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CONVERGENCE THEOREMS FOR A HYBRID PAIR OF SINGLE-VALUED AND MULTI-VALUED NONEXPANSIVE MAPPING IN CAT(0) SPACES

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Abstract. In this paper, we present a new mixed type iterative process for approximating the common fixed points of single-valued nonexpansive mapping and multi-valued nonexpansive mapping in a CAT(0) space. We demonstrate strong and weak convergence theorems for the new iterative process in CAT(0) spaces, as well as numerical results to support our theorem.

1. Introduction

A geodesic metric space is a metric space $\mathcal{X}$ such that every two points of $\mathcal{X}$ are joined by a geodesic; see for more details in [2, 3, 8, 13].

One of the special spaces of geodesic metric spaces is a CAT(0) space (for further details on the subject, the reader is referred to [3, 9, 15, 20]). The

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useful inequality of CAT(0) space is (CN) inequality [4], that is, if z, x, y are points in a CAT(0) space and if y_0 is the midpoint of the segment $[x, y]$, then the CAT(0) inequality implies

$$d^2(z, y_0) \leq \frac{1}{2}d^2(z, x) + \frac{1}{2}d^2(z, y) - \frac{1}{4}d^2(x, y). \qquad (CN)$$

Kirk [10] obtained this result in the nonlinear setting of CAT(0) spaces in 2003. Every nonexpansive map does not have to be contracted. In this case, the study of fixed points on nonexpansive maps is more difficult but more important than the study of contraction maps (see [1, 11, 14, 16, 17, 18, 19]).

Let $\mathcal{E}$ be a nonempty closed convex subset of a metric space $(\mathcal{X}, d)$ and let $\mathcal{S} : \mathcal{E} \to \mathcal{E}$ be a single-valued mapping. An element $p \in \mathcal{E}$ is called a fixed point of $\mathcal{S}$ if $p = \mathcal{S}p$. The set of all fixed points of $\mathcal{S}$ is denoted by $\mathscr{F}(\mathcal{S})$, that is, $\mathscr{F}(\mathcal{S}) = \{p \in \mathcal{E} : p = \mathcal{S}p\}$.

A single-valued mapping $\mathcal{S} : \mathcal{E} \to \mathcal{E}$ is said to be

- nonexpansive if

$$d(\mathcal{S}x, \mathcal{S}y) \leq d(x, y), \quad \forall x, y \in \mathcal{E}.$$

- semi-compact if for any sequence $\{x_n\}$ in $\mathcal{E}$ with

$$\lim_{n\to\infty} d(x_n, \mathcal{S}x_n) = 0,$$

  there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $p \in \mathcal{E}$.

Let $\mathcal{X} := (\mathcal{X}, d)$ be a metric space and let $\mathcal{E}$ be a nonempty closed convex subset of $\mathcal{X}$. Denote by $\mathcal{KC}(\mathcal{E})$ ($\mathcal{BC}(\mathcal{E})$, respectively) the set of all nonempty compact convex subsets (nonempty bounded and closed subsets, respectively) of $\mathcal{X}$. Let $\mathcal{H}$ be the Pompeiu-Hausdorff distance on $\mathcal{BC}(\mathcal{E})$, that is, for $\mathcal{A}$, $\mathcal{B} \in \mathcal{BC}(\mathcal{E})$

$$\mathcal{H}(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{x\in\mathcal{A}} \operatorname{dist}(x, \mathcal{B}), \sup_{y\in\mathcal{B}} \operatorname{dist}(y, \mathcal{A})\right\},$$

where $\operatorname{dist}(x, \mathcal{E}) = \inf\{d(x, y) : y \in \mathcal{E}\}$ is the distance from a point x to a subset $\mathcal{E}$.

A point $p \in \mathcal{E}$ is called a fixed point of a multi-valued mapping $\mathcal{T} : \mathcal{E} \to \mathcal{BC}(\mathcal{E})$ if $p \in \mathcal{T}p$.

A multi-valued mapping $\mathcal{T} : \mathcal{E} \to \mathcal{BC}(\mathcal{E})$ is said to be

- nonexpansive if

$$\mathcal{H}(\mathcal{T}x, \mathcal{T}y) \leq d(x, y), \quad \forall x, y \in \mathcal{E}.$$

- hemi-compact if for any sequence $\{x_n\}$ in $\mathcal{E}$ with
$$\lim_{n\to\infty} \text{dist}(x_n, \mathcal{T}x_n) = 0,$$
there exists a strongly convergent subsequence $\{x_{n_i}\}$ to $p \in \mathcal{E}$.

2. Preliminaries

Let $\{x_n\}$ be a bounded sequence in a metric space $\mathcal{X}$. For $x \in \mathcal{X}$, we set
$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$
The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by
$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in \mathcal{X}\},$$
and the asymptotic center $AC(\{x_n\})$ of $\{x_n\}$ is the set
$$AC(\{x_n\}) = \{x \in \mathcal{X} : r(x, \{x_n\}) = r(\{x_n\})\}.$$
It is known that in a complete CAT(0) space, $AC(\{x_n\})$ consists of exactly one point (see [6, Proposition 7]).

We now give the definition of Δ-convergence of a sequence .

Definition 2.1. ([10, 12]) A sequence $\{x_n\}$ in a metric space $\mathcal{X}$ is said to be Δ-convergent to $x \in \mathcal{X}$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.2. ([10]) *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 2.3. ([5]) *If $\mathcal{E}$ is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in $\mathcal{E}$, then the asymptotic center of $\{x_n\}$ is in $\mathcal{E}$.*

Lemma 2.4. ([7]) *Let $\mathcal{X}$ be a CAT(0) space.*

(i) *For $x, y, z \in \mathcal{X}$ and $t \in [0, 1]$, we have*
$$d((1-t)x \oplus ty, z) \le (1-t)d(x, z) + td(y, z).$$
(ii) *For $x, y, z \in \mathcal{X}$ and $t \in [0, 1]$, we have*
$$d^2((1-t)x \oplus ty, z) \le (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y).$$

Lemma 2.5. ([7]) *Let $\mathcal{E}$ be a nonempty closed convex subset of a complete CAT(0) space $\mathcal{X}$ and $\mathcal{S} : \mathcal{E} \to \mathcal{E}$ be a nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in $\mathcal{E}$ such that $\lim_{n\to\infty} d(x_n, \mathcal{S}x_n) = 0$ and Δ-$\lim_{n\to\infty} x_n = p$ then $p = \mathcal{S}p$.*

Lemma 2.6. ([7]) *Let $\{x_n\}$ be a sequence in a CAT(0) space $\mathcal{X}$ with $AC(\{x_n\}) = \{x\}$. If $\{u_n\}$ is a subsequence of $\{x_n\}$ with $AC(\{u_n\}) = \{u\}$ and $\{d(x_n, u)\}$ converges, then $x = u$.*

3. Main Results

Firstly, denote $\mathscr{F} = \mathscr{F}(\mathcal{S}) \cap \mathscr{F}(\mathcal{T})$ is the set of all common fixed points of the mappings $\mathcal{S}$ and $\mathcal{T}$.

Theorem 3.1. *Let $\mathcal{E}$ be a nonempty closed convex subset of a complete CAT(0) space $\mathcal{X}$. Let $\mathcal{S} : \mathcal{E} \to \mathcal{E}$ be a single-valued nonexpansive mapping and $\mathcal{T} : \mathcal{E} \to \mathcal{BC}(\mathcal{E})$ be a multi-valued nonexpansive mapping. Suppose that $\mathscr{F}$ is nonempty and $\mathcal{T}p = \{p\}$ for all $p \in \mathscr{F}$. The sequence $\{x_n\}$ is defined by*

$$\begin{cases} x_1 \in \mathcal{E}, \\ g_n = \gamma_n x_n \oplus (1-\gamma_n) w_n, \quad w_n \in \mathcal{T}x_n, \\ h_n = \beta_n g_n \oplus (1-\beta_n) z_n, \quad z_n \in \mathcal{T}g_n, \\ x_{n+1} = \alpha_n \mathcal{S}h_n \oplus (1-\alpha_n)\mathcal{S}g_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequence in $[0,1]$ such that satisfying

(a) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Then, we have the following:

(i) $\lim_{n\to\infty} d(x_n, p)$ *exists for all* $p \in \mathscr{F}$;
(ii) $\lim_{n\to\infty} d(x_n, \mathcal{S}x_n) = 0$;
(ii) $\lim_{n\to\infty} \operatorname{dist}(x_n, \mathcal{T}x_n) = 0$.

Proof. (i) Let $p \in \mathscr{F}$, by $\mathcal{T}p = \{p\}$, we get

$$\begin{aligned} d(g_n, p) &= d(\gamma_n x_n \oplus (1-\gamma_n) w_n, p) \\ &\leq \gamma_n d(x_n, p) + (1-\gamma_n) d(w_n, p) \\ &\leq \gamma_n d(x_n, p) + (1-\gamma_n) \operatorname{dist}(p, \mathcal{T}x_n) \\ &\leq \gamma_n d(x_n, p) + (1-\gamma_n) \mathcal{H}(\mathcal{T}p, \mathcal{T}x_n) \\ &\leq \gamma_n d(x_n, p) + (1-\gamma_n) d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{3.2}$$

Also

$$\begin{aligned} d(h_n, p) &= d(\beta_n g_n \oplus (1-\beta_n) z_n, p) \\ &\le \beta_n d(g_n, p) + (1-\beta_n) d(z_n, p) \\ &\le \beta_n d(g_n, p) + (1-\beta_n) \mathrm{dist}(p, \mathcal{T} g_n) \\ &\le \beta_n d(g_n, p) + (1-\beta_n) \mathcal{H}(\mathcal{T} p, \mathcal{T} g_n) \\ &\le \beta_n d(g_n, p) + (1-\beta_n) d(g_n, p) \\ &= d(g_n, p). \end{aligned} \tag{3.3}$$

Using (3.1) and (3.2), we obtain that

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n \mathcal{S} h_n \oplus (1-\alpha_n) \mathcal{S} g_n, p) \\ &\le \alpha_n d(\mathcal{S} h_n, p) + (1-\alpha_n) d(\mathcal{S} g_n, p) \\ &\le \alpha_n d(\mathcal{S} h_n, \mathcal{S} p) + (1-\alpha_n) d(\mathcal{S} g_n, \mathcal{S} p) \\ &\le \alpha_n d(h_n, p) + (1-\alpha_n) d(g_n, p) \\ &\le \alpha_n d(g_n, p) + (1-\alpha_n) d(g_n, p) \\ &= d(g_n, p) \\ &\le d(x_n, p). \end{aligned} \tag{3.4}$$

This implies that $\{d(x_n, p)\}$ is decreasing and bounded below, thus $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \mathscr{F}$.

(ii) By (i), we put

$$\lim_{n\to\infty} d(x_n, p) = \lambda, \quad \text{for some } \lambda. \tag{3.5}$$

From (3.4), we get

$$d(x_{n+1}, p) \le d(g_n, p) \le d(x_n, p).$$

So,

$$\lim_{n\to\infty} d(x_{n+1}, p) \le \lim_{n\to\infty} d(g_n, p) \le \lim_{n\to\infty} d(x_n, p).$$

This implies that

$$\lim_{n\to\infty} d(g_n, p) = \lambda. \tag{3.6}$$

Using condition (a) and (3.4), we get

$$\begin{aligned} d(x_{n+1}, p) &\le \alpha_n d(g_n, p) + (1-\alpha_n) d(g_n, p) \\ &\le \alpha_n d(x_n, p) + (1-\alpha_n) d(g_n, p) \\ &= \alpha_n d(x_n, p) + d(x_n, p) - d(x_n, p) + (1-\alpha_n) d(g_n, p) \\ &= d(x_n, p) - (1-\alpha_n) d(x_n, p) + (1-\alpha_n) d(g_n, p). \end{aligned}$$

Then

$$(1-\alpha_n) d(x_n, p) \le d(x_n, p) - d(x_{n+1}, p) + (1-\alpha_n) d(g_n, p)$$

and change it as

$$d(x_n, p) \leq \frac{1}{1-\alpha_n}\left[d(x_n, p) - d(x_{n+1}, p)\right] + d(g_n, p).$$

This implies that

$$\lambda = \liminf_{n\to\infty} d(x_n, p) \leq \liminf_{n\to\infty} d(g_n, p). \tag{3.7}$$

From (3.2), we get

$$\limsup_{n\to\infty} d(g_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = \lambda. \tag{3.8}$$

Hence, by (3.7) and (3.8), we get

$$\lim_{n\to\infty} d(g_n, p) = \lambda. \tag{3.9}$$

Again, by (3.4), we obtain that

$$d(x_{n+1}, p) \leq \alpha_n d(h_n, p) + (1-\alpha_n) d(g_n, p)$$

and also

$$d(h_n, p) \geq \frac{1}{\alpha_n}\left[d(x_{n+1}, p) - (1-\alpha_n) d(g_n, p)\right].$$

This implies that

$$\liminf_{n\to\infty} d(h_n, p) \geq \liminf_{n\to\infty} \left\{\frac{1}{\alpha_n}\left[d(x_{n+1}, p) - (1-\alpha_n) d(g_n, p)\right]\right\} = \lambda. \tag{3.10}$$

From (3.2), we get

$$\limsup_{n\to\infty} d(h_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = \lambda. \tag{3.11}$$

Hence, by (3.10) and (3.11), we get

$$\lim_{n\to\infty} d(h_n, p) = \lambda. \tag{3.12}$$

Using Lemma 2.4 (ii), $\mathcal{T}p = \{p\}$ and (3.1), we get

$$\begin{aligned} d^2(g_n, p) &= d^2(\gamma_n x_n \oplus (1-\gamma_n) w_n, p) \\ &\leq \gamma_n d^2(x_n, p) + (1-\gamma_n) d^2(w_n, p) - \gamma_n(1-\gamma_n) d^2(x_n, w_n) \\ &\leq \gamma_n d^2(x_n, p) + (1-\gamma_n) \mathrm{dist}^2(p, \mathcal{T}x_n) - \gamma_n(1-\gamma_n) d^2(x_n, w_n) \\ &\leq \gamma_n d^2(x_n, p) + (1-\gamma_n) \mathcal{H}^2(\mathcal{T}p, \mathcal{T}x_n) - \gamma_n(1-\gamma_n) d^2(x_n, w_n) \\ &\leq \gamma_n d^2(x_n, p) + (1-\gamma_n) d^2(x_n, p) - \gamma_n(1-\gamma_n) d^2(x_n, w_n) \\ &= d^2(x_n, p) - \gamma_n(1-\gamma_n) d^2(x_n, w_n). \end{aligned} \tag{3.13}$$

Also,

$$\begin{aligned} d^2(h_n,p) &= d^2(\beta_n g_n \oplus (1-\beta_n)z_n, p) \\ &\leq \beta_n d^2(g_n,p) + (1-\beta_n)d^2(z_n,p) - \beta_n(1-\beta_n)d^2(g_n,z_n) \\ &\leq \beta_n d^2(g_n,p) + (1-\beta_n)\mathrm{dist}^2(p,\mathcal{T}g_n) - \beta_n(1-\beta_n)d^2(g_n,z_n) \\ &\leq \beta_n d^2(g_n,p) + (1-\beta_n)\mathcal{H}^2(\mathcal{T}p,\mathcal{T}g_n) - \beta_n(1-\beta_n)d^2(g_n,z_n) \\ &\leq \beta_n d^2(g_n,p) + (1-\beta_n)d^2(g_n,p) - \beta_n(1-\beta_n)d^2(g_n,z_n) \\ &= d^2(g_n,p) - \beta_n(1-\beta_n)d^2(g_n,z_n) \\ &\leq d^2(x_n,p) - \beta_n(1-\beta_n)d^2(g_n,z_n). \end{aligned} \tag{3.14}$$

Using (3.13) and (3.14), we obtain that

$$\begin{aligned} d^2(x_{n+1},p) &= d^2(\alpha_n \mathcal{S}h_n \oplus (1-\alpha_n)\mathcal{S}g_n, p) \\ &\leq \alpha_n d^2(\mathcal{S}h_n,p) + (1-\alpha_n)d^2(\mathcal{S}g_n,p) - \alpha_n(1-\alpha_n)d(\mathcal{S}h_n,\mathcal{S}g_n) \\ &\leq \alpha_n d^2(h_n,p) + (1-\alpha_n)d^2(g_n,p) - \alpha_n(1-\alpha_n)d(\mathcal{S}h_n,\mathcal{S}g_n) \\ &\leq \alpha_n d^2(g_n,p) + (1-\alpha_n)d^2(g_n,p) - \alpha_n(1-\alpha_n)d(\mathcal{S}h_n,\mathcal{S}g_n) \\ &= d^2(g_n,p) - \alpha_n(1-\alpha_n)d(\mathcal{S}h_n,\mathcal{S}g_n) \\ &\leq d^2(x_n,p) - \alpha_n(1-\alpha_n)d(\mathcal{S}h_n,\mathcal{S}g_n). \end{aligned} \tag{3.15}$$

Using condition (a)-(c), (3.5), (3.9) and (3.12), we get

$$\begin{aligned} 0 &\leq \gamma_n(1-\gamma_n)d^2(x_n,w_n) \\ &\leq d^2(x_n,p) - d^2(g_n,p) \\ &\to 0 \quad \text{as} \quad n\to\infty, \end{aligned}$$

$$\begin{aligned} 0 &\leq \beta_n(1-\beta_n)d^2(g_n,z_n) \\ &\leq d^2(x_n,p) - d^2(h_n,p) \\ &\to 0 \quad \text{as} \quad n\to\infty \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \alpha_n(1-\alpha_n)d(\mathcal{S}h_n,\mathcal{S}g_n) \\ &\leq d^2(x_n,p) - d^2(x_{n+1},p) \\ &\to 0 \quad \text{as} \quad n\to\infty. \end{aligned}$$

Hence, we obtain that

$$\lim_{n\to\infty} d(x_n,w_n) = \lim_{n\to\infty} d(g_n,z_n) = \lim_{n\to\infty} d(\mathcal{S}h_n,\mathcal{S}g_n) = 0. \tag{3.16}$$

Indeed, because $g_n = \gamma_n x_n \oplus (1-\gamma_n) w_n$ and $h_n = \beta_n g_n \oplus (1-\beta_n) z_n$, we obtain that

$$\begin{aligned} d(g_n, x_n) &= d(\gamma_n x_n \oplus (1-\gamma_n) w_n, x_n) \\ &\le \gamma_n d(x_n, x_n) + (1-\gamma_n) d(w_n, x_n) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.17}$$

Also,

$$\begin{aligned} d(h_n, x_n) &= d(\beta_n g_n \oplus (1-\beta_n) z_n, x_n) \\ &\le \beta_n d(g_n, x_n) + (1-\beta_n) d(z_n, x_n) \\ &\le \beta_n d(g_n, x_n) + (1-\beta_n)\left[d(z_n, g_n) + d(g_n, x_n)\right] \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.18}$$

By the nonexpansiveness of $\mathcal{S}$, (3.16), (3.17) and (3.18) shows that

$$\begin{aligned} d(x_n, \mathcal{S}x_n) &\le d(x_n, g_n) + d(\mathcal{S}g_n, \mathcal{S}h_n) + d(\mathcal{S}h_n, \mathcal{S}x_n) \\ &\le d(x_n, g_n) + d(\mathcal{S}h_n, \mathcal{S}g_n) + d(h_n, x_n) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.19}$$

Hence, we have

$$\lim_{n\to\infty} d(x_n, \mathcal{S}x_n) = 0.$$

(iii) Because of nonexpansiveness of $\mathcal{T}$, also from (3.16) and (3.17), we get

$$\begin{aligned} \operatorname{dist}(x_n, \mathcal{T}x_n) &\le d(x_n, g_n) + \operatorname{dist}(g_n, \mathcal{T}g_n) + \mathcal{H}(\mathcal{T}g_n, \mathcal{T}x_n) \\ &\le d(x_n, g_n) + \operatorname{dist} d(g_n, \mathcal{T}g_n) + d(g_n, x_n) \\ &= 2d(x_n, g_n) + \operatorname{dist}(g_n, \mathcal{T}g_n) \\ &= 2d(x_n, g_n) + d(g_n, z_n) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} \operatorname{dist}(x_n, \mathcal{T}x_n) = 0.$$

This completes the proof. □

Theorem 3.2. *Let $\mathcal{E}$ be a nonempty closed convex subset of a complete CAT(0) space $\mathcal{X}$. Let $\mathcal{S} : \mathcal{E} \to \mathcal{E}$ be a single-valued nonexpansive mapping and $\mathcal{T} : \mathcal{E} \to \mathcal{KC}(\mathcal{E})$ be a multi-valued nonexpansive mapping. Suppose that $\mathscr{F}$ is nonempty and $\mathcal{T}p = \{p\}$ for all $p \in \mathscr{F}$. For $x_1 \in \mathcal{E}$, the sequence $\{x_n\}$ generated by* (3.1), *where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequence in $[0, 1]$ such that satisfying*

(a) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(b) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(c) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$.

Then the sequence $\{x_n\}$ is Δ-convergent to a point in $\mathscr{F}$.

Proof. Let $\omega_\Delta(x_n) := \bigcup AC(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Let $q \in \omega_\Delta(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $AC(\{u_n\}) = \{q\}$. Using Lemma 2.2 and Lemma 2.3, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that

$$\Delta - \lim_{n\to\infty} v_n = v \in \mathcal{E}. \tag{3.20}$$

From Theorem 3.1 (ii), we have

$$\lim_{n\to\infty} d(v_n, \mathcal{S}v_n) = 0.$$

Then, by the nonexpansiveness of $\mathcal{S}$, it implies by Lemma 2.5 that $v = \mathcal{S}v$. Thus, we get

$$v \in \mathscr{F}(\mathcal{S}). \tag{3.21}$$

Since $\mathcal{T}$ is compact valued, for each $n \in \mathbb{N}$, there exist $r_n \in \mathcal{T}v_n$ and $\vartheta_n \in \mathcal{T}v$ such that $d(v_n, r_n) = \text{dist}(v_n, \mathcal{T}v_n)$ and $d(r_n, \vartheta_n) = \text{dist}(r_n, \mathcal{T}v)$. By Theorem 3.1 (iii), it follows that

$$\lim_{n\to\infty} d(v_n, r_n) = 0.$$

By the compactness of $\mathcal{T}v$, so there exists a subsequence $\{\vartheta_{n_i}\}$ of $\{\vartheta_n\}$ such that $\lim_{i\to\infty} \vartheta_{n_i} = \vartheta \in \mathcal{T}v$. Then we have

$$\begin{aligned}
\limsup_{i\to\infty} d(v_{n_i}, \vartheta) &\le \limsup_{i\to\infty}(d(v_{n_i}, r_{n_i}) + d(r_{n_i}, \vartheta_{n_i}) + d(\vartheta_{n_i}, \vartheta)) \\
&\le \limsup_{i\to\infty}(d(v_{n_i}, r_{n_i}) + \text{dist}(r_{n_i}, \mathcal{T}v) + d(\vartheta_{n_i}, \vartheta)) \\
&\le \limsup_{i\to\infty}(d(v_{n_i}, r_{n_i}) + \mathcal{H}(\mathcal{T}v_{n_i}, \mathcal{T}v) + d(\vartheta_{n_i}, \vartheta)) \\
&\le \limsup_{i\to\infty}(d(v_{n_i}, r_{n_i}) + d(v_{n_i}, v) + d(\vartheta_{n_i}, \vartheta)) \\
&= \limsup_{i\to\infty} d(v_{n_i}, v).
\end{aligned}$$

By (3.20) and the uniqueness of asymptotic centers, we obtain $v = \vartheta \in \mathcal{T}v$. Thus, by (3.21), we have

$$v \in \mathscr{F}(\mathcal{S}) \cap \mathscr{F}(\mathcal{T}) = \mathscr{F}.$$

It follows by Theorem 3.1 and Lemma 2.5, we have $\omega_\Delta(x_n) \subseteq \mathscr{F}$.

Next, we show that $\{x_n\}$ is Δ-convergent to a point in $\mathscr{F}$. Suppose that $\{u_n\}$ is a subsequence of $\{x_n\}$ with $AC(\{u_n\}) = \{u^*\}$ and $AC(\{x_n\}) = \{x\}$. Since $u^* \in \omega_\Delta(x_n) \subseteq \mathscr{F}$ and $\{d(x_n, u^*)\}$ converges, it implies from Lemma 2.6 that $x = u^*$, which shows that $\omega_\Delta(x_n)$ consists of exactly one point. This implies that $\{x_n\}$ is Δ-convergent to a point in $\mathscr{F}$. □

Theorem 3.3. *Let $\mathcal{E}$ be a nonempty closed convex subset of a complete CAT(0) space $\mathcal{X}$. Let $\mathcal{S} : \mathcal{E} \to \mathcal{E}$ be a single-valued nonexpansive mapping and $\mathcal{S} : \mathcal{E} \to \mathcal{BC}(\mathcal{E})$ be a multi-valued nonexpansive mapping. Suppose that $\mathscr{F}$ is nonempty and $\mathcal{T}p = \{p\}$ for all $p \in \mathscr{F}$. For $x_1 \in \mathcal{E}$, the sequence $\{x_n\}$ generated by (3.1), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequence in $[0, 1]$ such that satisfying*

(a) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

If $\mathcal{S}$ is semi-compact or $\mathcal{T}$ is hemi-compact, then the sequence $\{x_n\}$ converges strongly to a point in $\mathscr{F}$.

Proof. Without loss of generality, we assume $\mathcal{T}$ is hemi-compact. Using Theorem 3.1 (iii), $\lim_{n\to\infty} \operatorname{dist}(x_n, \mathcal{T}x_n) = 0$. Then, by hemi-compactness of $\mathcal{T}$, there exists a subsequence $\{\delta_n\}$ of $\{x_n\}$ which converges strongly to p in $\mathcal{E}$.

Using Theorem 3.1, we have $\lim_{n\to\infty} d(\delta_n, \mathcal{S}\delta_n) = 0$, $\lim_{n\to\infty} \operatorname{dist}(\delta_n, \mathcal{T}\delta_n) = 0$. It follows from nonexpansiveness of $\mathcal{S}$ that $p = \mathcal{S}p$. Then, we get

$$p \in \mathscr{F}(\mathcal{S}). \tag{3.22}$$

By nonexpansiveness of $\mathcal{T}$, we obtain that

$$\begin{aligned} \operatorname{dist}(p, \mathcal{T}p) &\leq d(p, \delta_n) + \operatorname{dist}(\delta_n, \mathcal{T}\delta_n) + \mathcal{H}(\mathcal{T}\delta_n, \mathcal{T}p) \\ &\leq 2d(p, \delta_n) + \operatorname{dist}(\delta_n, \mathcal{T}\delta_n) \\ &\to \infty \quad \text{as} \quad n \to \infty. \end{aligned}$$

This implies that $\operatorname{dist}(p, \mathcal{T}p) = 0$, that is, $p \in \mathcal{T}p$. Hence, $p \in \mathscr{F}(\mathcal{T})$. Thus, using (3.22), we get

$$p \in \mathscr{F}(\mathcal{S}) \cap \mathscr{F}(\mathcal{T}) = \mathscr{F}.$$

Using double extract subsequence principle, we can conclude that the sequence $\{x_n\}$ converges strongly to a point p in $\mathscr{F}$. □

4. Numerical example

Let $\mathcal{X} = \mathbb{R}^2$ be a Euclidean metric space and $\mathcal{E} = \{x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^2 : 0 \leq x^{(1)}, x^{(2)} \leq 1\}$. For each $x = (x^{(1)}, x^{(2)}) \in \mathcal{E}$, we define mappings $\mathcal{S}$ and $\mathcal{T}$ on $\mathcal{E}$ as follows:

$$\mathcal{S}x = \left(\frac{3x^{(1)} + 2}{7}, \frac{x^{(2)} + 2}{3} \right) \quad \text{and} \quad \mathcal{T}x = \{x^{(1)}\} \times \left[\frac{x^{(2)} + 3}{4}, 1 \right],$$

which $\mathcal{S}$ and $\mathcal{T}$ are nonexpansive.

Let $x_n = (x_n^{(1)}, x_n^{(2)})$, $g_n = (g_n^{(1)}, g_n^{(2)})$ and $h_n = (h_n^{(1)}, h_n^{(2)})$ are points in $\mathbb{R}^2$. We took $w_n = \left(x_n^{(1)}, \frac{x_1^{(2)} + 3}{4} \right) \in \mathcal{T}x_n$, $\alpha_n = \frac{4n}{\sqrt{64n^2 + 5}}$, $\beta_n = \frac{3n - 1}{\sqrt{36n^2 + 10}}$

and $\gamma_n = \dfrac{5n-1}{\sqrt{49n^2+15}}$ in iterative scheme (3.1) with the initial point $x_1 = (0.1, 0.1)$, we have numerical results in Table 1.

Table 1. Numerical results of iterative scheme (3.1)

n	$x_n = (x_n^{(1)}, x_n^{(2)})$	$\|x_n - x_{n-1}\|_2$
1	(0.1, 0.1)	–
2	(0.328571, 0.846359)	0.780574
3	(0.426531, 0.968569)	0.156625
4	(0.468513, 0.993228)	0.048689
5	(0.486506, 0.998507)	0.018751
6	(0.494217, 0.999666)	0.007798
7	(0.497521, 0.999925)	0.003315
8	(0.498938, 0.999983)	0.001418
9	(0.499545, 0.999996)	0.000607
10	(0.499805, 0.999999)	0.000260
11	(0.499916, 1.000000)	0.000111
12	(0.499964, 1.000000)	0.000048
13	(0.499985, 1.000000)	0.000020
14	(0.499993, 1.000000)	0.000009
15	(0.499997, 1.000000)	0.000004
16	(0.499999, 1.000000)	0.000002
17	(0.499999, 1.000000)	0.000001

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