Nonlinear Functional Analysis and Applications
Vol. 18, No. 1 (2013), pp. 51-66

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

WEAK SOLUTIONS OF NONLINEAR p-q LAPLACIAN EQUATION

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Abstract. This paper deals the existence of weak solutions of degenerate parabolic p-q Laplacian equation with Dirichlet boundary condition using Galerkin's approximation method and semi-discretization process.

1. Introduction

In this paper, we consider the following degenerate parabolic p-q Laplacian equation with Dirichlet boundary condition as follows:

$$\begin{cases} u_t - \Delta_p u + a(x)\,|u|^{p-2}\,u - \Delta_q u + b(x)\,|u|^{q-2}\,u \\ \quad = f(u) + g(x,t) \text{ in } Q_T, \\ u = 0 \text{ on } \Gamma_T, \\ u(x,0) = u_0(x) \text{ on } \Omega, \end{cases} \tag{1.1}$$

[0]Received October 9, 2012. Revised January 28, 2013.
[0]2000 Mathematics Subject Classification: 35D30, 35K65.
[0]Keywords: p-q Laplacian, weak solution, Galerkin's method.

where $\Delta_r u = \nabla \cdot (|\nabla u|^{r-2} \nabla u)$ for some integer r, $Q_T = \Omega \times (0,T)$, $\Gamma_T = \partial\Omega \times (0,T)$, $p, q > 2$, $q < p$, Ω is open bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega \in C^1$ and $u_0(x)$ denotes the initial data. Assume that $a(x), b(x) \geq 0$. For some elliptic type equations related to the above equations one can see [12, 13].

The p-q Laplacian equation has been receiving increasing attention during the last decades, for example, one can see [12]-[15] and the references therein. In particular Afrouzi et al. [1] established the existence and multiplicity of solutions of some p-q Laplacian system using Ekeland's variational principle, the mountain pass theorem and the saddle point theorem. Furthermore, Afrouzi et al. [2] studied the existence results for a class of p-q Laplacian system in which the proof depends on the local minimization method. Afrouzi and Rasouli [3] proved the existence of nontrivial nonnegative solutions to a multi-parameter nonlinear elliptic system. Cherfils and Il'yasov [6] analyzed a family of stationary nonlinear equations of p-q Laplacian with Dirichlet boundary conditions which have a wide spectrum of applications in many areas of science. Figueiredo proved the existence of positive solutions to the class of elliptic problems with critical growth on $\mathbb{R}^n$ in [9]. He and Li [10] established the existence of a nontrivial solution to the elliptic problem without the assumption of the Ambrosetti-Rabinowitz condition. The existence of at least three weak solutions is established for a class of quasilinear elliptic systems involving the p-q Laplacian with Dirichlet boundary condition by Li and Tang [11] and also Li et al. established the three solutions for p-q biharmonic systems in [14]. Li and Zhang [13] studied the existence of multiple solutions for the nonlinear elliptic problem of p-q Laplacian type involving the critical Sobolev exponent. Rasouli et al. [15] obtained the existence of positive weak solution for a class of p-q Laplacian system with sign-changing weight by the method of sub-super solutions. It is convenient to mention that the present method and the problem is an extension our previous result in [5]. In contrast to the above mentioned results, in this work, we consider the degenerate parabolic equations with p-q Laplacian and establish the existence of weak solutions using semi-discretization process.

We now recall some function spaces to be used in this work. We suppose that if X is a Banach space, then $L^p(a,b;X)$ denotes the space of all measurable functions u from (a,b) to X such that $\|u(\cdot)\|_X$ belongs to $L^p(a,b)$. Throughout this work, we use the generic constant C instead of different constants. Finally, the results of the paper are organized as follows. In section 2, we establish the existence of weak solutions of steady-state problem of (1.1) using Galerkin's approximation method. In section 3, we prove the existence of weak solutions of the given problem (1.1) using semi-discretization process.

2. Steady-State Case

In this section, we consider the steady-state case of the original problem (1.1) and establish the existence of weak solutions using Galerkin's approximation method.

The steady-state version of the given parabolic problem (1.1) is as follows:

$$\begin{cases} -\Delta_p u + a(x)\,|u|^{p-2}\,u - \Delta_q u + b(x)\,|u|^{q-2}\,u = f(u) + g(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.1}$$

Definition 2.1. A function $u \in W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega)$, is said to be a weak solution of (2.1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi dx + \int_\Omega a(x)|u|^{p-2}u\phi dx + \int_\Omega |\nabla u|^{q-2}\nabla u\nabla\phi dx$$
$$+ \int_\Omega b(x)|u|^{q-2}u\phi dx = \int_\Omega f(u)\phi dx + \int_\Omega g(x)\phi dx,$$

holds for each $\phi \in W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega)$ with $f(u) \in L^{p'}(\Omega)$ and $g(x) \in L^{q'}(\Omega)$ where p', q' respectively denotes the Hölder conjugate of p and q.

Lemma 2.2. ([4, 8]) *Let $F : \mathbb{R}^K \to \mathbb{R}^K$ ($K \in \mathbb{N}$) be a continuous function such that $\langle F(r), r\rangle \geq 0$ on $|r| = \rho$. Then there exists $z \in \bar{B}_\rho(0)$ such that $F(z) = 0$ for sufficiently large ρ.*

Theorem 2.3. *Under the assumption for some $a_0, b_0 > 0$ such that $a(x) \geq a_0$, and $b(x) \geq b_0$ and further assume that there exists positive constants M_1, M_2 and m such that $|f(s)| \leq M_1|s|^m + M_2$, where either $m < p/p'$ or $m \leq p/p'$ with M_1 small enough. the steady-state problem* (2.1) *has a weak solution u in the sense of Definition* 2.1.

Proof. In order to prove the existence of solutions of the steady-state problem (2.1), we use the Galerkin's method of approximate solutions (see [4]). The material presented here is standard (see [7, 8]), and we have included it just for the sake of completeness. To use the Galerkin's method, we are in need of the specific basis. Let $r > 0$ be such that $r < p^* = \dfrac{Np}{N-p}$ and $s \in N$. Then

$$W_0^{s,2}(\Omega) \subset W_0^{1,p}(\Omega) \subset L^r(\Omega) \subset (W_0^{s,2})'(\Omega)$$

with continuous and dense inclusions. Now, let us introduce the spectral problem, find $w \in W_0^{s,2}(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} (w,\phi)_{W_0^{s,2}(\Omega)} &= \lambda(w,\phi)_{L^2(\Omega)}, \quad \text{for all } \phi \in W_0^{s,2}(\Omega), \\ w &= 0 \text{ on } \partial\Omega. \end{aligned}$$

where $(\cdot,\cdot)_{W_0^{s,2}(\Omega)}$ and $(\cdot,\cdot)_{L^2(\Omega)}$ denotes the inner products of $W_0^{s,2}(\Omega)$ and $L^2(\Omega)$ respectively. The above problem gives a sequence of non-decreasing eigenvalues $\{\lambda_l\}_{l=1}^{\infty}$ and a sequence of corresponding eigenfunctions $\{e_l\}_{l=1}^{\infty}$, forming an orthogonal basis in $W_0^{s,2}(\Omega)$ and orthonormal basis in $L^2(\Omega)$ (see [16].

For each $n \in \mathbb{N}$, define the subspace $V_n = span\{e_1, \cdots, e_n\}$. It is well known that $(V_n, \|\cdot\|)$ and $(\mathbb{R}^n, |\cdot|)$ are isometrically isomorphic by the natural linear map $T : V_n \to \mathbb{R}^n$ given by $z = \sum_{i=1}^{n} d_i e_i \to T(z) = d = (d_1, \cdots, d_n)$ (see [4]). So $\|z\| = |T(z)| = |d|$, where $|\cdot|$ and $\|\cdot\|$ denote the usual norms in $\mathbb{R}^N$ and $V_n(\Omega)$ respectively. We look for the function $u_n \in W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega)$ of the form

$$u_n = \sum_{l=1}^{n} d_{n,l} e_l(x),$$

where we need to determine the co-efficients $r_{n,l}$ so that, for $k = 1, 2, \cdots, n$.

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla e_k dx + \int_\Omega a(x)|u_n|^{p-2}u_n e_k dx + \int_\Omega |\nabla u_n|^{q-2}\nabla u_n \nabla e_k dx + \int_\Omega b(x)|u_n|^{q-2}u_n e_k dx = \int_\Omega f(u_n)e_k dx + \int_\Omega g(x)e_k dx \quad \text{in } \Omega. \tag{2.2}$$

Using Hölder's, Poincaré's and Young's inequalities and from the assumption of $f(u)$, we get

$$\begin{aligned} \left|\int_\Omega f(u)u dx\right| &\le \|f(u)\|_{L^{p'}(\Omega)}\|u\|_{L^p(\Omega)} \\ &\le c\|f(u)\|_{L^{p'}(\Omega)}\|\nabla u\|_{L^p(\Omega)} \\ &\le c(\epsilon)\|f(u)\|_{L^{p'}(\Omega)} + \epsilon\|\nabla u\|^p_{L^p(\Omega)} \\ &\le c(\epsilon)(\int_\Omega M_1|u|^m + M_2)^{p'} dx + \epsilon\|\nabla u\|^p_{L^p(\Omega)} \\ &\le c(\epsilon)M_1^{p'}\int_\Omega |u|^{mp'}dx + c(\epsilon) + \epsilon\|\nabla u\|^p_{L^p(\Omega)} \\ &\le c(\epsilon) + \epsilon\|\nabla u\|^p_{L^p(\Omega)} \end{aligned} \tag{2.3}$$

Now let us consider the following function $G : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$G(d) = (h_1(d), \cdots, h_n(d))$$

where

$$\begin{aligned}
h_k(d) = \int_\Omega \Bigg[\Big| \sum_{l=1}^{n} d_{n,l} \nabla e_l(x) \Big|^{p-2} \sum_{l=1}^{n} d_{n,l} \nabla e_l(x) \nabla e_k \\
+ a(x) \Big| \sum_{l=1}^{n} d_{n,l} e_l(x) \Big|^{p-2} \sum_{l=1}^{n} d_{n,l} e_l(x) e_k \\
+ \Big| \sum_{l=1}^{n} d_{n,l} \nabla e_l(x) \Big|^{q-2} \sum_{l=1}^{n} d_{n,l} \nabla e_l(x) \nabla e_k \\
+ b(x) \Big| \sum_{l=1}^{n} d_{n,l} e_l(x) \Big|^{q-2} \sum_{l=1}^{n} d_{n,l} e_l(x) e_k \\
- f\Big(\sum_{l=1}^{n} d_{n,l} e_l(x) \Big) e_k - g e_k \Bigg] dx,
\end{aligned}$$

for each point $d = (d_1, \cdots, d_n) \in \mathbb{R}^n$. Then from (2.3), we get

$$\begin{aligned}
\langle G(d), d \rangle \geq \int_\Omega \Bigg(\Big| \sum_{l=1}^{n} d_{n,l} \nabla e_l(x) \Big|^{p} + a_0 \Big| \sum_{l=1}^{n} d_{n,l} e_l(x) \Big|^{p} + \Big| \sum_{l=1}^{n} d_{n,l} \nabla e_l(x) \Big|^{q} \\
+ b_0 \Big| \sum_{l=1}^{n} d_{n,l} e_l(x) \Big|^{q} - c(\epsilon)(\|g\|_{L^{q'}}(\Omega) + 1) \Bigg) dx.
\end{aligned}$$

This shows that $\big\langle G(d), d \big\rangle \geq 0$ if $|d| = \rho$ provided $\rho > 0$ sufficiently large enough. Hence it follows from the Lemma 2.2, that for each $n \in N$, there exists $u_n \in V_n$ satisfying $F(u_n) = 0, \|u_n\| \leq \rho$. This proves that, given absolutely continuous co-efficients $b_{n,l}$, we set $\phi_n = \sum_{l=1}^{n} b_{n,l} e_l(x)$ such that

$$\begin{aligned}
\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla \phi_n dx + \int_\Omega a(x) |u_n|^{p-2} u_n \phi_n dx + \int_\Omega |\nabla u_n|^{q-2} \nabla u_n \nabla \phi_n dx \\
+ \int_\Omega b(x) |u_n|^{q-2} u_n \phi_n dx = \int_\Omega f(u_n) \phi_n dx + \int_\Omega g(x) \phi_n dx \quad \text{in } \Omega. \qquad (2.4)
\end{aligned}$$

holds with $\|u_n\| \leq \rho$, for all $n \in N$. Taking $\phi_n = u_n$ in (2.4) and using the Poincare's, Young's inequalities with the assumption of $f(u)$ we get,

$$\|u_n\|_{W_0^{1,p}(\Omega)} + \|u_n\|_{L^p(\Omega)} + \|u_n\|_{W_0^{1,q}(\Omega)} + \|u_n\|_{L^q(\Omega)} \leq c.$$

Let us assume that $u \in W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega)$ be the weak limit of $\{u_n\}$, then ther exists a subsequence which is also denoted by $\{u_n\}$ such that,

$$\begin{aligned}
&u_n \to u \text{ in } \ L^s(\Omega) \text{ for } 1 < s < \frac{2N}{N-2},\\
&u_n \rightharpoonup u \text{ weakly in } \ W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega),\\
&u_n \rightharpoonup u \text{ weakly in } \ L^p(\Omega) \cap L^q(\Omega),\\
&|\nabla u_n|^{p-2}\nabla u_n + |\nabla u_n|^{q-2}\nabla u_n \rightharpoonup \zeta_1 \text{ weakly in } L^{p'}(\Omega) \cap L^{q'}(\Omega).
\end{aligned}$$

Recall the monotonicity property of p-Laplacian operator and by adopting the technique proved in [8], one can easily obtain that $\zeta_1 = |\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u$. Then taking limit as $n \to \infty$, we get

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi dx + \int_\Omega a(x)|u|^{p-2}u\phi dx + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla \phi dx\\
&\quad + \int_\Omega b(x)|u|^{q-2}u\phi dx = \int_\Omega f(u)\phi dx + \int_\Omega g(x)\phi dx \ \text{ in } \Omega.
\end{aligned} \tag{2.5}$$

Equation (2.5) holds for all functions $W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega)$, as the function ϕ is dense in the space. This proves that u is a weak solution of the steady-state equation (2.1). □

3. Parabolic Case

In this section, first we consider the semi-discretized problem of the original problem (1.1) and establish existence of a weak solution. By using that, one can show the existence of a weak solution of the given problem (1.1).

The semi-discretized problem of the given parabolic problem (1.1) is as follows:

$$\begin{cases}
\frac{1}{h}(u_k - u_{k-1}) - \nabla \cdot (|\nabla u_k|^{p-2}\nabla u_k) + a(x)\,|u_k|^{p-2}\,u_k\\
-\nabla \cdot (|\nabla u_k|^{q-2}\nabla u_k) + b(x)\,|u_k|^{q-2}\,u_k = f(u_{k-1}) + g \text{ in } \Omega,\\
u_k = 0 \quad \text{on } \partial\Omega,
\end{cases} \tag{3.1}$$

where $u_k = u(x, kh)$, $h = T/n$ and $k = 1, 2, \cdots, n$.

Definition 3.1. A function u is defined as a weak solution for the problem (1.1) provided $u \in C([0,T]; L^2(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega)) \cap L^q(0,T; W_0^{1,q}(\Omega))$, $u_t \in L^{p'}(0,T; W^{-1,p'}(\Omega))$, and, for any $\phi \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^q(0,T; W_0^{1,q}(\Omega))$,

$$\int_0^T \langle \frac{\partial u}{\partial t}, \phi \rangle dt = -\int_{Q_T} |\nabla u|^{p-2}\nabla u \nabla \phi dxdt - \int_{Q_T} a(x)|u|^{p-2}u\phi dxdt$$
$$- \int_{Q_T} |\nabla u|^{q-2}\nabla u \nabla \phi dxdt - \int_{Q_T} b(x)|u|^{q-2}u\phi dxdt$$
$$+ \int_{Q_T} (f(u)+g)\phi dxdt$$

holds (where $\langle \cdot, \cdot \rangle$ denotes the pair $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$).

Theorem 3.2. *Assume that the conditions of Theorem* 2.3 *are satisfied with* $u_0 \in L^2(\Omega) \cap W_0^{1,p}(\Omega)$ *and* $f(\cdot) \in C^1(\mathbb{R})$. *Then the problem* (1.1) *has a weak solution in the sense of Definition* 3.1.

Lemma 3.3. *Assume* u_k *is a unique weak solution for the semi-discretized problem* (3.1). *Then there exists a constant* $C > 0$ *such that*

$$\frac{1}{h}\int_\Omega |u_k|^2 dx + \int_\Omega |\nabla u_k|^p dx + 2\int_\Omega a(x)|u_k|^p dx + \int_\Omega |\nabla u_k|^q dx$$
$$+2\int_\Omega b(x)|u_k|^q dx \leq \frac{1}{2}\int_\Omega |\nabla u_{k-1}|^p dx + \frac{1}{h}\int_\Omega |u_{k-1}|^2 dx + C, \quad (3.2)$$

and

$$\int_\Omega |u_i|^2 dx + \frac{h}{2}\sum_{k=1}^{i}\int_\Omega |\nabla u_k|^p dx + \frac{h}{2}\int_\Omega |\nabla u_i|^p dx$$
$$+2h\sum_{k=1}^{i}\int_\Omega a(x)|u_k|^p dx + h\sum_{k=1}^{i}\int_\Omega |\nabla u_k|^q dx + 2h\sum_{k=1}^{i}\int_\Omega b(x)|u_k|^q dx$$
$$\leq \int_\Omega (|u_0|^2 + \frac{h}{2}|\nabla u_0|^p) dx + TC, \quad (3.3)$$

for $k, i = 1, 2, \cdots, n$.

Proof. As in the steady-state case, one can show that there exists a weak solution $u_k \in W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^2(\Omega)$ satisfying

$$\frac{1}{h}\int_\Omega (u_k - u_{k-1})\phi dx + \int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla \phi dx + \int_\Omega a(x)|u_k|^{p-2}u_k \phi dx$$
$$+ \int_\Omega |\nabla u_k|^{q-2}\nabla u_k \nabla \phi dx + \int_\Omega b(x)|u_k|^{q-2}u_k\phi dx$$
$$= \int_\Omega (f(u_{k-1}) + g)\phi dx, \quad (3.4)$$

for each $\phi \in C_0^\infty(\Omega)$. Then taking $\phi = u_k$ as the test function in the above equation, one can obtain

$$\frac{1}{h}\int_\Omega |u_k|^2 dx + \int_\Omega |\nabla u_k|^p dx + \int_\Omega a(x)|u_k|^p dx + \int_\Omega |\nabla u_k|^q dx + \int_\Omega b(x)|u_k|^q dx$$
$$\leq \frac{1}{2h}\int_\Omega |u_k|^2 dx + \frac{1}{2h}\int_\Omega |u_{k-1}|^2 dx + \int_\Omega f(u_{k-1})u_k dx + \int_\Omega g u_k dx. \tag{3.5}$$

Then, by using Hölder's, Poincaré's and Young's inequalities and from the assumption of $f(u)$, we get

$$\Big|\int_\Omega f(u_{k-1})u_k dx\Big| \leq \frac{1}{2}\|\nabla u_k\|^p_{L^p(\Omega)} + \frac{1}{4}\|\nabla u_{k-1}\|^p_{L^p(\Omega)} + C.$$

Using the above said inequalities and the assumption of g, it is easy to obtain the following estimate

$$\Big|\int_\Omega g u_k dx\Big| \leq \frac{1}{2}\|\nabla u_k\|^q_{L^q(\Omega)} + \frac{1}{2}\|g\|_{L^{q'}(\Omega)}.$$

Substituting the above two estimates in (3.5), one can easily obtain (3.2).
To prove (3.3), take summation from $k = 1$ to i on both sides of the inequality (3.2) to get

$$\frac{1}{h}\sum_{k=1}^{i}\int_\Omega |u_k|^2 dx + \sum_{k=1}^{i}\int_\Omega |\nabla u_k|^p dx + 2\sum_{k=1}^{i}\int_\Omega a(x)|u_k|^p dx$$
$$+\sum_{k=1}^{i}\int_\Omega |\nabla u_k|^q dx + 2\sum_{k=1}^{i}\int_\Omega b(x)|u_k|^q dx$$
$$\leq \frac{1}{h}\sum_{k=1}^{i}\int_\Omega |u_{k-1}|^2 dx + \frac{1}{2}\sum_{k=1}^{i}\int_\Omega |\nabla u_{k-1}|^p dx + \sum_{k=1}^{i} C,$$

for $i \in \{1, 2, \cdots, n\}$. Noticing that $ih \leq T$ and using simple calculation leads to (3.3).

To prove the uniqueness of the weak solution, let us assume that v_1 and v_2 are two solutions of (3.1). Then the difference of two solutions satisfying the following equation,

$$\frac{1}{h}\int_\Omega (v_1 - v_2)\phi dx + \int_\Omega (|\nabla v_1|^{p-2}\nabla v_1 - |\nabla v_2|^{p-2}\nabla v_2)\nabla\phi dx$$
$$+\int_\Omega (|\nabla v_1|^{q-2}\nabla v_1 - |\nabla v_2|^{q-2}\nabla v_2)\nabla\phi dx + \int_\Omega a(x)(|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2)\phi dx$$
$$+\int_\Omega b(x)(|v_1|^{q-2}v_1 - |v_2|^{q-2}v_2)\phi dx$$
$$= 0,$$

for each $\phi \in C_0^\infty(\Omega)$. Then, taking $\phi = v_1 - v_2$ in the above equality and using the monotonicity inequality, one can get, $v_1 = v_2$ a.e in Ω. □

Definition 3.4. ([17]) Let us define the first kind of approximate solution as follows:

$$w^{(n)}(x,t) = \sum_{k=1}^{n} \chi_k(t) u_k(x), \tag{3.6}$$

where $\chi_k(t)$ is the characteristic function of the time interval $((k-1)h, kh]$, for $k = 1, 2, \cdots, n$.

Lemma 3.5. *Let $u_0 \in L^2(\Omega) \cap W_0^{1,p}(\Omega)$ and $f(\cdot) \in C^1(\mathbb{R})$. Then the approximate solution* (3.6) *satisfies the following estimate*

$$\begin{aligned} &\|w^{(n)}\|_{L^\infty(0,T;L^2(\Omega))} + \|w^{(n)}\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|w^{(n)}\|_{L^q(0,T;W_0^{1,q}(\Omega))} \\ &+\|f(w^{(n)})\|_{L^{p'}(Q_T)} + \|w^{(n)}\|_{L^p(0,T;L^p(\Omega))} + \|w^{(n)}\|_{L^q(0,T;L^q(\Omega))} \\ &+\||\nabla w^{(n)}|^{p-2}\nabla w^{(n)}\|_{L^{p'}(Q_T)} + \||\nabla w^{(n)}|^{q-2}\nabla w^{(n)}\|_{L^{q'}(Q_T)} \le C. \end{aligned} \tag{3.7}$$

Proof. For any $t \in (0,T)$, there exists some $k \in \{1, 2, \cdots, n\}$ such that $t \in ((k-1)h, kh]$. By using the definition of approximate solution (3.6) and from steady-state case one can have $\|w^{(n)}\|^2_{L^2(\Omega)} = \sum_{k=1}^{n} \|u_k(x)\|^2_{L^2(\Omega)} \le C$ which leads to $\|w^{(n)}\|_{L^\infty(0,T;L^2(\Omega))} \le C$. Taking $i = n$ in the inequality (3.3), we obtain

$$\begin{aligned} &\int_\Omega |u_n|^2 dx + \frac{h}{2}\sum_{k=1}^{n}\int_\Omega |\nabla u_k|^p dx + \frac{h}{2}\int_\Omega |\nabla u_n|^p dx \\ &\quad +2h\sum_{k=1}^{n}\int_\Omega a(x)|u_k|^p dx + h\sum_{k=1}^{n}\int_\Omega |\nabla u_k|^q dx + 2h\sum_{k=1}^{n}\int_\Omega b(x)|u_k|^q dx \\ &\le \int_\Omega |u_0|^2 dx + \frac{h}{2}\int_\Omega |\nabla u_0|^p dx + TC. \end{aligned}$$

From the above inequality,

$$\int_{Q_T} |\nabla w^{(n)}|^p dxdt = \sum_{k=1}^{n}\int_\Omega |\nabla u_k|^p dx \le C.$$

The above inequality shows the following result,

$$\|w^{(n)}\|_{L^p(0,T;W_0^{1,p}(\Omega))} \le C.$$

Similarly one can prove

$$\|w^{(n)}\|_{L^p(0,T;L^p(\Omega))} \le C, \qquad \|w^{(n)}\|_{L^q(0,T;W_0^{1,q}(\Omega))} \le C$$

and

$$\|w^{(n)}\|_{L^q(0,T;L^q(\Omega))} \leq C.$$

From the assumption of $f(u)$ it is easy to understand that

$$\|f(w^{(n)})\|_{L^{p'}(Q_T)} \leq C.$$

Taking $\phi = u_k$ as test function in (3.4) and the definition of $w^{(n)}$, we get

$$\begin{aligned}
&\frac{1}{2}\int_{Q_T} |w^{(n)}|^2 dxdt - \frac{1}{2}\int_{\Omega} |u_0|^2 dx + \int_{Q_T} |\nabla w^{(n)}|^{p-2}\nabla w^{(n)}\nabla w^{(n)} dxdt \\
&\quad + \int_{Q_T} a(x)|w^{(n)}|^p dxdt + \int_{Q_T} |\nabla w^{(n)}|^{q-2}\nabla w^{(n)}\nabla w^{(n)} dxdt \\
&\quad + \int_{Q_T} b(x)|w^{(n)}|^q dxdt \\
&= \int_{Q_T} f(w^{(n)})w^{(n)} dxdt + \int_{Q_T} g w^{(n)} dxdt.
\end{aligned}$$

Hence, from the above inequality, it is easy to obtain the results

$$\||\nabla w^{(n)}|^{p-2}\nabla w^{(n)}\|_{L^{p'}(Q_T)} \leq C \text{ and } \||\nabla w^{(n)}|^{q-2}\nabla w^{(n)}\|_{L^{q'}(Q_T)} \leq C.$$

□

Definition 3.6. ([17]) The second kind of approximate solution is defined as follows:

$$u^{(n)}(x,t) = \sum_{k=1}^{n} \chi_k(t)[\lambda_k(t)u_k(x) + (1-\lambda_k(t))u_{k-1}(x)], \tag{3.8}$$

where $\lambda_k(t) = \begin{cases} \frac{t}{h} - (k-1), & t \in ((k-1)h, kh], \\ 0, & \text{otherwise.} \end{cases}$

Lemma 3.7. *Let $u_0 \in L^2(\Omega) \cap W_0^{1,p}(\Omega)$ and $f(\cdot) \in C^1(\mathbb{R})$. Then there exists a constant $C > 0$ such that the following estimate*

$$\left\|\frac{\partial u^{(n)}}{\partial t}\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \left\|u^{(n)}\right\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{3.9}$$

holds, for $u^{(n)}$ in (3.8).

Proof. Differentiating the second kind of approximate solution with respect to t, we get

$$\frac{\partial u^{(n)}}{\partial t} = \frac{1}{h}\sum_{k=1}^{n} \chi_k(u_k(x) - u_{k-1}(x)). \tag{3.10}$$

Substituting (3.10) in (3.1), we get

$$\begin{aligned}
&\langle \frac{\partial u^{(n)}}{\partial t}, \phi\rangle \\
&= -\sum_{k=1}^{n} \chi_k(t) \int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla\phi dx - \sum_{k=1}^{n} \chi_k(t) \int_\Omega a(x)|u_k|^{p-2}u_k\phi dx \\
&\quad -\sum_{k=1}^{n} \chi_k(t) \int_\Omega |\nabla u_k|^{q-2}\nabla u_k \nabla\phi dx - \sum_{k=1}^{n} \chi_k(t) \int_\Omega b(x)|u_k|^{q-2}u_k\phi dx \\
&\quad +\sum_{k=1}^{n} \chi_k(t) \int_\Omega f(u_{k-1})\phi dx + \sum_{k=1}^{n} \chi_k(t) \int_\Omega g\phi dx.
\end{aligned}$$

For any $\phi \in C_0^\infty(\Omega)$, from the Lemma 3.5, we get

$$\Big\|\frac{\partial u^{(n)}}{\partial t}\Big\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le C.$$

Then, from the definition of the second kind of approximate solution $u^{(n)}$ and (3.3), we have

$$\begin{aligned}
&\|u^{(n)}\|^r_{L^r(0,T;L^2(\Omega))} \\
&\le C^r \int_0^T \Big(\int_\Omega |u^{(n)}|^2 \mathrm{d}x\Big)^{r/2} \mathrm{d}t \\
&= C^r \int_0^T \Big(\int_\Omega \Big|\sum_{k=1}^{n} \chi_k(t)[\lambda_k(t)u_k(x) + (1-\lambda_k(t))u_{k-1}(x)]\Big|^2 \mathrm{d}x\Big)^{r/2} \mathrm{d}t \\
&= C^r \sum_{k=1}^{n} \int_{(k-1)h}^{kh} \Big(\int_\Omega \Big|[\lambda_k(t)u_k(x) + (1-\lambda_k(t))u_{k-1}(x)]\Big|^2 \mathrm{d}x\Big)^{r/2} \mathrm{d}t \\
&\le C^r \sum_{k=1}^{n} h\Big(\int_\Omega (|u_k(x)|^2 + |u_{k-1}(x)|^2)\mathrm{d}x\Big)^{r/2} \\
&\le C^{r+\frac{r}{2}} T,
\end{aligned}$$

where $C > 0$ is independent of $r > 1$. Therefore one can have

$$\|u^{(n)}\|_{L^\infty(0,T;L^2(\Omega))} \quad = \quad \lim_{r\to\infty} \|u^{(n)}\|_{L^r(0,T;L^2(\Omega))} \le C.$$

□

Proof of Theorem 3.2. By the Lemma 3.5, there exists a subsequence of $w^{(n)}$ (which is also denoted by $w^{(n)}$), $\zeta \in L^{p'}(Q_T) \cap L^{q'}(Q_T)$ such that

$$w^{(n)} \rightharpoonup u \text{ weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega))$$

and

$$|\nabla w^{(n)}|^{p-2}\nabla w^{(n)} + |\nabla w^{(n)}|^{q-2}\nabla w^{(n)} \rightharpoonup \zeta \text{ weakly in } L^{p'}(Q_T) \cap L^{q'}(Q_T),$$

as $n \to \infty$. From the Lemma 3.7, we can find an integer $s > 0$ such that $W^{-1,p'}(\Omega) \hookrightarrow H^{-s}(\Omega)$ (for more details see [18]) and from Aubin's type lemma with the compact imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega)$, we can conclude that there exists a subsequence of $u^{(n)}$ (which is also denoted by $u^{(n)}$) such that

$$\begin{aligned}
\frac{\partial u^{(n)}}{\partial t} &\rightharpoonup \frac{\partial \rho}{\partial t} \text{ weakly in } L^{p'}(0,T;W^{-1,p'}(\Omega)),\\
u^{(n)} &\rightharpoonup \rho \text{ weakly* in } L^{\infty}(0,T;L^2(\Omega)),\\
u^{(n)} &\to \rho \text{ strongly in } C([0,T];L^2(\Omega)), \text{ and}\\
u^{(n)} &\to \rho \text{ a.e in } Q_T,
\end{aligned}$$

as $n \to \infty$. Further, from the definitions (3.6) and (3.8), we have

$$\begin{aligned}
&\Big|\int_{Q_T}(w^{(n)} - u^{(n)})\phi \mathrm{d}x\mathrm{d}t\Big|\\
&= \Big|\int_{Q_T}\sum_{k=1}^{n}\chi_k(t)(1-\lambda_k(t))(u_k - u_{k-1})\phi \mathrm{d}x\mathrm{d}t\Big|\\
&\leq h\int_0^T\Big(\Big|\int_\Omega |\nabla w^{(n)}|^{p-2}\nabla w^{(n)}\nabla\phi dx\Big| + \Big|\int_\Omega a(x)|w^{(n)}|^{p-2}w^{(n)}\phi dx\Big|\\
&\quad + \Big|\int_\Omega |\nabla w^{(n)}|^{q-2}\nabla w^{(n)}\nabla\phi dx\Big| + \Big|\int_\Omega b(x)|w^{(n)}|^{q-2}w^{(n)}\phi dx\Big|\\
&\quad + \Big|\int_\Omega f(w^{(n)})\phi dx\Big| + \Big|\int_\Omega f(u_o)\phi dx\Big| + \Big|\int_\Omega g\phi dx\Big|\Big)dt
\end{aligned}$$

for any $\phi \in C_0^\infty(Q_T)$. From Lemma 3.5, we get

$$\Big|\int_{Q_T}(w^{(n)} - u^{(n)})\phi \mathrm{d}x\mathrm{d}t\Big| \leq Ch \to 0 \text{ as } n \to \infty.$$

This shows that $\rho = u$ a.e in Q_T. Therefore $f(w^{(n)}) \to f(u)$ $a.e$ in Q_T due to the continuity of f. Also we have $\|f(w^{(n)})\|_{L^{p'}(Q_T)} < \infty$, by Lemma 3.5, so we get

$$f(w^{(n)}) \rightharpoonup f(u) \text{ weakly in } L^{p'}(Q_T).$$

From (3.10) and (3.6), we have

$$\begin{aligned}
&\int_0^T < \frac{\partial u^{(n)}}{\partial t}, \phi > dt \\
&\leq -\int_{Q_T} |\nabla w^{(n)}|^{p-2} \nabla w^{(n)} \nabla \phi dxdt - \int_{Q_T} a(x)|w^{(n)}|^{p-2} w^{(n)} \phi dxdt \\
&\quad - \int_{Q_T} |\nabla w^{(n)}|^{q-2} \nabla w^{(n)} \nabla \phi dxdt - \int_{Q_T} b(x)|w^{(n)}|^{q-2} w^{(n)} \phi dxdt \\
&\quad + \int_{Q_T} (f(w^{(n)}) + g) \phi dxdt + h \int_\Omega f(u_0) \phi dx - \int_{(n-1)h}^T \int_\Omega f(u_n) \phi dxdt,
\end{aligned}$$

for any $\phi \in C_0^\infty(Q_T)$. Letting $n \to \infty$, we get

$$\begin{aligned}
\int_0^T \langle \frac{\partial u}{\partial t}, \phi \rangle dt = &-\int_{Q_T} \zeta \nabla \phi dxdt - \int_{Q_T} a(x)|u|^{p-2} u \phi dxdt \\
&- \int_{Q_T} b(x)|u|^{q-2} u \phi dxdt + \int_{Q_T} f(u) \phi dxdt \\
&+ \int_{Q_T} g \phi dxdt.
\end{aligned}$$

Next we will show that $\zeta = |\nabla u|^{p-2} \nabla u + |\nabla u|^{q-2} \nabla u$. For that take $\phi = u$ as the test function in (3.4), to obtain

$$\begin{aligned}
&\frac{1}{h} \int_\Omega (u_k - u_{k-1}) u dx + \int_\Omega |\nabla u_k|^{p-2} \nabla uk \nabla u dx + \int_\Omega a(x)|u_k|^{p-2} u_k u dx \\
&\quad + \int_\Omega |\nabla u_k|^{q-2} \nabla u_k \nabla u dx + \int_\Omega b(x)|u_k|^{q-2} u_k u dx \\
&= \int_\Omega f(u_{k-1}) u dx + \int_\Omega g u dx.
\end{aligned}$$

Multiply by $\chi_k(t)$, take summation on both sides over the limits $k = 1$ to n and use the definition of $w^{(n)}$ to get

$$\begin{aligned}
&\frac{1}{h} \int_{Q_T} (w^{(n)} - u_0) u dxdt + \int_{Q_T} |\nabla w^{(n)}|^{p-2} \nabla w^{(n)} \nabla u dxdt \\
&\quad + \int_{Q_T} a(x)|w^{(n)}|^{p-2} w^{(n)} u dxdt + \int_{Q_T} |\nabla w^{(n)}|^{q-2} \nabla w^{(n)} \nabla u dxdt \\
&\quad + \int_{Q_T} b(x)|w^{(n)}|^{q-2} w^{(n)} u dxdt \\
&= \int_{Q_T} f(w^{(n)}) u dxdt + h \int_\Omega f(u_0) u_0 dx
\end{aligned}$$

$$- \int_{(n-1)h}^{T} \int_{\Omega} f(u_n) u_n dxdt + \int_{Q_T} gudxdt. \tag{3.11}$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned} &\frac{1}{2}\int_{\Omega} |u(x,T)|^2 dx - \frac{1}{2}\int_{\Omega} |u_0|^2 dx + \int_{Q_T} \zeta \nabla u dxdt \\ &\quad + \int_{Q_T} a(x)|u|^p dxdt + \int_{Q_T} b(x)|u|^q dxdt \\ &= \int_{Q_T} f(u) u dxdt + \int_{Q_T} gudxdt. \end{aligned} \tag{3.12}$$

Putting $u = w^{(n)}$ in (3.11), one can obtain

$$\begin{aligned} &\frac{1}{2}\int_{\Omega} |w^{(n)}(x,T)|^2 dx - \frac{1}{2}\int_{\Omega} |u_0|^2 dx + \int_{Q_T} |\nabla w^{(n)}|^p dxdt \\ &\quad + \int_{Q_T} a(x)|w^{(n)}|^p dxdt + \int_{Q_T} |\nabla w^{(n)}|^q dxdt + \int_{Q_T} b(x)|w^{(n)}|^q dxdt \\ &= \int_{Q_T} (f(w^{(n)}) + g) w^{(n)} dxdt + h \int_{\Omega} f(u_0) u_0 dx \\ &\quad - \int_{(n-1)h}^{T} \int_{\Omega} f(u_n) u_n dxdt. \end{aligned} \tag{3.13}$$

Consider the elementary inequality [7] as follows:

$$\int_0^T \int_{\Omega} \Big[((|\alpha|^{p-2}\alpha - |\beta|^{p-2}\beta)(\alpha - \beta)) + ((|\gamma|^{q-2}\gamma - |\delta|^{q-2}\delta)(\gamma - \delta)) \Big] dxdt \\ \geq 0,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}^n$.

In the above inequality, substitute $\alpha = \gamma = \nabla w^{(n)}$ and $\beta = \delta = \nabla(u - \epsilon\phi)$, to get

$$\begin{aligned} &\int_{Q_T} (|\nabla w^{(n)}|^{p-2} \nabla w^{(n)} - |\nabla(u - \epsilon\phi)|^{p-2} \nabla(u - \epsilon\phi))(\nabla w^{(n)} - \nabla(u - \epsilon\phi)) \\ &\quad + (|\nabla w^{(n)}|^{q-2} \nabla w^{(n)} - |\nabla(u - \epsilon\phi)|^{q-2} \nabla(u - \epsilon\phi))(\nabla w^{(n)} - \nabla(u - \epsilon\phi)) dxdt \\ &\geq 0. \end{aligned}$$

Further, doing simple calculations and using (3.13), we obtain

$$
\begin{aligned}
&-\frac{1}{2}\int_\Omega |w^{(n)}(x,T)|^2dx + \frac{1}{2}\int_\Omega |u_0|^2dx - \int_{Q_T} a(x)|w^{(n)}|^p dxdt \\
&-\int_{Q_T} b(x)|w^{(n)}|^q dxdt + \int_{Q_T} f(w^{(n)})w^{(n)}dxdt + h\int_\Omega f(u_0)u_0dx \\
&-\int_{(n-1)h}^T\int_\Omega f(u_n)u_ndxdt + \int_{Q_T} gw^{(n)}dxdt \\
&-\int_{Q_T}(|\nabla(u-\epsilon\phi)|^{p-2}\nabla(u-\epsilon\phi) + |\nabla(u-\epsilon\phi)|^{q-2}\nabla(u-\epsilon\phi))\nabla w^{(n)}dxdt \\
&+\int_{Q_T}|\nabla(u-\epsilon\phi)|^p dxdt + \int_{Q_T}|\nabla(u-\epsilon\phi)|^q dxdt \\
&-\int_{Q_T}(|\nabla w^{(n)}|^{p-2}\nabla w^{(n)} + |\nabla w^{(n)}|^{q-2}\nabla w^{(n)})\nabla(u-\epsilon\phi)dxdt \geq 0.
\end{aligned}
$$

Taking limit as $n\to\infty$ in the above inequality, using (3.12) and simple calculation lead to

$$
\begin{aligned}
&-\int_{Q_T}\zeta\nabla udxdt - \int_{Q_T}|\nabla(u-\epsilon\phi)|^p dxdt - \int_{Q_T}|\nabla(u-\epsilon\phi)|^q dxdt \\
&+\int_{Q_T}|\nabla(u-\epsilon\phi)|^p dxdt + \int_{Q_T}|\nabla(u-\epsilon\phi)|^q dxdt \\
&+\epsilon\int_{Q_T}|\nabla(u-\epsilon\phi)|^{p-2}\nabla(u-\epsilon\phi)\nabla\phi dxdt + \int_{Q_T}\zeta\nabla udxdt \\
&+\epsilon\int_{Q_T}|\nabla(u-\epsilon\phi)|^{q-2}\nabla(u-\epsilon\phi)\nabla\phi dxdt - \epsilon\int_{Q_T}\zeta\nabla\phi dxdt \leq 0.
\end{aligned}
$$

Taking ϵ sufficiently small and simplifying, we get

$$\int_{Q_T}(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u - \zeta)\nabla\phi dxdt \leq 0.$$

For any $\phi\in C_0^\infty(Q_T)$, we get

$$|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u = \zeta \text{ a.e in } Q_T.$$

□

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