\varphi(d(z,Tz)) + \varphi(d(u,Tu))),$$

it implies that

$$\varphi(d(z,u)) \le p(\varphi(d(z,z)) + \varphi(d(u,u))),$$

which is a contradiction. Hence, u = z. This completes the proof.

**Example 3.12.** Let (X, d) a bounded compact rectangular *b*-metric space, where  $X = [0, \infty[$  and

$$d(x,y) = \begin{cases} (x+y)^2, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Define  $T: X \to X$  by

$$Tx = \begin{cases} \frac{1}{3}, & \text{if } 0 \le x \le 2, \\ \frac{1}{x}, & \text{if } x > 2. \end{cases}$$

Then, for  $\varphi(t) = 3t$ , we have

$$d(Tx, Ty) < \frac{1}{2}(d(x, Tx) + d(y, Ty)).$$

For  $x \neq y$  and x, y > 2, we have

$$d(Tx, Ty) = \left(\frac{1}{x} + \frac{1}{y}\right)^2 < 1$$

and

$$\frac{1}{2}(d(x,Tx) + d(y,Ty)) = \frac{1}{2}\bigg(\big(x + \frac{1}{x}\big)^2 + \big(y + \frac{1}{y}\big)^2\bigg) > 1.$$

Similary, for  $0 \le x \le 2$  and y > 2, we have

$$d(Tx, Ty) = \left(\frac{1}{3} + \frac{1}{y}\right)^2$$

and

$$\frac{1}{2}(d(x,Tx) + d(y,Ty)) = \frac{1}{2}\left((x+\frac{1}{3})^2 + (y+\frac{1}{y})^2\right) > \left(\frac{1}{3} + \frac{1}{y}\right)^2.$$

Thus, T has a unique fixed point x = 3.

Garai et al. [8] defined T-orbitally compact metric spaces and derived a fixed point result for the same. The definition of T-orbitally compactness can be extended to rectangular b-metric spaces as follows.

**Definition 3.13.** Let (X,d) be a rectangular *b*-metric space and T be a self-mapping on X. The orbit of T at  $x \in X$  is defined as

$$O_x(T) = \{x, Tx, T^2x, T^3x, ...\}.$$

If every sequence in  $O_x(T)$  has a convergent subsequence for all  $x \in X$ , X is said to be T-orbitally compact.

It is easy to see that every compact rectangular b-metric space is T-orbitally compact. Also the bounded compactness and T-orbitally compactness are totally independent. Moreover, T-orbitally compactness of X does not give to be complete.

**Theorem 3.14.** Let (X,d) be a T-orbitally compact rectangular b-metric space and T satisfying (3.4) with  $p < \frac{1}{2}$  and bp < 1. Then T has a unique fixed point u and

$$\lim_{n \to \infty} T^n x = u, \ \forall x \in X.$$

*Proof.* Let  $x_0 \in X$  be arbitrary but fixed, and consider the iterative sequence  $\{x_n\}$ , where  $x_n = T^n x_0$  for each  $n \in \mathbb{N}$ . We denote  $d_n = d(x_n, x_{n+1})$  for  $n \in \mathbb{N}$ . Then, by (3.4) we have

$$\varphi(d_n) \le p(\varphi(d_{n-1}) + \varphi(d_n)),$$

it implies that

$$(1-p)\varphi(d_n) \le p\varphi(d_{n-1}).$$

Since  $1-p \ge p$ ,  $p < \frac{1}{2}$  and  $\varphi$  is strictly increasing, we get  $d_n < d_{n-1}$ , this show that  $\{d_n\}$  is a strictly decreasing sequence of non negative real numbers and hence convergent. Since X is T-orbitally compact, so  $\{x_n\}$  has a convergent subsequence  $\{x_{n_k}\}$  with  $\lim_k x_{n_k} = u$ 

$$\lim_{k} d_{n_k} = \lim_{k} d(x_{n_k}, x_{n_{k+1}}) = d(\lim_{k} x_{n_k}, \lim_{k} x_{n_{k+1}}) = 0.$$

Therefore,  $\lim_{n\to\infty} d_n = 0$ .

We have for  $n, m \in \mathbb{N}$ ,

$$\varphi(d(x_n, x_m)) \le p(\varphi(d(x_{n-1}, x_n)) + \varphi(d(x_{m-1}, x_m)))$$

$$= p(\varphi(d_{n-1}) + \varphi(d_{m-1}))$$

$$\to 0 \quad \text{as } n, m \to \infty,$$

this implies  $d(x_n, x_m) \to 0$  as  $n \to \infty$ . This means that  $\{x_n\}$  is a Cauchy sequence and  $x_n \to u$  as  $n \to \infty$ . Also we have

$$\varphi(d(u,Tu)) \leq \varphi(b(d(u,x_n) + d(x_n,x_{n+1}) + d(x_{n+1},Tu))) 
\leq b\varphi(d(u,x_n)) + bp[\varphi(d(x_{n-1},x_n)) + \varphi(d(x_n,x_{n+1})) 
+ \varphi(d(x_n,x_{n+1}) + \varphi(d(u,Tu)))].$$

This implies that

$$(1 - bp)\varphi(d(u, Tu)) \le b\varphi(d(u, x_n)) + bp[\varphi(d(x_{n-1}, x_n)) + \varphi(d(x_n, x_{n+1})) + \varphi(d(x_n, x_{n+1}))]$$

$$\to 0 \quad \text{as } n \to \infty.$$

Therefore, Tu = u.

Next, let  $u^*$  be an another fixed point of T. Then, we have

$$d(u, u^*) = d(Tu, Tu^*) < \frac{1}{2}(d(u, Tu) + d(Tu^*, Tu^*)) < 0,$$

which is a contradiction. Hence, T has a unique fixed point.

Let us point out that Theorem 3.14 does not hold for  $p \ge \frac{1}{2}$ .

To find a solution we assume that T is an asymptotically regular mapping, that is,  $\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0$ .

**Theorem 3.15.** Let (X,d) be a complete rectangular b-metric space and  $T: X \to X$  be an asymptotically regular mapping satisfying (3.4) for some p with bp < 1. Then T has a unique fixed point.

*Proof.* Let  $x \in X$  and define the sequence  $x_n = T^n x$ ,  $n \in \mathbb{N}$ . Since T is an asymptotically regular mapping, we get for m > n,

$$\varphi(d(T^{n+1}x, T^{m+1}x)) \le p(\varphi(d(T^nx, T^{n+1}x)) + \varphi(d(T^mx, T^{m+1}x)))$$

$$\to 0 \quad \text{as } n \to \infty.$$

it implies that

$$d(T^{n+1}x, T^{m+1}x) \to 0$$
 as  $n \to \infty$ .

Thus  $\{x_n\}$  is a Cauchy sequence and convergent in X with  $\lim_{n\to\infty} x_n = u$ . Hence, we have

$$\begin{split} \varphi(d(u,Tu)) & \leq \varphi(b[d(u,T^nx) + d(T^nx,T^{n+1}x) + d(T^{n+1}x,Tu)]) \\ & \leq b\varphi(d(u,T^nx)) + b\varphi(d(T^nx,T^{n+1}x)) + b\varphi(d(T^{n+1}x,Tu))) \\ & \leq b\varphi(d(u,T^nx)) + bp[\varphi(d(T^{n-1}x,T^nx)) + \varphi(d(T^nx,T^{n+1}x)) \\ & + \varphi(d(T^nx,T^{n+1}x)) + \varphi(d(u,Tu)), \end{split}$$

this implies that

$$(1 - bp)\varphi(d(u, Tu)) \le b\varphi(d(u, T^n x))$$
$$+ bp[\varphi(d(T^{n-1}x, T^n x)) + 2\varphi(d(T^n x, T^{n+1} x)))].$$

When  $n \to \infty$ , we obtain d(u, Tu) = 0. Therefore, u is a fixed point of T. Let  $u^*$  be an another fixed point of T. Then

$$d(u, u^*) = d(Tu, Tu^*) < P(d(u, Tu) + d(Tu^*, Tu^*)) = 0,$$

which is a contradiction. Hence T has a unique fixed point.

**Example 3.16.** Let (X,d) be a complete rectangular b-metric space and  $T: X \to X$  be an asymptotically regular mapping satisfying  $Tx = \frac{x}{3}$  for all  $x \in X$  and  $d(x,y) = |x-y|^2$ , b=2 and  $p < \frac{1}{2}$ . Then for  $\varphi(t) = \sqrt{t}$ , we have |x-y| < 2(|x|+|y|). Therefore, T has a unique fixed point x=0.

**Theorem 3.17.** Let (X,d) be a complete rectangular b-metric space and  $T: X \to X$  be an asymptotically regular mapping satisfying:

$$\varphi(d(Tx, Ty)) \le p[\varphi(d(x, y)) + \varphi(d(x, Tx)) + \varphi(d(y, Ty))], \ \forall x, y \in X$$

for some p with bp < 1. Then T has a unique fixed point.

*Proof.* Let  $x \in X$  and define the sequence  $x_n = T^n x$ ,  $n \in \mathbb{N}$ . Since T is an asymptotically regular mapping, we get for m > n,

$$\begin{split} \varphi(d(T^{n+1}x,T^{m+1}x)) &\leq p(\varphi(d(T^nx,T^mx)) + \varphi(d(T^nx,T^{n+1}x)) \\ &+ \varphi(d(T^mx,T^{m+1}x))) \\ &\to 0 \quad \text{as } n \to \infty. \end{split}$$

Thus,  $\{x_n\}$  is a Cauchy sequence and convergent in X with  $\lim_{n\to\infty} x_n = u$ . Also, we have

$$\begin{split} \varphi(d(u,Tu)) & \leq \varphi(b[d(u,T^{n}x) + d(T^{n}x,T^{n+1}x) + d(T^{n+1}x,Tu)]) \\ & \leq b\varphi(d(u,T^{n}x)) + b\varphi(d(T^{n}x,T^{n+1}x)) + b\varphi(d(T^{n+1}x,Tu))) \\ & \leq b\varphi(d(u,T^{n}x)) + bp[\varphi(d(T^{n-1}x,T^{n}x)) \\ & + \varphi(d(T^{n-1}x,T^{n}x)) + \varphi(d(T^{n}x,T^{n+1}x)) \\ & + \varphi(d(T^{n}x,u)) + \varphi(d(T^{n}x,T^{n+1}x)) + \varphi(d(u,Tu))], \end{split}$$

this implies that

$$(1 - bp)\varphi(d(u, Tu)) \le b(1 + p)\varphi(d(u, T^n x)) + 2bp[\varphi(d(T^{n-1} x, T^n x)) + 2\varphi(d(T^n x, T^{n+1} x)))].$$

When  $n \to \infty$ , we obtain d(u, Tu) = 0. Therefore, u is a fixed point of T. Let  $u^*(\neq u)$  be an another fixed point of T. Then

$$d(u, u^*) = d(Tu, Tu^*) < P(d(u, Tu) + d(Tu^*, Tu^*)) = 0,$$

which is a contradiction. Hence, T has a unique fixed point.

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