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ASYMPTOTIC POINTWISE CONTRACTIONS IN MODULAR METRIC SPACES

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Abstract. Kirk and Xu introduced the concept of asymptotic pointwise contractions and proved the existence of a fixed point of asymptotic pointwise contractions in Banach space. V. V. Chistyakov proposed the concept of modular metric spaces. In this paper, the existence of fixed points for asymptotic pointwise contractions is studied in modular metric spaces. Moreover, the fixed point theorem for asymptotic pointwise nonexpansive mappings in modular metric spaces is also studied. The results improve and extend the results of Kirk and Xu [W. A. Kirk, Hongkun Xu, Asymptotic pointwise contractions, Nonlinear Anal., 69(2008)4706-4712] to modular metric spaces.

1. Introduction

In [1, 2], Kirk and Xu introduced the following concepts of asymptotic pointwise contractions and gave the theorem of existence of a fixed point of asymptotic pointwise contractions in a Banach space.

Definition 1.1. Let K be a weakly compact convex subset of a Banach space. A mapping $S : K \to K$ is said to be an asymptotic pointwise contraction if

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there exists a function $\alpha : K \to [0, 1)$ such that, for each integer $n \geq 1$,

$$\|S^n x - S^n y\| \leq \alpha_n \|x - y\|, \forall x, y \in K. \tag{1.1}$$

where $\alpha_n \to \alpha$ pointwise on K.

Definition 1.2. Let K be a weakly compact convex subset of a Banach space. A mapping $S : K \to K$ is said to be pointwise asymptotically nonexpansive if, for each integer $n \geq 1$,

$$\|S^n x - S^n y\| \leq \alpha_n \|x - y\|, \forall x, y \in K. \tag{1.2}$$

here $\alpha_n \to 1$ pointwise on K.

Theorem 1.3. *Let K be a weakly compact convex subset of a Banach space and let $S : K \to K$ be an asymptotic pointwise contraction. Then S has a unique fixed point $z \in K$, and for each $x \in K$, the sequence of Picard iterates, $\{T^n x\}$ converges in norm to z.*

In [3, 4, 5, 6], Vyacheslav V. Chistyakov introduced the following concepts of metric modular and modular metric spaces.

Definition 1.4. Let X be a nonempty set. A function $w : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a metric modular on X (or simply a modular if no ambiguity arises) if it satisfies the following three axioms:

(i) given $x, y \in X$, $w_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
(ii) $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;
(iii) $w_{\lambda+\mu}(x, y) \leq w_\lambda(x, z) + w_\mu(y, z)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If, instead of (iii), we have conditions (i)(ii) and the following (iii')

(iii') $w_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda+\mu} w_\lambda(x, z) + \frac{\mu}{\lambda+\mu} w_\mu(y, z)$ $\forall \lambda, \mu > 0, x, y, z \in X$, then w is said to a convex (metric) modular on X.

If, instead of (i), we have only the condition

(i') $w_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$, then w is said to be a (metric) pseudomodular on X.

Motivated by the above ideas, we investigate fixed point theory of asymptotic pointwise contractions in modular metric spaces and obtain the existence and unique of fixed point.

2. Preliminaries

Throughout the paper, we assumed that X is a nonempty set.

Definition 2.1. Let (X, w) be a modular metric space.

(1) The sequence $\{x_n\}$ in X_w is said to be w-convergent to $x \in X$ if
$$w_\lambda(x_n, x) \to 0$$
for all $\lambda > 0$ as $n \to \infty$.

(2) The sequence $\{x_n\}$ in X_w is called $w-$Cauchy sequence if
$$w_\lambda(x_n, x_m) \to 0$$
for all $\lambda > 0$ as $n, m \to \infty$.

(3) A subset C of X_w is called $w-$closed if the $w-$limit of a $w-$convergent sequence of C always belongs to C.

(4) A subset C of X_w is called $w-$complete if any $w-$Cauchy sequence in C is $w-$convergent and its $w-$limit is in C.

(5) A subset C of X_w is called $w-$bounded if
$$\delta_w(C) = \sup_{x,y \in C} w_\lambda(x, y) < +\infty,$$
where $\delta_w(C)$ is called the $w-$diameter of C.

Definition 2.2. The metric modular w has the Fâtou property if and only if
$$w_\lambda(x, y) \leq \liminf w_\lambda(x_n, y),$$
for all $y \in X_w$ and $\lambda > 0$, where $\{x_n\}$ $w-$convergent to x.

Definition 2.3. Let C be a nonempty subset of a modular metric space X_w, the self$-$map $T : C \to C$ is said to be a $w-$pointwise contractions if there exists a function $\alpha : C \to [0, 1)$ such that, for all $x, y \in C$
$$w_\lambda(Tx, Ty) \leq \alpha(x) w_\lambda(x, y).$$

Definition 2.4. Let (X, w) be a modular metric space and C be a nonempty subset of X_w. The self$-$map $T : C \to C$ is said to be a $w-$asymptotic pointwise contractions if there exists a function $\alpha_n : C \to [0, 1)$ such that, for each integer $n \geq 1$,
$$w_\lambda(T^n x, T^n y) \leq \alpha_n(x) w_\lambda(x, y), \forall x, y \in C$$
where $\alpha_n \to \alpha$ pointwise on C.

3. Main Results

Theorem 3.1. *Let (X, w) be a modular metric space. Assume that w is convex and C is a $w-$sequentially compact convex subset of X_w. Let $f : C \to C$ be a $w-$asymptotic pointwise contraction. If a sequence $f^n(x)$ is $w-$bounded, then f has a unique fixed point.*

Proof. For a fixed $x \in C$, define the following function g as

$$g(p) = \limsup_{n\to\infty} w_\lambda(f^n(x), p), p \in C. \tag{3.1}$$

Since C is a w-sequential compact convex subset of X_w, the following asymptotic center of the sequence $f^n(x)$ relative to C

$$A_C(f^n(x)) = \{p \in C : g(p) = \min_{t\in C} g(f(t))\} \tag{3.2}$$

is a closed convex nonempty subset of C.

In the following we will show that f satisfies the property

$$g(f^m(p)) \leq \alpha_m(p)g(p), p \in C.$$

Indeed, for fixed $x \in X$,

$$\begin{aligned} g(f^m(p)) &= \limsup_{n\to\infty} w_\lambda(f^n(x), f^m(p)) \\ &= \limsup_{n\to\infty} w_\lambda(f^{n+m}(x), f^m(p)) \\ &= \limsup_{n\to\infty} w_\lambda(f^m f^n(x), f^m(p)) \\ &\leq \limsup_{n\to\infty} \alpha_m(p) w_\lambda(f^n, p) \\ &= \alpha_m(p)g(p). \end{aligned} \tag{3.3}$$

Taking $p \in A_C(f^n(x))$ and since $f^m(p) \in C$, we have, for $m \geq 1$,

$$g(p) \leq g(f^m(p)) \leq \alpha_m(p)g(p). \tag{3.4}$$

Taking the limit in (3.4) as $m \to \infty$, we can obtain

$$g(p) \leq \alpha(p)g(p). \tag{3.5}$$

Since $\alpha(p) \in [0, 1)$, by (3.5), we have $g(p) = 0$.

For each $p \in C$, since $\alpha_n(p) \to \alpha(p) < 1$ and (3.4), we get

$$g(f^m(p)) = 0, \forall m \geq 1$$

that is $g(f(p)) = 0$.

Thus, we obtain $f^n(x) \to p$ and $f^n(x) \to f(p)$. Hence

$$f(p) = p.$$

Therefore, p is a fixed point of f.

In the following, we will prove that f has a unique fixed point. Indeed, assume that $q \in C$ is another fixed point of f, then, for all $n \geq 1$,

$$w_\lambda(p,q) = w_\lambda(f^n(p), f^n(q)) \leq \alpha_n(p) w_\lambda(p,q). \tag{3.6}$$

Taking the limit of (3.6) as $n \to \infty$, we have

$$w_\lambda(p,q) \leq \alpha(p) w_\lambda(p,q).$$

Since $\alpha(p) < 1$, we obtain $p = q$. Hence, f has a unique fixed point. This completes the proof. □

Definition 3.2. Let C be a subset of a modular metric space X_w. A mapping $T : C \to C$ is said to be $w-$pointwise asymptotically nonexpansive, if there exist a sequence of mappings $\alpha_n : C \to [0,1)$ for each integer $n \geq 1$, such that

$$w_\lambda(T^n x, T^n y) \leq \alpha_n(x) w_\lambda(x,y), \forall x, y \in C,$$

where $\alpha_n \to 1$ pointwise on C.

Remark 3.3. It is not hard to see that if C is bounded, then $w-$pointwise asymptotically nonexpansive mapping T is asymptotically nonexpansive type [7]; that is there exists a sequence $\{k_n\}$ of positive numbers with the property $k_n \to 1$ as $n \to \infty$ and such that

$$w_\lambda(T^n x, T^n y) \leq k_n w_\lambda(x,y)$$

for all n and $x, y \in C$. Therefore, it is immediately that an asymptotically nonexpansive mapping is pointwise asymptotically nonexpansive.

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References

[1] W.A. Kirk, *Asymptotic pointwise contrations*, in: Plenary Lecture, the 8th International Conference on Fixed Point Theory and Its Applications, Chiang Mai University, Thailand, July, (2007), 16-22.

[2] W.A. Kirk and Hong-kun Xu, *Asymptotic pointwise contractions, Nonlinear Anal.*, **69** (2008), 4706–4712.

[3] V.V. Chistyakov, *Metric modulars and their aplication*, Dokl. Math. **73** (2006), 32–35.

[4] V.V. Chistyakov, *Modular metric spaces generated by F-modulars*, Folia Math., **14** (2008), 3–25.

[5] V.V. Chistyakov, *Modular metric spaces,I:Basic concepts*, Nonlinear Anal., **72** (2010), 1–14.

[6] V.V. Chistyakov, *Modular metric spaces, II: Application to superposition operators*, Nonlinear Anal., **72** (2010), 15–30.
[7] W.A. Kirk, *Fixed point theorems for non-Lipschitzian mappings of asymptotically non-expansive type*, Israel J. Math., **17** (1974), 339–346.