**Nonlinear Functional Analysis and Applications**
**Vol. 27, No. 3 (2022), pp. 691-700**
**ISSN: 1229-1595(print), 2466-0973(online)**

https://doi.org/10.22771/nfaa.2022.27.03.15
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# IMPROVEMENT AND GENERALIZATION OF A THEOREM OF T. J. RIVLIN

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**Abstract.** Let $p(z)$ be a polynomial of degree $n$ having no zero inside the unit circle. Then for $0 < r \leq 1$, the well-known inequality due to Rivlin [Amer. Math. Monthly., 67 (1960) 251-253] is

$$\max_{|z|=r} |p(z)| \geq \left(\frac{r+1}{2}\right)^n \max_{|z|=1} |p(z)|.$$

In this paper, we generalize as well as sharpen the above inequality. Also our results not only generalize, but also sharpen some known results proved recently.

## 1. Introduction

Let $p(z)$ be a polynomial of degree $n$. We denote $M(p,r) = \max_{|z|=r} |p(z)|$, $r > 0$ and $\|p\| = \max_{|z|=1} |p(z)|$. Then we have the well-known inequalities

$$M(p',1) \leq n\|p\| \tag{1.1}$$

[0]Received December 21, 2021. Revised February 12, 2022. Accepted February 21, 2022.
[0]2020 Mathematics Subject Classification: 15A18, 30C10, 30C15, 30A10.
[0]Keywords: Inequalities, polynomials, zeros, maximum modulus.
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and

$$M(p,R) \leq R^n \|p\|, \quad R \geq 1. \tag{1.2}$$

Inequalities (1.1) and (1.2) are known as Bernstein's inequalities and have been serving as the beginning of a considerable literature in approximation theory.

Several papers and research monographs have been published on this subject (see, for example Milovanović et al. [11], Rahman [13], Rahman and Schmeisser [14, 15] and Govil and Mohapatra [7].)

For polynomials $p(z)$ of degree $n$ not vanishing in $|z| < 1$, inequalities (1.1) and (1.2) have been replaced by

$$M(p',1) \leq \frac{n}{2}\|p\| \tag{1.3}$$

and

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right)\|p\|, \quad R \geq 1. \tag{1.4}$$

Both the inequalities (1.3) and (1.4) are best possible and equality holds for the polynomials having all their zeros on $|z| = 1$.

If we apply inequality (1.2) to the polynomial $P(z) = z^n p(1/z)$, we get the following inequality.

**Theorem 1.1.** *If $p(z)$ is a polynomial of degree $n$, then for $0 < r \leq 1$,*

$$M(p,r) \geq r^n \|p\|. \tag{1.5}$$

Equality holds in (1.5) for $p(z) = \alpha z^n$, $\alpha$ being a complex number.

The above result is due to Varga [17] who attributes it to Zarantonello. It was shown by Govil et al. [10] that inequalities (1.1), (1.2) and (1.5) are all equivalent.
The analogue of inequality (1.5) for the polynomials not vanishing in the interior of a unit circle was proved in 1960 by Rivlin [16] who in fact proved the following result.

**Theorem 1.2.** *If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < 1$, then for $0 < r \leq 1$,*

$$M(p,r) \geq \left(\frac{r+1}{2}\right)^n \|p\|.$$

*This inequality is sharp and equality holds for $p(z) = \left(\frac{\alpha+\beta z}{2}\right)^n$, where $|\alpha| = |\beta| = 1$.*

Another result which generalized the above result was proved by Govil [4] which is stated as

**Theorem 1.3.** *If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < 1$, then for $0 < r \le R \le 1$,*

$$M(p,r) \ge \left(\frac{1+r}{1+R}\right)^n M(p,R). \tag{1.6}$$

*This result is best possible and equality attains for the polynomial $p(z) = (\frac{1+z}{1+R})^n$.*

In the literature, there have been many extensions of inequality (1.6) (see. for example Govil et al. [10], Govil and Qazi [9] and Qazi [12]). Recently, Govil and Nwaeze [8] proved the following refinement of inequality (1.6).

**Theorem 1.4.** *If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < k$, $k \ge 1$, then for $0 < r \le R \le 1$,*

$$M(p,r) \ge \frac{(1+r)^n}{(1+r)^n+(R+k)^n-(r+k)^n}\left\{M(p,R)+n\min_{|z|=k}|p(z)|\ln\left(\frac{R+k}{r+k}\right)\right\}. \tag{1.7}$$

Govil and Hans [6] sharpened Theorem 1.4 by proving the following result.

**Theorem 1.5.** *If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < k$, $k \ge 1$, then for $0 < r < R \le 1$,*

$$M(p,r) \ge \frac{(1+r)^n}{(1+r)^n+nh(n)}\left\{M(p,R)+n\min_{|z|=k}|p(z)|\ln\left(\frac{R+k}{r+k}\right)\right\}, \tag{1.8}$$

*where*

$$h(n)=\sum_{m=0}^{n-1}(-1)^m(k-1)^m\left\{\frac{(1+R)^{n-m}-(1+r)^{n-m}}{n-m}\right\}+(-1)^n(k-1)^n\ln\left(\frac{R+k}{r+k}\right). \tag{1.9}$$

## 2. Lemmas

For the proofs of the theorems, we will use the following lemmas. The first lemma is due to Govil [5].

**Lemma 2.1.** *If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < k$, $k \ge 1$, then*

$$\max_{|z|=1}|p'(z)| \le \frac{n}{1+k}\left\{\max_{|z|=1}|p(z)| - \min_{|z|=k}|p(z)|\right\}. \tag{2.1}$$

The next lemma is due to Dewan [1, 2].

**Lemma 2.2.** *If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then for $0 < r \leq R \leq 1$,*

$$\max_{|z|=r} |p(z)| \geq \left(\frac{k+r}{k+R}\right)^n \max_{|z|=R} |p(z)|. \tag{2.2}$$

**Lemma 2.3.** *Let $h(n) = \int_r^R \frac{(1+t)^n}{K+t} dt$ for $n \geq 0$. Then*

$$h(n) = \sum_{m=0}^{n-1} (-1)^m (k-1)^m \left\{\frac{(1+R)^{n-m} - (1+r)^{n-m}}{n-m}\right\} + (-1)^n (k-1)^n \ln\left(\frac{R+k}{r+k}\right).$$

Lemma 2.3 is due to Govil and Hans [6].

**Lemma 2.4.** *If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then for $0 < r \leq R \leq 1$,*

$$\max_{|z|=r} |p(z)| \geq \left(\frac{r+k}{R+k}\right)^n \max_{|z|=R} |p(z)| + \left\{1 - \left(\frac{r+k}{R+k}\right)^n\right\} \min_{|z|=k} |p(z)|. \tag{2.3}$$

*Proof.* Since $p(z)$ has no zero in $|z| < k$, $k \geq 1$, for $0 < t \leq 1$, $P(z) = p(tz)$ has no zero in $|z| < k/t$, $k/t \geq 1$.

Using Lemma 2.1 to $P(z)$, we get

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + \frac{k}{t}} \left\{\max_{|z|=1} |P(z)| - \min_{|z|=\frac{k}{t}} |P(z)|\right\}, \tag{2.4}$$

which gives for $|z| = 1$

$$|p'(tz)| \leq \frac{n}{k+t} \left\{M(p,t) - \min_{|z|=k} |p(z)|\right\}. \tag{2.5}$$

For $0 < r < R \leq 1$, $\theta \in [0, 2\pi)$, we have

$$|p(Re^{i\theta}) - p(re^{i\theta})| \leq \int_r^R |p'(te^{i\theta})| dt, \tag{2.6}$$

which implies

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \int_r^R |p'(te^{i\theta})| dt. \tag{2.7}$$

Applying (2.5) to inequality (2.7), we have

$$M(p,R) \leq M(p,r) + \int_r^R \frac{n}{k+t} M(p,t) dt - \int_r^R \frac{n}{k+t} \min_{|z|=k} |p(z)| dt. \tag{2.8}$$

Let

$$\phi(R) = M(p,r) + \int_r^R \frac{n}{k+t} \left\{M(p,t) - \min_{|z|=k} |p(z)|\right\} dt. \tag{2.9}$$

Then

$$\phi'(R) = \frac{n}{k+R}\left\{M(p,R) - \min_{|z|=k}|p(z)|\right\}. \tag{2.10}$$

From (2.8), we have

$$M(p,R) \leq \phi(R). \tag{2.11}$$

Applying (2.11) to (2.10), we get

$$\phi'(R) - \left(\frac{n}{k+R}\right)\phi(R) + \left(\frac{n}{k+R}\right)\min_{|z|=k}|p(z)| \leq 0. \tag{2.12}$$

Multiplying (2.12) by $(k+R)^{-n}$, we have

$$\frac{d}{dR}\left[\left\{\phi(R) - \min_{|z|=k}|p(z)|\right\}(k+R)^{-n}\right] \leq 0. \tag{2.13}$$

From (2.13), we can conclude that the function

$$\Psi(R) = \left\{\phi(R) - \min_{|z|=k}|p(z)|\right\}(k+R)^{-n}$$

is a non-increasing function of $R$ in $(0,1]$. Hence, for $0 < r \leq R \leq 1$, we have

$$\Psi(r) \geq \Psi(R), \tag{2.14}$$

which is equivalent to

$$\phi(r) - \min_{|z|=k}|p(z)| \geq \left(\frac{k+r}{k+R}\right)^n\left\{\phi(R) - \min_{|z|=k}|p(z)|\right\}. \tag{2.15}$$

Since $\phi(r) = M(p,r)$ and $\phi(R) \geq M(p,R)$, we get

$$M(p,r) \geq \left(\frac{k+r}{k+R}\right)^n M(p,R) + \left\{1 - \left(\frac{k+r}{k+R}\right)^n\right\}\min_{|z|=k}|p(z)|. \tag{2.16}$$

This completes the proof of Lemma 2.4. □

**Lemma 2.5.** *For* $0 < r \leq R \leq 1$*,* $k \geq 1$*,* $n \geq 0$*,*

$$\frac{(1+r)^n}{(1+r)^n + nh(n)} \leq \left(\frac{k+r}{k+R}\right)^n, \tag{2.17}$$

*where* $h(n)$ *is as defined in* (1.9)*.*

*Proof.* For $0 < r \leq t \leq 1$ and $k \geq 1$, we have

$$\frac{k+t}{k+r} \leq \frac{1+t}{1+r},$$

from which it follows that

$$\frac{n}{k+t}\left(\frac{k+t}{k+r}\right)^n \leq \frac{n}{k+t}\left(\frac{1+t}{1+r}\right)^n. \tag{2.18}$$

Integrating (2.18) with respect to $t$ from $r$ to $R$ , we have

$$\frac{n}{(k+r)^n}\left\{\frac{(k+R)^n-(k+r)^n}{n}\right\} \leq \frac{n}{(1+r)^n}\int_r^R \frac{(1+t)^n}{k+t}dt. \tag{2.19}$$

Using Lemma 2.3 for the value of integration, inequality (2.19) becomes

$$\frac{n}{(k+r)^n}\left\{\frac{(k+R)^n-(k+r)^n}{n}\right\} \leq \frac{n}{(1+r)^n}h(n), \tag{2.20}$$

where $h(n)$ is as defined in (1.9). Which is equivalent to

$$\frac{(1+r)^n}{(1+r)^n+nh(n)} \leq \left(\frac{k+r}{k+R}\right)^n. \tag{2.21}$$

This completes the proof. □

**Lemma 2.6.** *For* $0<r\leq R\leq 1$*,* $k\geq 1$*,* $n\geq 0$*,*

$$n\ln\left(\frac{R+k}{r+k}\right) \leq \frac{(k+R)^n-(k+r)^n}{(k+r)^n}. \tag{2.22}$$

*Proof.* For $0<r\leq t\leq 1$,

$$r+k\leq t+k,$$

which is equivalent to

$$\frac{n}{t+k} \leq \left(\frac{n}{k+t}\right)\left(\frac{t+k}{r+k}\right)^n. \tag{2.23}$$

Integrating both sides of inequality (2.23) with respect to $t$ from $r$ to $R$, we get

$$n\ln\left(\frac{R+k}{r+k}\right) \leq \frac{(k+R)^n-(k+r)^n}{(k+r)^n}, \tag{2.24}$$

which is inequality (2.22) of Lemma 2.6. □

## 3. Main results

In this paper, under the same hypothesis, we first prove the following result which sharpens Theorem 1.5 due to Govil and Hans [6]. More precisely, we prove

**Theorem 3.1.** *If* $p(z)$ *is a polynomial of degree* $n$ *having no zero in* $|z|<k$*,* $k\geq 1$*, then for* $0<r<R\leq 1$*,*

$$M(p,r) \geq \left(\frac{k+r}{R+k}\right)^n\left\{M(p,R)+n\min_{|z|=k}|p(z)|\ln\left(\frac{R+k}{r+k}\right)\right\}. \tag{3.1}$$

*Proof.* Let $0 < r < R \le 1$ and $\theta \in [0, 2\pi)$. Then we have

$$|p(Re^{i\theta}) - p(re^{i\theta})| = |\int_r^R e^{i\theta} p'(te^{i\theta}) dt|, \tag{3.2}$$

which implies

$$|p(Re^{i\theta})| \le |p(re^{i\theta})| + \left| \int_r^R e^{i\theta} p'(te^{i\theta})) dt \right|. \tag{3.3}$$

Since $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < k$, $k \ge 1$, the polynomial $p(tz)$ has no zero in $|z| < k/t$. Further, if $0 < t \le 1$, then $1/t \ge 1$ and $k/t \ge 1$, therefore on applying Lemma 2.1 to the polynomial $p(tz)$, we have for $|z| = 1$

$$|p'(tz)| \le \frac{n}{k+t} \left\{ M(p,t) - \min_{|z|=k} |p(z)| \right\}. \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$|p(Re^{i\theta})| \le |p(re^{i\theta})| + \int_r^R \frac{n}{k+t} M(p,t) dt - n \min_{|z|=k} |p(z)| \int_r^R \frac{1}{k+t} dt, \tag{3.5}$$

which implies

$$M(p,R) \le M(p,r) + \int_r^R \frac{n}{k+t} M(p,t) dt - n \min_{|z|=k} |p(z)| \int_r^R \frac{1}{k+t} dt. \tag{3.6}$$

Using Lemma 2.2 for the value of $M(p,t)$ in (3.6), we have

$$\begin{aligned} M(p,R) &\le M(p,r) + \int_r^R \left(\frac{n}{k+t}\right) \left(\frac{k+t}{k+r}\right)^n M(p,r) dt \\ &\quad - n \min_{|z|=k} |p(z)| \int_r^R \frac{1}{k+t} dt, \end{aligned}$$

which on simplification, we have

$$\begin{aligned} M(p,R) &\le M(p,r) + \frac{(k+R)^n - (k+r)^n}{(k+r)^n} M(p,r) \\ &\quad + n \ln \left(\frac{k+R}{k+r}\right) \min_{|z|=k} |p(z)| \\ &= \frac{(k+R)^n}{(k+r)^n} M(p,r) + n \ln \left(\frac{k+R}{k+r}\right) \min_{|z|=k} |p(z)|, \end{aligned}$$

and the proof of Theorem 3.1 is completed. □

Further, we prove the following result which is a refinement of Theorem 3.1.

**Theorem 3.2.** *If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then for $0 < r < R \leq 1$,*

$$M(p,r) \geq \left(\frac{k+r}{R+k}\right)^n \left[M(p,R) + \left\{\frac{(k+R)^n - (k+r)^n}{(k+r)^n}\right\} \min_{|z|=k} |p(z)|\right]. \quad (3.7)$$

*Proof.* Proceeding in the similar steps as that of the proof of Theorem 3.1 till inequality (3.6), we have

$$M(p,R) \leq M(p,r) + \int_r^R \frac{n}{k+t} M(p,t)dt - n \min_{|z|=k} |p(z)| \int_r^R \frac{1}{k+t} dt. \quad (3.8)$$

Using Lemma 2.4 for the value of $M(p,t)$ in (3.8), we have

$$\begin{aligned} M(p,R) \leq\ & M(p,r) + \int_r^R \left(\frac{n}{k+t}\right)\left(\frac{t+k}{r+k}\right)^n M(p,r)dt \\ & + \int_r^R \left\{\frac{n}{k+t} - \left(\frac{n}{k+t}\right)\left(\frac{t+k}{r+k}\right)^n\right\} \min_{|z|=k} |p(z)| dt \\ & - \int_r^R \frac{n}{k+t} \min_{|z|=k} |p(z)| dt, \end{aligned}$$

which on simplification, we have

$$\begin{aligned} M(p,R) &\leq \left(\frac{R+k}{r+k}\right)^n M(p,r) - \frac{n}{(r+k)^n} \min_{|z|=k} |p(z)| \int_r^R (k+t)^{n-1} dt \\ &= \left(\frac{R+k}{r+k}\right)^n M(p,r) - \left\{\frac{(k+R)^n - (k+r)^n}{(k+r)^n}\right\} \min_{|z|=k} |p(z)|, \end{aligned}$$

and the proof of Theorem 3.2 is completed. □

**Remark 3.3.** In the proof of Theorem 1.5, Govil and Hans [6] obtained the expression for $h(n)$ by constructing a seperate lemma [6, Lemma 2] which was obtained by routing through a reduction approach. In the same paper [6, Remark 3], they further discussed for $k > 1$, an alternative expression for $h(n)$ in terms of Lerch function and its suitablility for symbolic and numerical manipulations were mentioned.

It is interesting to note that in our theorems all the concerns mentioned in the above remark have been completely avoided.

**Remark 3.4.** By Lemma 2.5, it is evident that Theorem 3.1 sharpens Theorem 1.5. Also for $k = 1$, Theorem 3.1 reduces to the following result due to Govil and Nwaeze [8, Corollary 2.1].

**Corollary 3.5.** *If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < 1$, then for $0 < r \leq R \leq 1$,*

$$M(p,r) \geq \left(\frac{1+r}{1+R}\right)^n \left\{M(p,R) + n \min_{|z|=1} |p(z)| \ln\left(\frac{R+1}{r+1}\right)\right\}.$$

**Remark 3.6.** The bound given by Corollary 3.5 is an improvement of Theorem 1.3 due to Govil [4]. Also for $R = 1$, Corollary 3.5 reduces to the following result of Govil and Nwaeze [8, Corollary 2.3] which further improves Theorem 1.2.

**Corollary 3.7.** *If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < 1$, then for $0 < r < 1$,*

$$M(p,r) \geq \left(\frac{1+r}{2}\right)^n \left\{\|p\| + n \min_{|z|=1} |p(z)| ln\left(\frac{2}{r+1}\right)\right\}.$$

**Remark 3.8.** By Lemma 2.6, it is obvious that Theorem 3.2 is an improvement of Theorem 3.1 and hence improved versions of above corollaries are followed.

**Acknowledgments**: The authors are thankful to the referees for their valuable suggestions.

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