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REMARKS ON SOME COUPLED FIXED POINT RESULTS IN PARTIAL METRIC SPACES

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Abstract. In this paper we have developed a new method of reducing coupled fixed point results in partial metric spaces to the respective results for mappings with one variable, even obtaining (in some cases) more general theorems. Our results generalize, improve, enrich and complement recently coupled fixed point results established by H. Alaeidizaji and V. Parvaneh [H. Alaeidizaji and V. Parvaneh, Coupled fixed point results in complete partial metric spaces, Internat. J. Math. Math. Sci., (2012), in press]. Also, by using our method several tripled fixed point results in partial metric spaces can be reduced to the fixed point results with one variable.

1. Introduction

Matthews [18] generalized the concept of a metric space introducing partial metric spaces. Based on the notion of partial metric spaces, Matthews [17], [18], Oltra and Valero [22], Kadelburg et al. [15], Di Bari et al. [12] obtained some fixed point theorems for mappings satisfying different contractive conditions. Recently, Alaeidizaji and Parvaneh [5], proved interesting coupled fixed point results in the context of complete partial metric space. For some coupled results see [8]-[11]. The aim of this paper is to continue the study of coupled fixed point but in the context of 0-complete partial metric spaces. For new results on partial metric spaces see [1]-[7], [12]-[16], [19], [23] and [24].

Consistent with Matthews [17], [18] and O'Neill [20], [21] the following definitions and results will be needed in the sequel.

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Definition 1.1. A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(**p**1)$x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(**p**2) $p(x, x) \leq p(x, y)$,

(**p**3) $p(x, y) = p(y, x)$,

(**p**4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

For a partial metric p on X, the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a (usual) metric on X. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open $p-$balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1.2. ([18], [19]) A sequence $\{x_n\}$ in a partial metric space (X, p) converges to $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$;

(**i**) a sequence $\{x_n\}$ in a partial metric space (X, p) is called Cauchy if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and finite);

(**ii**) a partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$;

(**iii**) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_p(x_0, \delta)) \subset B_p(fx_0, \varepsilon)$.

Lemma 1.3. ([18], [19]) *Let (X, p) be a partial metric space. Then:*

(**1**) *The sequence $\{x_n\}$ is a Cauchy in a partial metric space (X, p) if and only if $\{x_n\}$ is a Cauchy in a metric space (X, p^s);*

(**2**) *A partial metric space (X, p) is complete if and only if a metric space (X, p^s) is complete; Moreover, $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if*

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Remark 1.4. (**1**) ([19]) Clearly, a limit of a sequence in a partial metric space does not need to be unique. Moreover, the function $p(\cdot, \cdot)$ does not need to be continuous in the sense that $x_n \to x$ and $y_n \to y$ implies $p(x_n, y_n) \to p(x, y)$. For example, if $X = [0, +\infty)$ and $p(x, y) = \max\{x, y\}$ for $x, y \in X$, then for $\{x_n\} = \{1\}, p(x_n, x) = x = p(x, x)$ for each $x \geq 1$ and so, e.g., $x_n \to 2$ and $x_n \to 3$ when $n \to \infty$.

(**2**) ([2]) However, if $p(x_n, x) \to p(x, x) = 0$ then $p(x_n, y) \to p(x, y)$ for all $y \in X$.

A sequence $\{x_n\}$ is called 0-Cauchy [12] if $\lim_{m,n} p(x_n, x_m) = 0$. The partial metric space (X, p) is called 0-complete [12], [23] if every 0-Cauchy sequence in X converges to a point $x \in X$ with respect to p and $p(x,x) = 0$. Clearly, every complete partial metric space is 0-complete. The converse need not be true (see [14]).

Definition 1.5. ([8], [11]) Let X be a non-empty set and $F : X^2 \to X$ be a mapping. An element $(x,y) \in X^2$ is called a coupled fixed point of F if $F(x,y) = x, F(y,x) = y$.

Note that if (x,y) is a coupled fixed point of F then (y,x) is coupled fixed point of F too.

The proof of the following Lemma is immediately.

Lemma 1.6. (**1**) *Let (X,p) be a partial metric space. If $P : X^2 \times X^2 \to \mathbb{R}^+$ defined by*

$$P(Y,V) = p(x,u) + p(y,v), Y = (x,y), V = (u,v) \in X^2$$

then $\left(X^2,P\right)$ is a new partial metric space. It is not hard to see that partial metric space $\left(X^2,P\right)$ is complete (resp. 0-complete) if and only if (X,p) is a complete (resp. 0-complete).

(**2**) *Mapping $F : X^2 \to X$ has a coupled fixed point if and only if mapping $T_F : X^2 \to X^2$ defined by $T_F(x,y) = (F(x,y), F(y,x))$ has a fixed point in X^2.*

2. Preliminaries

In [5] Alaedizaji and Parvaneh proved the following results and formulated as Theorem 2.4. and Corollary 2.6.-2.15.

Theorem 2.1. *Let (X,p) be a complete partial metric space, and $F : X^2 \to X$ be a mapping such that*

$$\begin{aligned} & p(F(x,y), F(u,v)) \\ & \leq \alpha_1 p(x,u) + \alpha_2 p(y,v) + \alpha_3 p(F(x,y), x) + \alpha_4 p(F(y,x), y) \\ & \quad + \alpha_5 p(F(x,y), u) + \alpha_6 p(F(y,x), v) + \alpha_7 p(F(u,v), x) \\ & \quad + \alpha_8 p(F(v,u), y) + \alpha_9 p(F(u,v), u) + \alpha_{10} p(F(v,u), v), \end{aligned} \tag{2.1}$$

for every pairs $(x,y), (u,v) \in X^2$, where $\alpha_i \geq 0$ and $\sum_{i=1}^{10} \alpha_i < 1$. Then, F has a unique coupled fixed point in X.

Corollary 2.2. *Let (X,p) be a complete partial metric space and $F : X^2 \to X$ be a mapping such that*

$$\begin{aligned} p(F(x,y), F(u,v)) \leq & \ \alpha_1 p(F(x,y), x) + \alpha_2 p(F(y,x), y) \\ & + \alpha_3 p(F(u,v), u) + \alpha_4 p(F(v,u), v), \end{aligned} \tag{2.2}$$

for every pairs $(x, y), (u, v) \in X^2$, *where* $\alpha_i \geq 0$ *and* $\sum_{i=1}^{4} \alpha_i < 1$. *Then,* F *has a unique coupled fixed point in* X.

Corollary 2.3. *Let* (X, p) *be a complete partial metric space and* $F : X^2 \to X$ *be a mapping such that*

$$\begin{aligned} p(F(x, y), F(u, v)) \leq\ & \alpha_1 p(F(x, y), u) + \alpha_2 p(F(y, x), v) \\ & + \alpha_3 p(F(u, v), x) + \alpha_4 p(F(v, u), y), \end{aligned} \tag{2.3}$$

for every pairs $(x, y), (u, v) \in X^2$, *where* $\alpha_i \geq 0$ *and* $\sum_{i=1}^{4} \alpha_i < 1$. *Then,* F *has a unique coupled fixed point in* X.

Corollary 2.4. *Let* (X, p) *be a complete partial metric space and* $F : X^2 \to X$ *be a mapping such that*

$$\begin{aligned} p(F(x, y), F(u, v)) \leq\ & \alpha_1 p(F(x, y), x) + \alpha_2 p(F(y, x), y) \\ & + \alpha_3 p(F(u, v), x) + \alpha_4 p(F(v, u), y), \end{aligned} \tag{2.4}$$

for every pairs $(x, y), (u, v) \in X^2$, *where* $\alpha_i \geq 0$ *and* $\sum_{i=1}^{4} \alpha_i < 1$. *Then,* F *has a unique coupled fixed point in* X.

Corollary 2.5. *Let* (X, p) *be a complete partial metric space and* $F : X^2 \to X$ *be a mapping such that*

$$\begin{aligned} p(F(x, y), F(u, v)) \leq\ & \alpha_1 p(F(x, y), u) + \alpha_2 p(F(y, x), v) \\ & + \alpha_3 p(F(u, v), u) + \alpha_4 p(F(v, u), v), \end{aligned} \tag{2.5}$$

for every pairs $(x, y), (u, v) \in X^2$, *where* $\alpha_i \geq 0$ *and* $\sum_{i=1}^{4} \alpha_i < 1$. *Then,* F *has a unique coupled fixed point in* X.

Corollary 2.6. *Let* (X, p) *be a complete partial metric space and* $F : X^2 \to X$ *be a mapping such that*

$$\begin{aligned} p(F(x, y), F(u, v)) \leq\ & \frac{k}{2}[p(x, u) + p(y, v)] \\ & + \frac{l}{2}[p(F(x, y), x) + p(F(y, x), y)] \\ & + \frac{r}{2}[p(F(x, y), u) + p(F(y, x), v)] \\ & + \frac{s}{2}[p(F(u, v), x) + p(F(v, u), y)] \\ & + \frac{t}{2}[p(F(u, v), u) + p(F(v, u), v)], \end{aligned} \tag{2.6}$$

for every pairs $(x, y), (u, v) \in X^2$, *where* $k, l, r, s, t \geq 0$ *and* $k+l+r+s+t < 1$. *Then,* F *has a unique coupled fixed point in* X.

Corollary 2.7. *Let* (X, p) *be a complete partial metric space and* $F : X^2 \to X$ *be a mapping such that*

$$\begin{aligned} p(F(x,y), F(u,v)) \leq {} & \frac{k}{2}\left[p(F(x,y), x) + p(F(y,x), y)\right] \\ & + \frac{l}{2}\left[p(F(u,v), u) + p(F(v,u), v)\right], \end{aligned} \tag{2.7}$$

for every pairs $(x, y), (u, v) \in X^2$, *where* $k, l \geq 0$ *and* $k + l < 1$. *Then,* F *has a unique coupled fixed point in* X.

Corollary 2.8. *Let* (X, p) *be a complete partial metric space and* $F : X^2 \to X$ *be a mapping such that*

$$\begin{aligned} p(F(x,y), F(u,v)) \leq {} & \frac{k}{2}\left[p(F(x,y), u) + p(F(y,x), v)\right] \\ & + \frac{l}{2}\left[p(F(u,v), x) + p(F(v,u), y)\right] \end{aligned} \tag{2.8}$$

for every pairs $(x, y), (u, v) \in X^2$, *where* $k, l \geq 0$ *and* $k + l < 1$. *Then,* F *has a unique coupled fixed point in* X.

Corollary 2.9. *Let* (X, p) *be a complete partial metric space and* $F : X^2 \to X$ *be a mapping such that*

$$\begin{aligned} p(F(x,y), F(u,v)) \leq {} & \frac{k}{2}\left[p(F(x,y), x) + p(F(y,x), y)\right] \\ & + \frac{l}{2}\left[p(F(u,v), x) + p(F(v,u), y)\right] \end{aligned} \tag{2.9}$$

for every pairs $(x, y), (u, v) \in X^2$, *where* $k, l \geq 0$ *and* $k + l < 1$. *Then,* F *has a unique coupled fixed point in* X.

Corollary 2.10. *Let* (X, p) *be a complete partial metric space and* $F : X^2 \to X$ *be a mapping such that*

$$\begin{aligned} & p(F(x,y), F(u,v)) \\ & \leq \frac{k}{2}\left[p(x,u) + p(y,v)\right] + \frac{l}{4}\left[p(F(x,y), x) + p(F(y,x), y)\right. \\ & \quad \left. + p(F(u,v), u) + p(F(v,u), v)\right] + \frac{r}{4}\left[p(F(x,y), u)\right. \\ & \quad \left. + p(F(y,x), v) + p(F(u,v), x) + p(F(v,u), y)\right] \end{aligned} \tag{2.10}$$

for every pairs $(x, y), (u, v) \in X^2$, *where* $k, l, r \geq 0$ *and* $k + l + r < 1$. *Then,* F *has a unique coupled fixed point in* X.

Corollary 2.11. *Let (X,p) be a complete partial metric space and $F: X^2 \to X$ be a mapping such that*

$$p\left(F\left(x,y\right),F\left(u,v\right)\right) \leq \frac{k}{2}\left[p\left(F\left(x,y\right),u\right)+p\left(F\left(y,x\right),v\right)\right] + \frac{l}{2}\left[p\left(F\left(u,v\right),u\right)+p\left(F\left(v,u\right),v\right)\right] \tag{2.11}$$

for every pairs $(x,y),(u,v) \in X^2$, where $k,l \geq 0$ and $k+l<1$. Then, F has a unique coupled fixed point in X.

3. Main Results

Our first result is the following Lemma which is crucial for the proof of Theorem 3.2. below. In fact, this is very known Hardy-Rogers theorem in the context of 0-complete partial metric spaces (for the proof see [15]). After that we will formulate the theorem which is inspired by Theorem 2.4. from [5] and is more general than it.

Lemma 3.1. *Let (X,d) be a 0-complete partial metric space. Suppose mappings $f: X \to X$ and that there exist nonnegative constants α_i satisfying $\sum_{i=1}^{5} \alpha_i < 1$ such that, for each $x,y \in X$*

$$p\left(fx,fy\right) \preceq \alpha_1 p\left(x,y\right)+\alpha_2 p\left(x,fx\right)+\alpha_3 p\left(y,fy\right) + \alpha_4 p\left(x,fy\right)+\alpha_5 p\left(y,fx\right). \tag{3.1}$$

Then f has a unique fixed point z in X such that $p(z,z)=0$.

The following result generalizes and extends Theorem 2.4. from [5].

Theorem 3.2. *Let (X,p) be a 0-complete partial metric space and $F: X^2 \to X$ be a mapping. Suppose that for any $x,y,u,v \in X$, the following condition*

$$\begin{aligned} &p\left(F\left(x,y\right),F\left(u,v\right)\right)+p\left(F\left(y,x\right),F\left(v,u\right)\right) \\ &\preceq b_1\left(p\left(F\left(x,y\right),x\right)+p\left(F\left(y,x\right),y\right)\right) \\ &\quad + b_2\left(p\left(F\left(u,v\right),u\right)+d\left(F\left(v,u\right),v\right)\right) \\ &\quad + b_3\left(p\left(F\left(u,v\right),x\right)+d\left(F\left(v,u\right),y\right)\right) \\ &\quad + b_4\left(p\left(F\left(x,y\right),u\right)+p\left(F\left(y,x\right),v\right)\right) \\ &\quad + b_5\left(p\left(x,u\right)+p\left(y,v\right)\right) \end{aligned} \tag{3.2}$$

holds, where $b_i, i=1,...,5$ are nonnegative real numbers such that $\sum_{i=1}^{5} b_i < 1$. Then F has a unique coupled fixed point $(x,x) \in X^2$ and $p(x,x)=0$.

Proof. Putting $b_5 = \alpha_1, b_2 = \alpha_2, b_1 = \alpha_3, b_4 = \alpha_4$ and $b_3 = \alpha_5$, by Lemma 1.6. (2) the condition (3.2) for all $Y = (x, y), V = (u, v) \in X^2$ become

$$\begin{aligned} P(T_F(Y), T_F(V)) \preceq\ & \alpha_1 P(Y, V) + \alpha_2 P(T_F(V), V) + \alpha_3 P(T_F(Y), Y) \\ & + \alpha_4 P(T_F(Y), V) + \alpha_5 P(T_F(V), Y), \end{aligned}$$

which is in fact the condition (3.1). Hence, all conditions of Lemma 3.1 are satisfied. It means that the mappings T_F has a unique fixed point $Y = (x, y)$ in X^2 such that $P(Y, Y) = 0$, that is, by Lemma 1.6. (2) F has a unique coupled fixed point $(x, y) \in X^2$. Since, (y, x) is coupled fixed point of F too, then $x = y$. Hence, (x, x) is a unique coupled fixed point of F and $p(x, x) = 0$. This completes the proof. □

Remark 3.3. Theorem 3.2. is more general than Theorem 2.4. from [5] since the contractive condition (3.1) implies (3.2) with $a_1 + a_2 = b_5, a_3 + a_4 = b_1, a_5 + a_6 = b_4, a_7 + a_8 = b_3$ and $a_9 + a_{10} = b_2$. The following example shows that generalization is proper.

Example 3.4. Let $X = [0, 1]$ be equipped with the partial metric p defined by $p(x, y) = \max\{x, y\}$ for $x, y \in X$. Let $F : X^2 \to X$ be given by $F(x, y) = \frac{1}{2}x$ for all $x, y \in X$. Finally, take $b_1 = b_2 = b_3 = b_4 = 0$ and $b_5 \in [0, 1)$. The contractive condition (2.1) is not satisfied. Indeed, taking $v = y = 0, u = 1$ and $x \in (0, 1)$, we have that (2,1) reduces to

$$p\left(\frac{x}{2}, \frac{1}{2}\right) \le \frac{b_5}{2}(p(x, 1) + p(0, 0)),$$

i.e., $1 \le b_5$, which is impossible for $b_5 \in [0, 1)$.

On the other hand, condition (3.2) is satisfied. To verify this consider the following possible cases (we denote $L =: p(F(x, y), F(u, v)) + p(F(y, x), F(v, u))$ and $R = b_5(p(x, u) + p(y, v))$).

(**1**) $0 \le x \le u$ and $0 \le y \le v$. Then

$$L = p\left(\frac{x}{2}, \frac{u}{2}\right) + p\left(\frac{y}{2}, \frac{v}{2}\right) = \frac{1}{2}(u + v) \le b_5(p(x, u) + p(y, v)) = R,$$

whenever $b_5 \in [\frac{1}{2}, 1)$.

(**2**) $0 \le x \le u$ and $0 \le v \le y$. Then

$$L = p\left(\frac{x}{2}, \frac{u}{2}\right) + p\left(\frac{y}{2}, \frac{v}{2}\right) = \frac{1}{2}(u + y) \le b_5(p(x, u) + p(y, v)) = R,$$

whenever $b_5 \in [\frac{1}{2}, 1)$.

(**3**) $0 \le u \le x$ and $0 \le y \le v$. Then

$$L = p\left(\frac{x}{2}, \frac{u}{2}\right) + p\left(\frac{y}{2}, \frac{v}{2}\right) = \frac{1}{2}(u + v) \le b_5(p(x, u) + p(y, v)) = R,$$

whenever $b_5 \in [\frac{1}{2}, 1)$.

(**4**) $0 \leq u \leq x$ and $0 \leq v \leq y$. Then

$$L = p\left(\frac{x}{2}, \frac{u}{2}\right) + p\left(\frac{y}{2}, \frac{v}{2}\right) = \frac{1}{2}(x+y) \leq b_5\left(p(x,u) + p(y,v)\right) = R,$$

whenever $b_5 \in [\frac{1}{2}, 1)$.

Hence, in all cases (3.2) is satisfied. □

Corollary 3.5. *Let (X,p) be a 0-complete partial metric space and $F : X^2 \to X$ be a mapping. Suppose that for any $x,y,u,v \in X$, the following condition*

$$\begin{aligned} &p\left(F(x,y), F(u,v)\right) + p\left(F(y,x), F(v,u)\right) \\ &\preceq b_1\left(p\left(F(x,y), x\right) + p\left(F(y,x), y\right)\right) \\ &\quad + b_2\left(p\left(F(u,v), u\right) + d\left(F(v,u), v\right)\right) \end{aligned} \tag{3.3}$$

holds, where b_1, b_2 are nonnegative real numbers such that $b_1 + b_2 < 1$. Then F has a unique coupled fixed point $(x,x) \in X^2$ and $p(x,x) = 0$.

Proof. By Lemma 1.6. (2) the condition (3.3) for all $Y = (x,y), V = (u,v) \in X^2$ become

$$P\left(T_F(Y), T_F(V)\right) \leq b_1 P\left(T_F(Y), Y\right) + b_2 P\left(T_F(V), V\right),$$

and the proof further follows from Theorem 3.2. □

Corollary 3.6. *Let (X,p) be a 0-complete partial metric space and $F : X^2 \to X$ be a mapping. Suppose that for any $x,y,u,v \in X$, the following condition*

$$\begin{aligned} &p\left(F(x,y), F(u,v)\right) + p\left(F(y,x), F(v,u)\right) \\ &\preceq b_1\left(p\left(F(x,y), u\right) + p\left(F(y,x), v\right)\right) \\ &\quad + b_2\left(p\left(F(u,v), x\right) + d\left(F(v,u), y\right)\right) \end{aligned} \tag{3.4}$$

holds, where b_1, b_2 are nonnegative real numbers such that $b_1 + b_2 < 1$. Then F has a unique coupled fixed point $(x,x) \in X^2$ and $p(x,x) = 0$.

Proof. In this case the condition (3.4) become

$$P\left(T_F(Y), T_F(V)\right) \leq b_1 P\left(T_F(Y), V\right) + b_2 P\left(T_F(V) Y\right),$$

and the proof follows. □

Corollary 3.7. *Let (X,p) be a 0-complete partial metric space and $F : X^2 \to X$ be a mapping. Suppose that for any $x,y,u,v \in X$, the following condition*

$$\begin{aligned} &p\left(F(x,y), F(u,v)\right) + p\left(F(y,x), F(v,u)\right) \\ &\preceq b_1\left(p\left(F(x,y), x\right) + p\left(F(y,x), y\right)\right) \\ &\quad + b_2\left(p\left(F(u,v), x\right) + p\left(F(v,u), y\right)\right) \end{aligned} \tag{3.5}$$

holds, where b_1, b_2 *are nonnegative real numbers such that* $b_1 + b_2 < 1$. *Then* F *has a unique coupled fixed point* $(x, x) \in X^2$ and $p(x, x) = 0$.

Proof. The condition (3.5) now has the form

$$P(T_F(Y), T_F(V)) \leq b_1 P(T_F(Y), Y) + b_2 P(T_F(V), Y),$$

and the proof follows from Theorem 3.2. □

Corollary 3.8. *Let* (X, p) *be a 0-complete partial metric space and* $F : X^2 \to X$ *be a mapping. Suppose that for any* $x, y, u, v \in X$, *the following condition*

$$\begin{aligned} &p(F(x,y), F(u,v)) + p(F(y,x), F(v,u)) \\ &\preceq b_1 (p(F(x,y), u) + p(F(y,x), v)) \\ &\quad + b_2 (p(F(u,v), u) + d(F(v,u), v)) \end{aligned} \tag{3.6}$$

holds, where b_1, b_2 *are nonnegative real numbers such that* $b_1 + b_2 < 1$. *Then* F *has a unique coupled fixed point* $(x, x) \in X^2$ and $p(x, x) = 0$.

Proof. By Lemma 1.6. (2) the condition (3.6) for all $Y = (x, y), V = (u, v) \in X^2$ become

$$P(T_F(Y), T_F(V)) \leq b_1 P(T_F(Y), V) + b_2 P(T_F(V), V),$$

and the proof follows. □

Corollary 3.9. *Let* (X, p) *be a 0-complete partial metric space and* $F : X^2 \to X$ *be a mapping. Suppose that for any* $x, y, u, v \in X$, *the following condition*

$$\begin{aligned} &p(F(x,y), F(u,v)) + p(F(y,x), F(v,u)) \\ &\preceq b_1 [p(x,u) + p(y,v)] + b_2 [p(F(x,y), x) + p(F(y,x), y)] \\ &\quad + b_3 [p(F(x,y), u) + p(F(y,x), v)] \\ &\quad + b_4 [p(F(u,v), x) + p(F(v,u), y)] \\ &\quad + b_5 [p(F(u,v), u) + p(F(v,u), v)], \end{aligned} \tag{3.7}$$

for every pairs $(x, y), (u, v) \in X^2$, where $b_1, b_2, b_3, b_4, b_5 \geq 0$ and $b_1 + b_2 + b_3 + b_4 + b_5 < 1$. Then, F has a unique coupled fixed point $(x, x) \in X^2$ and $p(x, x) = 0$.

Proof. According to Lemma 1.6. (2) the condition (3.7) for all $Y = (x, y), V = (u, v) \in X^2$ become

$$\begin{aligned} &P(T_F(Y), T_F(V)) \\ &\leq b_1 P(Y, V) + b_2 P(T_F(Y), Y) + b_3 P(T_F(Y), V) \\ &\quad + b_4 P(T_F(V), Y) + b_5 P(T_F(V), V), \end{aligned}$$

that is., the condition (3.1). Hence, the proof again follows. □

Corollary 3.10. *Let (X,p) be a complete partial metric space and $F: X^2 \to X$ be a mapping such that*

$$\begin{aligned} &p(F(x,y),F(u,v))+p(F(y,x),F(v,u)) \\ &\le b_1[p(F(x,y),x)+p(F(y,x),y)] \\ &\quad + b_2[p(F(u,v),u)+p(F(v,u),v)], \end{aligned} \tag{3.8}$$

for every pairs $(x,y),(u,v) \in X^2$, where $b_1,b_2 \ge 0$ and $b_1+b_2<1$. Then, F has a unique coupled fixed point $(x,x) \in X^2$ and $p(x,x)=0$.

Proof. In this case we obtain the following condition

$$P(T_F(Y),T_F(V)) \le b_1 P(T_F(Y),Y)+b_2 P(T_F(V),V),$$

from which the proof follows. □

Corollary 3.11. *Let (X,p) be a complete partial metric space and $F: X^2 \to X$ be a mapping such that*

$$\begin{aligned} &p(F(x,y),F(u,v))+p(F(y,x),F(v,u)) \\ &\le b_1[p(F(x,y),u)+p(F(y,x),v)] \\ &\quad + b_2[p(F(u,v),x)+p(F(v,u),y)] \end{aligned} \tag{3.9}$$

for every pairs $(x,y),(u,v) \in X^2$, where $b_1,b_2 \ge 0$ and $b_1+b_2<1$. Then, F has a unique coupled fixed point $(x,x) \in X^2$ and $p(x,x)=0$.

Proof. In this case the new condition is

$$P(T_F(Y),T_F(V)) \le b_1 P(T_F(Y),V)+b_2 P(T_F(V),Y),$$

from which the proof follows. □

Corollary 3.12. *Let (X,p) be a complete partial metric space and $F: X^2 \to X$ be a mapping such that*

$$\begin{aligned} &p(F(x,y),F(u,v))+p(F(y,x),F(v,u)) \\ &\le b_1[p(F(x,y),x)+p(F(y,x),y)] \\ &\quad + b_2[p(F(u,v),x)+p(F(v,u),y)] \end{aligned} \tag{3.10}$$

for every pairs $(x,y),(u,v) \in X^2$, where $b_1,b_2 \ge 0$ and $b_1+b_2<1$. Then, F has a unique coupled fixed point $(x,x) \in X^2$ and $p(x,x)=0$.

Proof. Now we have

$$P(T_F(Y),T_F(V)) \le b_1 P(T_F(Y),Y)+b_2 P(T_F(V),Y),$$

that is., the proof follows by Theorem 3.2. □

Corollary 3.13. *Let (X, p) be a complete partial metric space and $F : X^2 \to X$ be a mapping such that*

$$\begin{aligned} & p(F(x,y), F(u,v)) + p(F(y,x), F(v,u)) \\ & \le b_1 [p(x,u) + p(y,v)] + b_2 [p(F(x,y), x) + p(F(y,x), y) \\ & \quad + p(F(u,v), u) + p(F(v,u), v)] + b_3 [p(F(x,y), u) \\ & \quad + p(F(y,x), v) + p(F(u,v), x) + p(F(v,u), y)] \end{aligned} \tag{3.11}$$

for every pairs $(x, y), (u, v) \in X^2$, where $b_1, b_2, b_3 \ge 0$ and $b_1 + b_2 + b_3 < 1$. Then, F has a unique coupled fixed point $(x, x) \in X^2$ and $p(x, x) = 0$.

Proof. According to Lemma 1.6. (2) the condition (3.11) for all $Y = (x, y), V = (u, v) \in X^2$ become

$$\begin{aligned} & P(T_F(Y), T_F(V)) \\ & \le b_1 P(Y, V) + b_2 [P(T_F(Y), Y) + P(T_F(V), V)] \\ & \quad + b_3 [P(T_F(Y), V) + P(T_F(V), Y)]. \end{aligned}$$

Hence the proof follows by Theorem 3.2. □

Corollary 3.14. *Let (X, p) be a complete partial metric space and $F : X^2 \to X$ be a mapping such that*

$$\begin{aligned} & p(F(x,y), F(u,v)) + p(F(y,x), F(v,u)) \\ & \le b_1 [p(F(x,y), u) + p(F(y,x), v)] \\ & \quad + b_2 [p(F(u,v), u) + p(F(v,u), v)] \end{aligned} \tag{3.12}$$

for every pairs $(x, y), (u, v) \in X^2$, where $b_1, b_2 \ge 0$ and $b_1 + b_2 < 1$. Then, F has a unique coupled fixed point $(x, x) \in X^2$ and $p(x, x) = 0$.

Proof. The condition (3.12) obtain the following form

$$P(T_F(Y), T_F(V)) \le b_1 P(T_F(Y), V) + b_2 P(T_F(V), V).$$

The proof follows by Theorem 3.2. □

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