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A GENERAL ITERATIVE METHOD FOR VARIATIONAL INCLUSION PROBLEMS AND FIXED POINT PROBLEMS IN HILBERT SPACES

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Abstract. In this paper, we introduce two iterative schemes by the general iterative method for finding a common element of the set of fixed points of a strictly pseudo-contractive mapping and the set of solutions of a variational inclusion for an $\alpha-$inverse-strongly monotone mapping and a maximal monotone mapping in a Hilbert space. Our results improve and extend the corresponding results announced by many others.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H, let $F : C \times C \to \mathbb{R}$ be a bifunction. Let $A : H \to H$ be a single-valued mapping and $M : H \to 2^H$ be a multivalued mapping. Then, we consider the following variational inclusion problem which is to find $u \in H$ such that

$$0 \in A(u) + M(u). \tag{1.1}$$

The set of solutions of the variational inclusion (1.1) is denoted by $VI(H, A, M)$.
Special Cases.

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(1) When M is a maximal monotone mapping and A is a strongly monotone and Lipschitz continuous mapping, problem (1.1) has been studied by Huang [8].

(2) If $M = \partial\phi$, where $\partial\phi$ denotes the subdifferential of a proper, convex and lower semi-continuous function $\phi : H \to \mathbb{R}\bigcup\{+\infty\}$, then problem (1.1) reduces to the following problem: find $u \in H$,such that

$$\langle A(u), v-u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{1.2}$$

which is called a nonlinear variational inequality and has been studied by many authors; see, for example, [2-3].

(3) If $M = \partial\delta_C$, where δ_C is the indicator function of C, then problem (1.1) reduces to the following problem: find $u \in C$, such that

$$\langle A(u), v-u\rangle \geq 0, \quad \forall v \in C, \tag{1.3}$$

which is the classical variational inequality; see, e.g.,[7,9] and the reference therein. A mapping $A : H \to H$ is called inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2, \quad \forall x, y \in H.$$

Such a mapping A is also called α-inverse-strongly monotone. If A is an α-inverse-strongly monotone mapping of H to H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that for all $x, y \in H$, and $\lambda > 0$,

$$\begin{aligned} \|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|(x-y)-\lambda(Ax-Ay)\|^2 \\ &= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle \\ &\quad +\lambda^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 + \lambda(\lambda-2\alpha)\|Ax-Ay\|^2. \end{aligned} \tag{1.4}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of H into H. See [9] for some examples of inverse-strongly monotone mappings.

A mapping T of C into itself is nonexpansive if $\|Tx-Ty\| \leq \|x-y\|, \forall x, y \in C$. Recently, Iiduka and Takahashi [9], Takahashi and Toyoda [15], Chen et al. [6] , Nadezhkina and Takahashi [13], Ceng and Yao [4], Yao and Yao [17] introduced many iterative methods for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality (1.3) for an α-inverse-strongly monotone mapping, they obtained some weak and strong convergence theorems.

Recall that a self-mapping $f : C \to C$ is a contraction on C if there is a constant $\beta \in (0,1)$ such that

$$\|f(x)-f(y)\| \leq \beta\|x-y\|, \quad \forall x, y \in C.$$

An operator B is strongly positive if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Bx, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.5}$$

In 2006, Marino and Xu [12] introduced the general iterative method and proved that for given $x_0 \in H$, the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)Tx_n, \quad n \in \mathbb{N},$$

where T is a self-nonexpansive mapping on H, f is a contraction of H into itself with $\beta \in (0,1)$ and $\{\alpha_n\} \subseteq (0,1)$ satisfies certain conditions, B is a strongly positive bounded linear operator on H, converges strongly to a fixed point x^* of T which is the unique solution to the following variational inequality:

$$\langle (B - \gamma f)x^*, x^* - x\rangle \leq 0, \quad \forall x \in F(T),$$

and is also the optimality condition for some minimization problem.

A mapping $S : C \to H$ is said to be $k-$strictly pseudo-contractive if there exists a constant $k \in [0,1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{1.6}$$

Note that the class of $k-$strict pseudo-contractions strictly includes the class of nonexpansive mappings. That is, S is nonexpansive if and only if S is 0-strictly pseudo-contractive. It is also said to be pseudo-contractive if $k = 1$. Clearly, the class of k-strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractions.

The set of fixed points of S is denoted by $F(S)$. Very recently, by using the general approximation method Liu [10] obtained two strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a $k-$strictly pseudo-contractive non-self mapping.

In this paper, motivated and inspired by the above results, we introduce two iteration schemes for finding an element of $VI(H, A, M) \bigcap F(S)$, where $S : H \to H$ is a k-strict pseudocontraction, and $A : H \to H$ is an inverse-strongly monotone mapping and then obtain two strong convergence theorems.

2. Preliminaries

Throughout this paper, we always let X be a real Banach space with dual space X^*, H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively.

It is also known that H satisfies Opial's condition [13], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

A set-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H, u \in Mx, v \in My$ imply $\langle x - y, u - v \rangle \geq 0$. A monotone mapping $M : H \to 2^H$ is maximal if the graph $G(M)$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, u) \in H \times H, \langle x - y, u - v \rangle \geq 0$ for every $(y, v) \in G(M)$ implies $u \in Mx$.

The following definitions and lemmas are useful for our paper.

Definition 2.1. ([14]) If M is a maximal monotone mapping on H, then the resolvent operator associated with M is defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}u, \quad \forall u \in H,$$

where $\lambda > 0$ is a constant and I is the identity operator.

Definition 2.2. ([14]) A single-valued operator $A : H \to H$ is said to be hemi-continuous if for any fixed $x, y, z \in H$, the function $t \to \langle A(x + ty), z \rangle$ is continuous at 0^+. It is well known that a continuous mapping must be hemi-continuous.

Definition 2.3. ([14]) A set-valued mapping $A : X \to 2^{X^*}$ is said to be bounded if $A(B)$ is bounded for every bounded subset B of X.

Lemma 2.4. ([11]) *The resolvent operator $J_{M,\lambda}$ is firmly nonexpansive, that is*

$$\langle J_{M,\lambda}u - J_{M,\lambda}v, u - v \rangle \geq \|J_{M,\lambda}u - J_{M,\lambda}v\|^2, \quad \forall u, v \in H.$$

Lemma 2.5. ([14]) *If $T : X \to 2^{X^*}$ is a maximal monotone mapping and $P : X \to X^*$ is a hemi-continuous bounded monotone operator with $D(P) = X$, then the sum $S = T + P$ is a maximal monotone mapping.*

Lemma 2.6. ([1]) *Let $S : C \to H$ be a k-strict pseudo-contraction. Define $T : C \to H$ by $Tx = \lambda x + (1 - \lambda)Sx$ for each $x \in C$. Then, as $\lambda \in [k, 1)$, T is a nonexpansive mapping such that $F(T) = F(S)$.*

Lemma 2.7. ([16]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty}\gamma_n=\infty$; (ii) $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n}\le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n|<\infty$.
Then $\lim_{n\to\infty}a_n=0$.

Lemma 2.8. ([5]) *The following inequality holds in a Hilbert space,*

$$\|x+y\|^2\le\|x\|^2+2\langle y,(x+y)\rangle,\quad \forall x,y\in H.$$

Lemma 2.9. *The function $u\in H$ is a solution of variational inclusion* (1.1) *if and only if $u\in H$ satisfies the relation*

$$u=J_{M,\lambda}[u-\lambda Au],$$

where $\lambda>0$ is a constant, M is a maximal monotone mapping and $J_{M,\lambda}=(I+\lambda M)^{-1}$ is the resolvent operator.

Proof. Using Definition 2.1, we can obtain the desired result. □

Lemma 2.10. ([12]) *Assume that B is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar\gamma>0$ and $0<\rho\le\|B\|^{-1}$. Then $\|I-\rho B\|\le 1-\rho\bar\gamma$.*

Lemma 2.11. ([12]) *Let H be a Hilbert space and $f:H\to H$ be a contraction with coefficient $0<\beta<1$, and B be a strongly positive linear bounded operator with coefficient $\bar\gamma>0$. Then, for $0<\gamma<\frac{\bar\gamma}{\beta}$,*

$$\langle x-y,(B-\gamma f)x-(B-\gamma f)y\rangle\ge(\bar\gamma-\gamma\beta)\|x-y\|^2,\quad\forall x,y\in H.$$

That is, $B-\gamma f$ is strongly monotone with coefficient $\bar\gamma-\gamma\beta$.

3. Main Results

Throughout the rest of this paper, we always assume that f is a contraction of H into itself with coefficient $\beta\in(0,1)$, and B is a strongly positive bounded linear operator with coefficient $\bar\gamma$ and $0<\gamma<\frac{\bar\gamma}{\beta}$. Let $\{J_{M,\lambda_n}\}$ be a sequence of mappings defined as Definition 2.1 and let A be an $\alpha-$inverse-strongly monotone mapping, where $\{\lambda_n\}\subset[a,b]$ for some a,b with $0<a<b<2\alpha$. Define a mapping $S_n:H\to H$ by $S_nx=\beta_nx+(1-\beta_n)Sx,\forall x\in H$, where $\beta_n\in[k,1)$. Then, by Lemma 2.6, S_n is nonexpansive.

Consider the following mapping G_n on H defined by

$$G_nx=\alpha_n\gamma f(x)+(I-\alpha_nB)S_nJ_{M,\lambda_n}(I-\lambda_nA)x,\ \ x\in H,n\in\mathbb{N},$$

where $\alpha_n \in (0,1)$. By (1.4), Lemmas 2.10 and 2.4, we have

$$\begin{aligned}\|G_nx - G_ny\| &\le \alpha_n\gamma\|f(x)-f(y)\| \\ &\quad +(1-\alpha_n\bar{\gamma})\|J_{M,\lambda_n}(I-\lambda_nA)x - J_{M,\lambda_n}(I-\lambda_nA)y\| \\ &\le \alpha_n\gamma\beta\|x-y\| + (1-\alpha_n\bar{\gamma})\|x-y\| \\ &= \big(1-\alpha_n(\bar{\gamma}-\gamma\beta)\big)\|x-y\|.\end{aligned}$$

Since $0 < 1-\alpha_n(\bar{\gamma}-\gamma\beta) < 1$, it follows that G_n is a contraction. Therefore, by the Banach contraction principle, G_n has a unique fixed point $x_n^f \in H$ such that

$$x_n^f = \alpha_n\gamma f(x_n^f) + (I-\alpha_nB)S_nJ_{M,\lambda_n}(I-\lambda_nA)x_n^f.$$

For simplicity we will write x_n for x_n^f provided no confusion occurs. Next we prove the convergence of $\{x_n\}$, while they claim the existence of the $q \in F(S)\bigcap VI(H,A,M)$ which solves the variational inequality

$$\langle (B-\gamma f)q, p-q\rangle \ge 0, \forall p \in F(S)\bigcap VI(H,A,M). \tag{3.1}$$

Theorem 3.1. *Let H be a real Hilbert space and let $M : H \to 2^H$ be a maximal monotone mapping. Let A be an $\alpha-$inverse-strongly monotone mapping of H into H and let S be a $k-$strictly pseudocontractive mapping on H such that $F(S)\bigcap VI(H,A,M) \neq \emptyset$. Let f be a contraction of H into itself with $\beta \in (0,1)$ and let B be a strongly positive bounded linear operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\beta}$. Let $\{x_n\}$ be sequence generated by*

$$\begin{cases} u_n = J_{M,\lambda_n}(x_n - \lambda_nAx_n), \\ y_n = \beta_nu_n + (1-\beta_n)Su_n, \\ x_n = \alpha_n\gamma f(x_n) + (I-\alpha_nB)y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.2}$$

where $y_n = S_nu_n, \{\lambda_n\} \subset [a,b]$ for some a,b with $0 < a < b < 2\alpha$. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0,1), \lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 \le k \le \beta_n \le \lambda < 1$ *and* $\lim_{n\to\infty} \beta_n = \lambda$,

then $\{x_n\}$ converges strongly to a point $q \in F(S)\bigcap VI(H,A,M)$, which solves the variational inequality (3.1).

Proof. First, we assume that $\alpha_n \in (0, \|B\|^{-1})$. By Lemma 2.10, we obtain $\|I-\alpha_nB\| \le 1-\alpha_n\bar{\gamma}$. Take $p \in F(S)\bigcap VI(H,A,M)$. Since $u_n = J_{M,\lambda_n}(x_n - \lambda_nAx_n)$ and $p = J_{M,\lambda_n}(p-\lambda_nAp)$, then, from (1.4) and Lemma 2.4, we know that, for any $n \in \mathbb{N}$,

$$\|u_n-p\|^2 \le \|x_n-p\|^2 + \lambda_n(\lambda_n-2\alpha)\|Ax_n-Ap\|^2 \le \|x_n-p\|^2. \tag{3.3}$$

Further, since $S_np = p$, we have

$$\|y_n-p\| = \|S_nu_n - S_np\| \le \|u_n-p\| \le \|x_n-p\|. \tag{3.4}$$

Thus, we have

$$\begin{aligned}\|x_n-p\| &= \|\alpha_n(\gamma f(x_n)-Bp)+(I-\alpha_nB)(y_n-p)\|\\ &\leq \alpha_n\|\gamma(f(x_n)-f(p))+(\gamma f(p)-Bp)\|+\|I-\alpha_nB\|\|y_n-p\|\\ &\leq \alpha_n\gamma\beta\|x_n-p\|+\alpha_n\|\gamma f(p)-Bp\|+(1-\alpha_n\bar{\gamma})\|x_n-p\|\\ &= \big(1-\alpha_n(\bar{\gamma}-\gamma\beta)\big)\|x_n-p\|+\alpha_n\|\gamma f(p)-Bp\|.\end{aligned}$$

It follows that $\|x_n-p\|\leq \dfrac{\|\gamma f(p)-Bp\|}{\bar{\gamma}-\gamma\beta}$. Hence $\{x_n\}$ is bounded and we also obtain that $\{u_n\},\{y_n\},\{Ax_n\}$ and $\{f(x_n)\}$ are bounded. We note that

$$\begin{aligned}\|u_n-y_n\| &\leq \|u_n-x_n\|+\|x_n-y_n\|\\ &= \|u_n-x_n\|+\alpha_n\|\gamma f(x_n)-By_n\|.\end{aligned} \tag{3.5}$$

Using Lemma 2.8, (3.3) and (3.4), we also have

$$\begin{aligned}\|x_n-p\|^2 &\leq \|(I-\alpha_nB)(y_n-p)\|^2+2\alpha_n\langle\gamma f(x_n)-Bp,x_n-p\rangle\\ &\leq (1-\alpha_n\bar{\gamma})^2\|u_n-p\|^2+2\alpha_n\langle\gamma f(x_n)-Bp,x_n-p\rangle\\ &\leq (1-\alpha_n\bar{\gamma})^2\big(\|x_n-p\|^2+\lambda_n(\lambda_n-2\alpha)\|Ax_n-Ap\|^2\big)\\ &\quad +2\alpha_n\langle\gamma f(x_n)-Bp,x_n-p\rangle\\ &\leq \|x_n-p\|^2+(1-\alpha_n\bar{\gamma})^2a(b-2\alpha)\|Ax_n-Ap\|^2\\ &\quad +2\alpha_n\langle\gamma f(x_n)-Bp,x_n-p\rangle,\end{aligned}$$

and hence

$$(1-\alpha_n\bar{\gamma})^2a(2\alpha-b)\|Ax_n-Ap\|^2\leq 2\alpha_n\|\gamma f(x_n)-Bp\|\|x_n-p\|.$$

Since $\alpha_n\to 0$, we have

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{3.6}$$

Using Lemma 2.4 and (1.4), we have

$$\begin{aligned}&\|u_n-p\|^2\\ &=\|J_{M,\lambda_n}(x_n-\lambda_nAx_n)-J_{M,\lambda_n}(p-\lambda_nAp)\|^2\\ &\leq\langle(x_n-\lambda_nAx_n)-(p-\lambda_nAp),u_n-p\rangle\\ &=\frac{1}{2}\big(\|(x_n-\lambda_nAx_n)-(p-\lambda_nAp)\|^2+\|u_n-p\|^2\\ &\quad-\|(x_n-u_n)-\lambda_n(Ax_n-Ap)\|^2\big)\\ &\leq\frac{1}{2}\big(\|x_n-p\|^2+\|u_n-p\|^2-\|(x_n-u_n)-\lambda_n(Ax_n-Ap)\|^2\big)\\ &=\frac{1}{2}\big(\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n\|^2-\lambda_n^2\|Ax_n-Ap\|^2\\ &\quad+2\lambda_n\langle x_n-u_n,Ax_n-Ap\rangle\big).\end{aligned}$$

So, we have

$$\begin{aligned}\|u_n-p\|^2\leq \|x_n-p\|^2-\|x_n-u_n\|^2-\lambda_n^2\|Ax_n-Ap\|^2\\ +2\lambda_n\langle x_n-u_n,Ax_n-Ap\rangle.\end{aligned} \tag{3.7}$$

Then, from Lemma 2.8, (3.4) and (3.7), we have

$$\begin{aligned}
&\|x_n - p\|^2 \\
&= \|(I - \alpha_n B)(y_n - p) + \alpha_n(\gamma f(x_n) - Bp)\|^2 \\
&\le (1 - \alpha_n\bar{\gamma})^2\|y_n - p\|^2 + 2\alpha_n\langle \gamma f(x_n) - Bp, x_n - p\rangle \\
&\le (1 - \alpha_n\bar{\gamma})^2\|u_n - p\|^2 + 2\alpha_n\gamma\langle f(x_n) - f(p), x_n - p\rangle \\
&\quad +2\alpha_n\langle \gamma f(p) - Bp, x_n - p\rangle \\
&\le (1 - \alpha_n\bar{\gamma})^2\big(\|x_n - p\|^2 - \|x_n - u_n\|^2 - \lambda_n^2\|Ax_n - Ap\|^2 \\
&\quad +2\lambda_n\langle x_n - u_n, Ax_n - Ap\rangle\big) + 2\alpha_n\gamma\beta\|x_n - p\|^2 \\
&\quad +2\alpha_n\|\gamma f(p) - Bp\|\|x_n - p\| \\
&= \big(1 - 2\alpha_n(\bar{\gamma} - \gamma\beta) + (\alpha_n\bar{\gamma})^2\big)\|x_n - p\|^2 - (1 - \alpha_n\bar{\gamma})^2\|x_n - u_n\|^2 \\
&\quad -(1 - \alpha_n\bar{\gamma})^2\lambda_n^2\|Ax_n - Ap\|^2 \\
&\quad +2\lambda_n(1 - \alpha_n\bar{\gamma})^2\langle x_n - u_n, Ax_n - Ap\rangle + 2\alpha_n\|\gamma f(p) - Bp\|\|x_n - p\| \\
&\le \|x_n - p\|^2 + \alpha_n^2\bar{\gamma}^2\|x_n - p\|^2 - (1 - \alpha_n\bar{\gamma})^2\|x_n - u_n\|^2 \\
&\quad -(1 - \alpha_n\bar{\gamma})^2\lambda_n^2\|Ax_n - Ap\|^2 \\
&\quad +2\lambda_n(1 - \alpha_n\bar{\gamma})^2\langle x_n - u_n, Ax_n - Ap\rangle + 2\alpha_n\|\gamma f(p) - Bp\|\|x_n - p\|,
\end{aligned}$$

and hence

$$\begin{aligned}
&(1 - \alpha_n\bar{\gamma})^2\|x_n - u_n\|^2 \\
&\le \alpha_n^2\bar{\gamma}^2\|x_n - p\|^2 + 2\lambda_n(1 - \alpha_n\bar{\gamma})^2\langle x_n - u_n, Ax_n - Ap\rangle \\
&\quad +2\alpha_n\|\gamma f(p) - Bp\|\|x_n - p\|.
\end{aligned}$$

Since $\|Ax_n - Ap\| \to 0$ and $\alpha_n \to 0$, it follows that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.8}$$

From (3.5), we know that

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.9}$$

Define $T : H \to H$ by $Tx = \lambda x + (1 - \lambda)Sx$. Then T is nonexpansive with $F(T) = F(S)$ by Lemma 2.6. Notice that

$$\|Tu_n - u_n\| \le \|Tu_n - y_n\| + \|y_n - u_n\| \le |\lambda - \beta_n|\|u_n - Su_n\| + \|y_n - u_n\|.$$

By (3.9) and $\beta_n \to \lambda$, we obtain that

$$\lim_{n\to\infty} \|Tu_n - u_n\| = 0. \tag{3.10}$$

Consider a subsequence $\{u_{n_i}\}$ of $\{u_n\}$. Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to q. Next, we show that $q \in F(S)\bigcap VI(H, A, M)$. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup q$. From $\|Tu_n - u_n\| \to 0$, we obtain $Tu_{n_i} \rightharpoonup q$. Let us show $q \in F(T)$. Assume $q \bar{\in} F(T)$. Since $u_{n_i} \rightharpoonup q$ and $q \neq Tq$, it follows from the Opial's condition that

$$\begin{aligned}\liminf_{n\to\infty}\|u_{n_i}-q\| &< \liminf_{n\to\infty}\|u_{n_i}-Tq\| \\ &\le \liminf_{n\to\infty}\big(\|u_{n_i}-Tu_{n_i}\|+\|Tu_{n_i}-Tq\|\big) \\ &\le \liminf_{n\to\infty}\|u_{n_i}-q\|.\end{aligned}$$

This is a contradiction. So, we get $q\in F(T)$ and hence $q\in F(S)$.

We shall show $q\in VI(H,A,M)$. Since A is $\frac{1}{\alpha}-$Lipschitz continuous monotone and $D(A)=H$, by Lemma 2.5, $M+A$ is a maximal monotone mapping. Let $(v,f)\in G(M+A)$. Since $f-Av\in Mv$ and $\frac{1}{\lambda_{n_i}}(x_{n_i}-u_{n_i}-\lambda_{n_i}Ax_{n_i})\in Mu_{n_i}$, we have

$$\langle v-u_{n_i},(f-Av)-\frac{1}{\lambda_{n_i}}(x_{n_i}-u_{n_i}-\lambda_{n_i}Ax_{n_i})\rangle\ge 0.$$

Therefore, we have

$$\begin{aligned}\langle v-u_{n_i},f\rangle &\ge \langle v-u_{n_i},Av+\tfrac{1}{\lambda_{n_i}}(x_{n_i}-u_{n_i}-\lambda_{n_i}Ax_{n_i})\rangle \\ &= \langle v-u_{n_i},Av-Ax_{n_i}\rangle+\langle v-u_{n_i},\tfrac{1}{\lambda_{n_i}}(x_{n_i}-u_{n_i})\rangle \\ &= \langle v-u_{n_i},Av-Au_{n_i}\rangle+\langle v-u_{n_i},Au_{n_i}-Ax_{n_i}\rangle \\ &\quad +\langle v-u_{n_i},\tfrac{1}{\lambda_{n_i}}(x_{n_i}-u_{n_i})\rangle \\ &\ge \langle v-u_{n_i},Au_{n_i}-Ax_{n_i}\rangle+\langle v-u_{n_i},\tfrac{1}{\lambda_{n_i}}(x_{n_i}-u_{n_i})\rangle.\end{aligned}$$

Let $i\to\infty$, we obtain $\langle v-q,f\rangle\ge 0$. Since $A+M$ is maximal monotone, we have $0\in Aq+Mq$ and hence $q\in VI(H,A,M)$. Therefore, $q\in F(S)\bigcap VI(H,A,M)$. On the other hand, we note that

$$x_n-q=\alpha_n(\gamma f(x_n)-Bq)+(I-\alpha_nB)(y_n-q).$$

It follows that

$$\begin{aligned}\|x_n-q\|^2 &= \alpha_n\langle\gamma f(x_n)-Bq,x_n-q\rangle+\langle(I-\alpha_nB)(y_n-q),x_n-q\rangle \\ &\le \alpha_n\langle\gamma f(x_n)-Bq,x_n-q\rangle+\|I-\alpha_nB\|\|y_n-q\|\|x_n-q\| \\ &\le \alpha_n\langle\gamma f(x_n)-Bq,x_n-q\rangle+(1-\alpha_n\bar{\gamma})\|x_n-q\|^2.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|x_n-q\|^2 &\le \tfrac{1}{\bar{\gamma}}\langle\gamma f(x_n)-Bq,x_n-q\rangle \\ &= \tfrac{1}{\bar{\gamma}}\big(\gamma\langle f(x_n)-f(q),x_n-q\rangle+\langle\gamma f(q)-Bq,x_n-q\rangle\big) \\ &\le \tfrac{1}{\bar{\gamma}}\big(\gamma\beta\|x_n-q\|^2+\langle\gamma f(q)-Bq,x_n-q\rangle\big).\end{aligned}$$

This implies that

$$\|x_n-q\|^2\le\frac{\langle\gamma f(q)-Bq,x_n-q\rangle}{\bar{\gamma}-\gamma\beta}.$$

In particular, we have

$$\|x_{n_i}-q\|^2\le\frac{\langle\gamma f(q)-Bq,x_{n_i}-q\rangle}{\bar{\gamma}-\gamma\beta}. \tag{3.11}$$

Since $x_{n_i} \rightharpoonup q$, it follows from (3.11) that $x_{n_i} \to q$ as $i \to \infty$. Next, we show that q solves the variational inequality (3.1). Since

$$\begin{aligned} x_n &= \alpha_n \gamma f(x_n) + (I - \alpha_n B) y_n \\ &= \alpha_n \gamma f(x_n) + (I - \alpha_n B) S_n J_{M,\lambda_n}(I - \lambda_n A) x_n, \end{aligned}$$

we have

$$(B - \gamma f) x_n = -\frac{1}{\alpha_n}(I - \alpha_n B)(I - S_n J_{M,\lambda_n}(I - \lambda_n A)) x_n.$$

It follows that for $p \in F(S) \bigcap VI(H, A, M)$,

$$\begin{aligned} &\langle (B - \gamma f) x_n, x_n - p\rangle \\ &= -\frac{1}{\alpha_n}\langle (I - \alpha_n B)(I - S_n J_{M,\lambda_n}(I - \lambda_n A)) x_n, x_n - p\rangle \\ &= -\frac{1}{\alpha_n}\langle (I - S_n J_{M,\lambda_n}(I - \lambda_n A)) x_n \\ &\quad -(I - S_n J_{M,\lambda_n}(I - \lambda_n A)) p, x_n - p\rangle \\ &\quad +\langle B(I - S_n J_{\lambda_n}(I - \lambda_n A)) x_n, x_n - p\rangle \\ &\le \langle B(I - S_n J_{M,\lambda_n}(I - \lambda_n A)) x_n, x_n - p\rangle. \end{aligned} \tag{3.12}$$

Since $I - S_n J_{M,\lambda_n}(I - \lambda_n A)$ is monotone $\big($i.e.$\langle x - y, (I - S_n J_{M,\lambda_n}(I - \lambda_n A))x - (I - S_n J_{M,\lambda_n}(I - \lambda_n A))y\rangle \ge 0$ for all $x, y \in H$. This is due to the nonexpansivity of $S_n J_{M,\lambda_n}(I - \lambda_n A)\big)$. Now replacing n in (3.12) with n_i and letting $i \to \infty$, we have

$$\begin{aligned} \langle (B - \gamma f) q, q - p\rangle &= \lim_{i\to\infty}\langle (B - \gamma f) x_{n_i}, x_{n_i} - p\rangle \\ &\le \lim_{i\to\infty}\langle B(x_{n_i} - y_{n_i}), x_{n_i} - p\rangle = 0. \end{aligned} \tag{3.13}$$

That is, $q \in F(S) \bigcap VI(H, A, M)$ is a solution of (3.1). To show that the sequence $\{x_n\}$ converges to q, assume $x_{n_k} \to \hat{x}$. By the same as the proof above, we have $\hat{x} \in F(S) \bigcap VI(H, A, M)$. Moreover, it follows from the inequality (3.13) that

$$\langle (B - \gamma f) q, q - \hat{x}\rangle \le 0. \tag{3.14}$$

Interchange q and $\hat{x}$ to obtain

$$\langle (B - \gamma f)\hat{x}, \hat{x} - q\rangle \le 0. \tag{3.15}$$

Adding these two inequalities yields

$$(\bar{\gamma} - \gamma\beta)\|q - \hat{x}\|^2 \le \langle q - \hat{x}, (B - \gamma f) q - (B - \gamma f)\hat{x}\rangle \le 0$$

by Lemma 2.11. Hence $q = \hat{x}$ and therefore $x_n \to q$ as $n \to \infty$. □

Theorem 3.2. *Let H be a real Hilbert space and let $M : H \to 2^H$ be a maximal monotone mapping. Let A be an $\alpha-$inverse-strongly monotone mapping of H into H and let S be a $k-$strictly pseudocontractive mapping on H such that $F(S) \bigcap VI(H, A, M) \neq \emptyset$. Let f be a contraction of H into itself with $\beta \in (0, 1)$ and let B be a strongly positive bounded linear operator on H with coefficient*

$\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\beta}$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} u_n = J_{M,\lambda_n}(x_n - \lambda_n Ax_n), \\ y_n = \beta_n u_n + (1-\beta_n)Su_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $y_n = S_n u_n$. If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(ii) $0 \leq k \leq \beta_n \leq \lambda < 1$ *and* $\lim_{n\to\infty} \beta_n = \lambda$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(iii) $\{\lambda_n\} \subset [a,b]$ *for some* a, b *with* $0 < a < b < 2\alpha$, $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$,

then $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $q \in F(S)\bigcap VI(H,A,M)$, which solves the variational inequality (3.1).

Proof. Since $\alpha_n \to 0$, we may assume that $\alpha_n \in (0, \|B\|^{-1})$. By Lemma 2.10, we obtain $\|I - \alpha_n B\| \leq 1 - \alpha_n \bar{\gamma}$. We now observe that $\{x_n\}$ is bounded. Indeed, pick any $p \in F(S)\bigcap VI(H,A,M)$ to obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - Bp) + (I - \alpha_n B)(y_n - p)\| \\ &\leq \alpha_n \|\gamma f(x_n) - Bp\| + \|I - \alpha_n B\|\|y_n - p\| \\ &\leq \alpha_n \gamma \|f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - Bp\| + (1 - \alpha_n \bar{\gamma})\|x_n - p\| \\ &\leq \alpha_n \gamma \beta \|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\| + (1 - \alpha_n \bar{\gamma})\|x_n - p\| \\ &= \big(1 - \alpha_n(\bar{\gamma} - \gamma\beta)\big)\|x_n - p\| + \alpha_n \|\gamma f(p) - Bp\|. \end{aligned}$$

It follows from induction that

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{1}{\bar{\gamma} - \gamma\beta}\|\gamma f(p) - Bp\|\}, \quad n \in \mathbb{N},$$

and hence $\{x_n\}$ is bounded. From (3.3) and (3.4), we also obtain that $\{u_n\}$ and $\{y_n\}$ are bounded. Next, we show that $\|x_{n+1} - x_n\| \to 0$. We have

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &= \|\alpha_n \gamma f(x_n) + (I - \alpha_n B)y_n - (\alpha_{n-1}\gamma f(x_{n-1}) + (I - \alpha_{n-1}B)y_{n-1})\| \\ &= \|\alpha_n \gamma f(x_n) - \alpha_n \gamma f(x_{n-1}) + \alpha_n \gamma f(x_{n-1}) - \alpha_{n-1}\gamma f(x_{n-1}) \\ &\quad + (I - \alpha_n B)y_n - (I - \alpha_n B)y_{n-1} + (I - \alpha_n B)y_{n-1} \\ &\quad - (I - \alpha_{n-1}B)y_{n-1}\| \\ &\leq \alpha_n \gamma \beta \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma \|f(x_{n-1})\| \\ &\quad + \|I - \alpha_n B\|\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|By_{n-1}\| \\ &\leq \alpha_n \gamma \beta \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma K \\ &\quad + (1 - \alpha_n \bar{\gamma})\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|K, \end{aligned} \tag{3.16}$$

where $K = \sup\{\|f(x_n)\| + \|By_n\| : n \in \mathbb{N}\} < \infty$. On the other hand, we note that

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|S_n u_n - S_{n-1}u_{n-1}\| \\ &\le \|S_n u_n - S_n u_{n-1}\| + \|S_n u_{n-1} - S_{n-1}u_{n-1}\| \\ &\le \|u_n - u_{n-1}\| + \|S_n u_{n-1} - S_{n-1}u_{n-1}\|. \end{aligned} \tag{3.17}$$

Putting $v_n = x_n - \lambda_n Ax_n$, from $u_{n+1} = J_{M,\lambda_{n+1}}v_{n+1}$ and $u_n = J_{M,\lambda_n}v_n$, we have

$$v_{n+1} - u_{n+1} \in \lambda_{n+1}Mu_{n+1} \tag{3.18}$$

and

$$v_n - u_n \in \lambda_n M u_n. \tag{3.19}$$

Since M is monotone, we have

$$\langle u_{n+1} - u_n, \frac{u_n - v_n}{\lambda_n} - \frac{u_{n+1} - v_{n+1}}{\lambda_{n+1}}\rangle \ge 0$$

and hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - v_n - \frac{\lambda_n}{\lambda_{n+1}}(u_{n+1} - v_{n+1})\rangle \ge 0.$$

Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\le \langle u_{n+1} - u_n, v_{n+1} - v_n + (1 - \frac{\lambda_n}{\lambda_{n+1}})(u_{n+1} - v_{n+1})\rangle \\ &\le \|u_{n+1} - u_n\|\big\{\|v_{n+1} - v_n\| + |1 - \frac{\lambda_n}{\lambda_{n+1}}|\|u_{n+1} - v_{n+1}\|\big\} \end{aligned}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\le \|v_{n+1} - v_n\| + \frac{1}{\lambda_{n+1}}|\lambda_{n+1} - \lambda_n|\|u_{n+1} - v_{n+1}\| \\ &\le \|v_{n+1} - v_n\| + \frac{1}{a}|\lambda_{n+1} - \lambda_n|L, \end{aligned} \tag{3.20}$$

where $L = \sup\{\|u_n - v_n\| : n \in N\}$. Since $I - \lambda_n A$ is nonexpansive, we also have

$$\begin{aligned} \|v_{n+1} - v_n\| &= \|x_{n+1} - \lambda_{n+1}Ax_{n+1} - (x_n - \lambda_n Ax_n)\| \\ &\le \|x_{n+1} - \lambda_{n+1}Ax_{n+1} - (x_n - \lambda_{n+1}Ax_n) \\ &\quad -\lambda_{n+1}Ax_n + \lambda_n Ax_n\| \\ &\le \|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}|\|Ax_n\|. \end{aligned} \tag{3.21}$$

From (3.21) and (3.20), we have

$$\|u_n - u_{n-1}\| \le \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\bigg(\frac{L}{a} + \|Ax_{n-1}\|\bigg). \tag{3.22}$$

Next, we estimate $\|S_n u_{n-1} - S_{n-1}u_{n-1}\|$. Notice that

$$\begin{aligned} \|S_n u_{n-1} - S_{n-1}u_{n-1}\| &= \|(\beta_n u_{n-1} + (1 - \beta_n)Su_{n-1}) \\ &\quad -(\beta_{n-1}u_{n-1} + (1 - \beta_{n-1})Su_{n-1})\| \\ &\le |\beta_n - \beta_{n-1}|\|u_{n-1} - Su_{n-1}\|. \end{aligned} \tag{3.23}$$

Substituting (3.22) and (3.23) into (3.17), we have

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\left(\frac{L}{a} + \|Ax_{n-1}\|\right) \\
&\quad + |\beta_n - \beta_{n-1}|\|u_{n-1} - Su_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1 + |\beta_n - \beta_{n-1}|M_1,
\end{aligned} \tag{3.24}$$

where M_1 is an appropriate constant such that

$$M_1 \ge \frac{L}{a} + \|Ax_{n-1}\| + \|u_{n-1} - Su_{n-1}\|, \quad \forall n \in \mathbb{N}.$$

From (3.16) and (3.24), we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\le \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K(\gamma + 1) \\
&\quad + (1 - \alpha_n\bar{\gamma})(\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1 + |\beta_n - \beta_{n-1}|M_1) \\
&\le [1 - \alpha_n(\bar{\gamma} - \gamma\beta)]\|x_n - x_{n-1}\| \\
&\quad + M(|\alpha_n - \alpha_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}|),
\end{aligned} \tag{3.25}$$

where $M = \max\{K(\gamma + 1), M_1\}$. Hence, by Lemma 2.7, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.26}$$

From (3.22), (3.24), $|\lambda_{n+1} - \lambda_n| \to 0$ and $|\beta_{n+1} - \beta_n| \to 0$, we have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0 \text{ and } \lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.27}$$

Since $x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n B)y_n$, it follows that

$$\begin{aligned}
\|x_n - y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\
&= \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - By_n\|.
\end{aligned}$$

From $\alpha_n \to 0$ and (3.26), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.28}$$

For $p \in F(S)\bigcap VI(H, A, M)$, from (3.3) and (3.4), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - Bp) + (I - \alpha_n B)(y_n - p)\|^2 \\
&\le (1 - \alpha_n\bar{\gamma})^2\|y_n - p\|^2 + \alpha_n^2\|\gamma f(x_n) - Bp\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n\bar{\gamma})\|\gamma f(x_n) - Bp\|\|y_n - p\| \\
&\le \|y_n - p\|^2 + \alpha_n^2\|\gamma f(x_n) - Bp\|^2 \\
&\quad + 2\alpha_n\|\gamma f(x_n) - Bp\|\|y_n - p\| \\
&\le \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Ap\|^2 \\
&\quad + \alpha_n^2\|\gamma f(x_n) - Bp\|^2 + 2\alpha_n\|\gamma f(x_n) - Bp\|\|y_n - p\| \\
&\le \|x_n - p\|^2 + a(b - 2\alpha)\|Ax_n - Ap\|^2 \\
&\quad + \alpha_n^2\|\gamma f(x_n) - Bp\|^2 + 2\alpha_n\|\gamma f(x_n) - Bp\|\|y_n - p\|,
\end{aligned} \tag{3.29}$$

and hence

$$\begin{aligned}&-a(b-2\alpha)\|Ax_n-Ap\|^2\\&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2\\&\quad+\alpha_n^2\|\gamma f(x_n)-Bp\|^2+2\alpha_n\|\gamma f(x_n)-Bp\|\|y_n-p\|.\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$, we have

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{3.30}$$

Using Lemma 2.4, we have

$$\begin{aligned}&\|u_n-p\|^2\\&=\|J_{M,\lambda_n}(x_n-\lambda_nAx_n)-J_{M,\lambda_n}(p-\lambda_nAp)\|^2\\&\le\langle(x_n-\lambda_nAx_n)-(p-\lambda_nAp),u_n-p\rangle\\&=\frac{1}{2}\big(\|(x_n-\lambda_nAx_n)-(p-\lambda_nAp)\|^2+\|u_n-p\|^2\\&\quad-\|(x_n-\lambda_nAx_n)-(p-\lambda_nAp)-(u_n-p)\|^2\big)\\&\le\frac{1}{2}\big(\|x_n-p\|^2+\|u_n-p\|^2-\|(x_n-u_n)-\lambda_n(Ax_n-Ap)\|^2\big)\\&=\frac{1}{2}\big(\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n\|^2\\&\quad-\lambda_n^2\|Ax_n-Ap\|^2+2\lambda_n\langle x_n-u_n,Ax_n-Ap\rangle\big).\end{aligned}$$

So, we have

$$\begin{aligned}\|u_n-p\|^2\le\|x_n-p\|^2-\|x_n-u_n\|^2-\lambda_n^2\|Ax_n-Ap\|^2\\+2\lambda_n\langle x_n-u_n,Ax_n-Ap\rangle.\end{aligned} \tag{3.31}$$

Then, from (3.4) and (3.31), we have

$$\begin{aligned}&\|x_{n+1}-p\|^2\\&=\|\alpha_n(\gamma f(x_n)-Bp)+(I-\alpha_nB)(y_n-p)\|^2\\&\le(1-\alpha_n\bar{\gamma})^2\|y_n-p\|^2+\alpha_n^2\|\gamma f(x_n)-Bp\|^2\\&\quad+2\alpha_n(1-\alpha_n\bar{\gamma})\|\gamma f(x_n)-Bp\|\|y_n-p\|\\&\le\|u_n-p\|^2+\alpha_n^2\|\gamma f(x_n)-Bp\|^2+2\alpha_n\|\gamma f(x_n)-Bp\|\|y_n-p\|\\&\le\|x_n-p\|^2-\|x_n-u_n\|^2-\lambda_n^2\|Ax_n-Ap\|^2+2\lambda_n\langle x_n-u_n,Ax_n-Ap\rangle\\&\quad+\alpha_n^2\|\gamma f(x_n)-Bp\|^2+2\alpha_n\|\gamma f(x_n)-Bp\|\|y_n-p\|.\end{aligned}$$

Since $\alpha_n \to 0, \|x_n - x_{n+1}\| \to 0$ and $\|Ax_n - Ap\| \to 0$, we obtain

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.32}$$

From (3.28) and (3.32), we have

$$\|u_n-y_n\|\le\|u_n-x_n\|+\|x_n-y_n\|\to 0,\ \text{as } n\to\infty. \tag{3.33}$$

Define $T : H \to H$ by $Tx = \lambda x + (1-\lambda)Sx$. Then T is nonexpansive with $F(T) = F(S)$ by Lemma 2.6. Notice that

$$\begin{aligned}\|Tu_n-u_n\|&\le\|Tu_n-y_n\|+\|y_n-u_n\|\\&\le|\lambda-\beta_n|\|u_n-Su_n\|+\|y_n-u_n\|.\end{aligned}$$

By (3.33) and $\beta_n \to \lambda$, we obtain that

$$\lim_{n\to\infty} \|Tu_n - u_n\| = 0. \tag{3.34}$$

Next, we show that $\limsup_{n\to\infty}\langle (B-\gamma f)q, q-x_n\rangle \le 0$, where q is the unique solution of the variational inequality (3.1). To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\langle (B-\gamma f)q, q-x_{n_i}\rangle = \limsup_{n\to\infty}\langle (B-\gamma f)q, q-x_n\rangle.$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. From (3.32) and (3.34), we obtain $x_{n_i} \rightharpoonup w$, and $Tu_{n_i} \rightharpoonup w$. By the same argument as in the proof of Theorem 3.1, we have $w \in F(S)\bigcap VI(H,A,M)$. Since q is the unique solution of the variational inequality (3.1), it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle (B-\gamma f)q, q-x_n\rangle &= \lim_{i\to\infty}\langle (B-\gamma f)q, q-x_{n_i}\rangle \\ &= \langle (B-\gamma f)q, q-w\rangle \le 0.\end{aligned} \tag{3.35}$$

From $x_{n+1}-q = \alpha_n(\gamma f(x_n) - Bq) + (I-\alpha_n B)(y_n - q)$, we have

$$\begin{aligned}&\|x_{n+1}-q\|^2\\ &\le \|(I-\alpha_n B)(y_n-q)\|^2 + 2\alpha_n\langle \gamma f(x_n) - Bq, x_{n+1}-q\rangle\\ &\le (1-\alpha_n\bar{\gamma})^2\|x_n-q\|^2 + 2\alpha_n\gamma\langle f(x_n)-f(q), x_{n+1}-q\rangle\\ &\quad +2\alpha_n\langle \gamma f(q) - Bq, x_{n+1}-q\rangle\\ &\le (1-\alpha_n\bar{\gamma})^2\|x_n-q\|^2 + 2\alpha_n\gamma\beta\|x_n-q\|\|x_{n+1}-q\|\\ &\quad +2\alpha_n\langle \gamma f(q) - Bq, x_{n+1}-q\rangle\\ &\le (1-\alpha_n\bar{\gamma})^2\|x_n-q\|^2 + \alpha_n\gamma\beta\big(\|x_n-q\|^2 + \|x_{n+1}-q\|^2\big)\\ &\quad +2\alpha_n\langle \gamma f(q) - Bq, x_{n+1}-q\rangle\\ &\le \big((1-\alpha_n\bar{\gamma})^2 + \alpha_n\gamma\beta\big)\|x_n-q\|^2\\ &\quad +\alpha_n\gamma\beta\|x_{n+1}-q\|^2 + 2\alpha_n\langle \gamma f(q) - Bq, x_{n+1}-q\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}&\|x_{n+1}-q\|^2\\ &\le \frac{1-2\alpha_n\bar{\gamma}+(\alpha_n\bar{\gamma})^2+\alpha_n\gamma\beta}{1-\alpha_n\gamma\beta}\|x_n-q\|^2 + \frac{2\alpha_n}{1-\alpha_n\gamma\beta}\langle \gamma f(q) - Bq, x_{n+1}-q\rangle\\ &= \Big(1-\frac{2(\bar{\gamma}-\gamma\beta)\alpha_n}{1-\alpha_n\gamma\beta}\Big)\|x_n-q\|^2\\ &\quad +\frac{(\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\beta}\|x_n-q\|^2 + \frac{2\alpha_n}{1-\alpha_n\gamma\beta}\langle \gamma f(q) - Bq, x_{n+1}-q\rangle\\ &\le \Big(1-\frac{2(\bar{\gamma}-\gamma\beta)\alpha_n}{1-\alpha_n\gamma\beta}\Big)\|x_n-q\|^2\\ &\quad +\frac{2(\bar{\gamma}-\gamma\beta)\alpha_n}{1-\alpha_n\gamma\beta}\Big\{\frac{(\alpha_n\bar{\gamma}^2)M^*}{2(\bar{\gamma}-\gamma\beta)} + \frac{1}{\bar{\gamma}-\gamma\beta}\langle \gamma f(q) - Bq, x_{n+1}-q\rangle\Big\}\\ &= (1-\gamma_n)\|x_n-q\|^2 + \gamma_n\delta_n,\end{aligned}$$

where

$$M^* = \sup\{\|x_n - q\|^2 : n \in \mathbb{N}\}, \quad \gamma_n = \frac{2(\bar{\gamma} - \gamma\beta)\alpha_n}{1 - \alpha_n \gamma\beta}$$

and

$$\delta_n = \frac{(\alpha_n \bar{\gamma}^2) M^*}{2(\bar{\gamma} - \gamma\beta)} + \frac{1}{\bar{\gamma} - \gamma\beta} \langle \gamma f(q) - Bq, x_{n+1} - q \rangle.$$

It is easily to see that $\gamma_n \to 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$ by (3.35). Hence, by Lemma 2.7, the sequence $\{x_n\}$ converges strongly to q. □

Remark 3.3. Theorem 3.2 improves Proposition 3.1 of [6] in the following senses:

(1) We generalize classical variational inequality (1.3) considered by [6] to variational inclusion (1.1).

(2) We generalize a nonexpansive mapping considered by [6] to a strictly pseudocontractive mapping.

(3) We generalize the iterative algorithm from viscosity approximation methods proposed by [6] to general iterative methods.

Remark 3.4. Theorems 3.1 and 3.2 are also development of the iterative algorithms of [10] in different directions.

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