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# AN ALGORITHM FOR SOLVING RESOLVENT INCLUSION PROBLEM

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**Abstract.** In this article, we put forward a new type of variational inclusion problem known as resolvent inclusion. An algorithm is given for approximating its solution. The convergence of the algorithm is explained with the help of an example and plots using Matlab.

## 1. Introduction and preliminaries

It is very old problem to find a zero of a function and it has great importance because it is used for modeling of a large number of mathematical and physical problems. Variational inclusion is actually a generalization of this problem, when multi-valued map is taken in place of single-valued. Given a Hilbert space $\mathcal{V}$ and the operator $M : \mathcal{V} \to 2^{\mathcal{V}}$, the inclusion problem is to find $v \in \mathcal{V}$ such that

$$0 \in M(v). \tag{1.1}$$

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Rockafellar [18] designed an algorithm for solving (1.1). Many researchers work on it and give different shapes to it like by decomposing $M$ into sum of two operators $M = \mathcal{C} + \mathcal{B}$, $\mathcal{C}$ is a single-valued map while $\mathcal{B}$ is a multi-valued map. In this case (1.1) takes the form:

$$0 \in \mathcal{C}(v) + \mathcal{B}(v), \ \forall \ v \in \mathcal{V}. \tag{1.2}$$

This type of problem is studied by many researchers and give different algorithms for solving it like Lions and Mercier [15], Passty [16] etc. This problem is of great importance as many problems of mechanics, economics, physics etc. can be modelled by using it. The problem (1.2) was also examined by many researchers in different spaces and for different types of mappings like, Fang and Huang [9] for $H$-monotone operators in Hilbert spaces and Fang and Huang [10] for $H$-accretive operators in Banach spaces. The problem was also generalized by many researchers like Abbas et al. [1], Adamu et al. [2], Adly [3], Ahmad and Ansari [4], Ahmad et al. [5], Ding [8], Huang [13], Khan [14] etc.

Ahmad et al. [7] introduced a new type of variational inclusion problem known as Yosida inclusion problem which uses yosida operator. Latest generalization of variational inclusion is Cayley inclusion, which was introduced by Ahmad et al. [6]. They use Cayley operator for its construction.

Ahmad's research has encouraged us to design a new variational inclusion problem known as resolvent inclusion which uses resolvent operator. First we give foundations for defining the problem. Throughout the paper, $\mathcal{V}$ is assumed to be a Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, respectively.

**Definition 1.1.** ( [3]) A single-valued mapping $\mathcal{C} : \mathcal{V} \to \mathcal{V}$ is said to be *monotone*, if it satisfies

$$0 \leq (\mathcal{C}(v) - \mathcal{C}(w), v - w), \ \forall \ v, w \in \mathcal{V},$$

it is called *strictly monotone*, if it satisfies

$$0 = (\mathcal{C}(v) - \mathcal{C}(w), v - w) \ \textit{iff} \ v = w,$$

and it is called *strongly monotone*, if it is monotone and for some positive constant $\alpha$, it satisfies

$$\alpha\|v - w\|^2 \leq (\mathcal{C}(v) - \mathcal{C}(w), v - w).$$

**Definition 1.2.** Let $\mathcal{C}, \mathcal{H} : \mathcal{V} \to \mathcal{V}$ be two single-valued maps such that for some positive constant $\beta$ and for all $v, w \in \mathcal{V}$ the following inequality is satisfied

$$\beta\|v - w\|^2 \leq (\mathcal{C}(v) - \mathcal{C}(w), \mathcal{H}(v) - \mathcal{H}(w)),$$

then $\mathcal{C}$ is known as *strongly monotone with respect to* $\mathcal{H}$.

**Definition 1.3.** ([3]) A multi-valued mapping $\mathcal{B} : \mathcal{V} \to 2^{\mathcal{V}}$ is said to be *monotone*, if it satisfies for all $x, y \in \mathcal{V}$, $v \in \mathcal{B}(x)$, $w \in \mathcal{B}(y)$, the following inequality

$$0 \leq (v - w, x - y),$$

it is called *strongly monotone*, if for some positive constant $\eta$, it satisfies the following inequality

$$\eta \|x - y\|^2 \leq (v - w, x - y),$$

and is called *maximal monotone*, if it is monotone and for identity mapping $I$, the following equation is satisfied

$$(I + n\mathcal{B})(\mathcal{V}) = \mathcal{V}, \ \forall \ n > 0.$$

**Definition 1.4.** ([3]) If $\mathcal{H} : \mathcal{V} \to \mathcal{V}$ be a single-valued map, then a multi-valued map $\mathcal{B} : \mathcal{V} \to 2^{\mathcal{V}}$ is said to $\mathcal{H}$-*monotone* if it is monotone and satisfies

$$(\mathcal{H} + n\mathcal{B})(\mathcal{V}) = \mathcal{V} \ for \ every \ n > 0,$$

and it is called *strongly* $\mathcal{H}$-*monotone*, if it is strongly monotone and satisfies

$$(\mathcal{H} + n\mathcal{B})(\mathcal{V}) = \mathcal{V}.$$

**Definition 1.5.** ([9]) Let $\mathcal{H} : \mathcal{V} \to \mathcal{V}$ be a strictly monotone operator and $\mathcal{B} : \mathcal{V} \to 2^{\mathcal{V}}$ be an $\mathcal{H}$-monotone operator. Then for all $v \in \mathcal{V}$ the resolvent operator is defined as:

$$\mathbb{R}^{\mathcal{H}}_{\mathcal{B},n}(v) = [\mathcal{H} + n\mathcal{B}]^{-1}(v). \tag{1.3}$$

Fang and Huang [9] had proved that the resolvent operator given by (1.3) is single- valued and also $\frac{1}{\eta}$ Lipschitzian continuous for $\mathcal{H}$ to be a strongly $\eta$-monotone operator.

Fang and Huang [9] had also proved that the resolvent operator given by (1.3) is single-valued and $\frac{1}{\eta}$-Lipschitzian continuous for $\mathbb{H}$ to be a strongly $\eta$ monotone operator.

## 2. Construction of a problem and its algorithm

In this section, we set up a resolvent inclusion problem and an algorithm for approximating its solution.

Let $\mathcal{C}, \mathbb{H} : \mathcal{V} \to \mathcal{V}$ be two single-valued operators and $\mathcal{B} : \mathcal{V} \to 2^{\mathcal{V}}$ be a muti-valued operator. Then resolvent variational inclusion problem consists in finding $v \in \mathcal{V}$ such that

$$0 \in \mathbb{R}^{\mathbb{H}}_{\mathcal{B},n}(v) + \mathcal{B}(v). \tag{2.1}$$

In short form, it can be written as;

$$0 \in \mathfrak{R}(v) + \mathcal{B}(v), \tag{2.2}$$

where $\mathfrak{R}(v) = \mathbb{R}^{\mathbb{H}}_{\mathcal{B},n}(v)$. If resolvent operator is equal to $\mathcal{C}$, then problem (2.1) becomes problem (1.2). It can be seen very easily that $v$ is a solution of problem (2.1) if and only if

$$v = \mathbb{R}^{\mathbb{H}}_{\mathcal{B},n}[\mathbb{H}(v) - n\mathfrak{R}(v)], \tag{2.3}$$

for details, see [11, 12, 17, 18].

**Algorithm 2.1.** *Based on* (2.3)*, we set an iterative algorithm for problem* (1.2) *as for $v_0 \in \mathcal{V}$, the iterative sequence is given by*

$$v_{m+1} = \mathbb{R}^{\mathbb{H}}_{\mathcal{B},n}[\mathbb{H}(v_m) - n\mathfrak{R}(v_m)], \ \forall \ m \geq 0. \tag{2.4}$$

## 3. Main result

Here, we establish the strong convergence of the sequence generated by iterative algorithm (2.4).

**Theorem 3.1.** *The sequence $\{v_m\}$ generated by algorithm* (2.4) *converges strongly and uniquely to the solution of problem* (2.1)*, if the following conditions are satisfied:*

- *$\mathcal{H} : \mathcal{V} \to \mathcal{V}$ is strongly $\alpha$-monotone and $\beta$-Lipschitz continuous so that $\mathfrak{R}$ is $\frac{1}{\alpha}$-Lipschitz continuous;*
- *$\mathfrak{R}$ is strongly $\gamma$-monotone w.r.t. $\mathcal{H}$;*
- *$\mathcal{B} : \mathcal{V} \to 2^{\mathcal{V}}$ is an $\mathcal{H}$-monotone operator;*
- *$0 \leq \theta < 1$, where $\theta = \frac{1}{\alpha}\sqrt{\beta^2 - 2n\gamma + \frac{n^2}{\alpha^2}}$.*

*Proof.* From (2.4), we can write

$$\begin{aligned} \|v_{m+1} - v_m\| &= \|\mathbb{R}^{\mathcal{H}}_{B,n}[\mathcal{H}(v_m) - n\mathfrak{R}(v_m)] - \mathbb{R}^{\mathcal{H}}_{B,n}[\mathcal{H}(v_{m-1}) - n\mathfrak{R}(v_{m-1})]\| \\ &\leq \frac{1}{\alpha}\|\mathcal{H}(v_m) - \mathcal{H}(v_{m-1}) - n(\mathfrak{R}(v_m) - \mathfrak{R}(v_{m-1}))\|. \end{aligned} \tag{3.1}$$

Since

$$\begin{aligned} &\|\mathcal{H}(v_m) - \mathcal{H}(v_{m-1}) - n(\mathfrak{R}(v_m) - \mathfrak{R}(v_{m-1}))\|^2 \\ &\leq \|\mathcal{H}(v_m) - \mathcal{H}(v_{m-1})\|^2 - 2n\left(\mathfrak{R}(v_m) - \mathfrak{R}(v_{m-1}), \mathcal{H}(v_m) - \mathcal{H}(v_{m-1})\right) \\ &\quad + n^2\|\mathfrak{R}(v_m) - \mathfrak{R}(v_{m-1})\|^2, \end{aligned} \tag{3.2}$$

using Lipschitz continuity of $\mathcal{H}$ and $\mathfrak{R}$, and strongly monotonicity of $\mathfrak{R}$ with respect to $\mathcal{H}$, we obtain

$$\|\mathcal{H}(v_m) - \mathcal{H}(v_{m-1}) - n(\mathfrak{R}(v_m) - \mathfrak{R}(v_{m-1}))\|^2 \leq (\beta^2 - 2n\gamma + \frac{n^2}{\alpha^2})\|v_m - v_{m-1}\|^2. \tag{3.3}$$

Combining (3.1) and (3.3), we obtain

$$\|v_{m+1} - v_m\| \leq \theta\|v_m - v_{m-1}\|, \tag{3.4}$$

where $\theta = \frac{1}{\alpha}\sqrt{\beta^2 - 2n\gamma + \frac{n^2}{\alpha^2}}$. Since $0 \leq \theta < 1$ and $\{v_m\}$ is a Cauchy sequence, so $v_m \to v$ as $m \to \infty$.

For showing uniqueness, consider $v^*$ as other solution of problem (2.1). Then by (2.3)

$$v^* = \mathbb{R}^{\mathcal{H}}_{\mathcal{B},n}[\mathcal{H}(v^*) - n\mathfrak{R}(v^*)].$$

So on similar lines as above we obtain

$$\|v - v^*\| \leq \theta\|v - v^*\|,$$

where $\theta$ is same as above, therefore $v = v^*$. That is $v$ is unique solution of (2.1). □

## 4. Validation of main result by a numerical example

**Example 4.1.** Let us take $\mathbb{R}$ as a real Hilbert space and define the following maps on it:

$$\mathcal{H} : \mathbb{R} \to \mathbb{R} \text{ by } \mathcal{H}(v) = 3v;$$
$$\mathcal{B} : \mathbb{R} \to 2^{\mathbb{R}} \text{ by } \mathcal{B}(v) = \{2v\}.$$

Then

$$(\mathcal{H}(v) - \mathcal{H}(w), v - w) = (3v - 3w, v - w) = 3\|v - w\|^2 \geq 2.97\|v - w\|^2.$$

This means that $\mathcal{H}$ is strongly monotone with constant $\alpha = 2.97$. Also

$$\|\mathcal{H}(v) - \mathcal{H}(w)\| = \|3v - 3w\| \leq 3.05\|v - w\|.$$

That is $\mathcal{H}$ is Lipschitz continuous with constant $\beta = 3.05$.
Now for $n = 1$, resolvent operator becomes

$$\mathbb{R}^{\mathcal{H}}_{\mathcal{B},n}(v) = [3v + 2v]^{-1} = \frac{1}{5}v,$$

which satisfies

$$\|\frac{1}{5}v - \frac{1}{5}w\| = 0.2\|v - w\| \leq \frac{1}{2.97}\|v - w\|.$$

That is, $\mathfrak{R}$ becomes $\frac{1}{\alpha}$ Lipschitz continuous. Now consider

$$\begin{aligned}(\mathfrak{R}(v) - \mathfrak{R}(w), \mathcal{H}(v) - \mathcal{H}(w)) &= \left(\frac{1}{5}v - \frac{1}{5}w, 3v - 3w\right)\\ &= \frac{3}{5}\|v - w\|^2\\ &\geq \frac{2}{5}\|v - w\|^2.\end{aligned}$$

Hence $\mathfrak{R}$ is strongly monotone with respect to $\mathcal{H}$ with constant $\delta = \frac{2}{5}$. So, we can use Theorem 3.1 as all its conditions are satisfied.

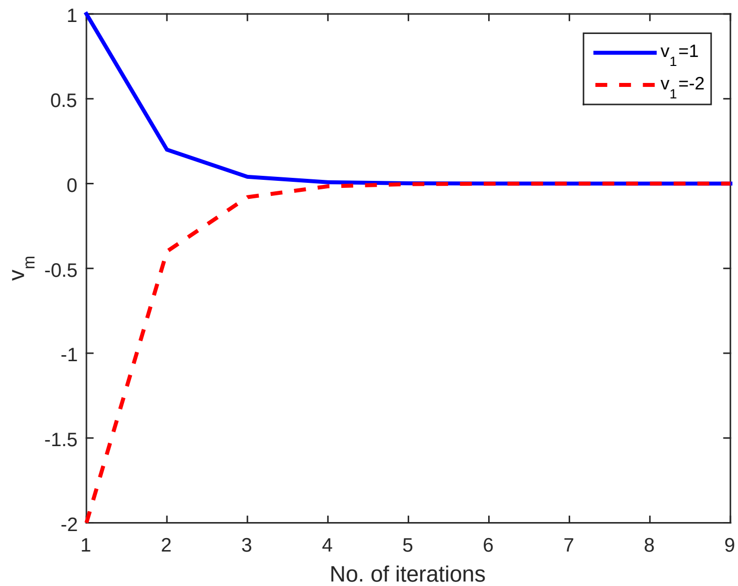


FIGURE 1. The convergence of $\{v_n\}$ with initial values $v_1 = 1$ and $-2$.

TABLE 1. The values of $\{v_n\}$ with initial value $v_1 = 1$ and $v_1 = -2$.

| No.of Iterations | initial value $v_1 = 1$ | initial value $v_1 = -2$ |
|---|---|---|
| 1 | 1.0000 | -2.0000 |
| 2 | 0.2000 | -0.4000 |
| 3 | 0.0400 | -0.0800 |
| 4 | 0.0080 | -0.0160 |
| 5 | 0.0016 | -0.0032 |
| 6 | 0.0003 | -0.0006 |
| 7 | 0.0001 | -0.0001 |
| 8 | 0.0000 | 0.0000 |

## 5. Conclusion

In this paper, we have introduced a resolvent variational inclusion problem. By a simple model problem, we have generated an iterative algorithm and established strong convergence of a solution for the resolvent variational inequality problems under suitable conditions. This papers opens the perspective of enhanced convergence analysis of simple iterative methods for the solution of resolvent variational inclusions.

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