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GENERALIZED QUASI-VARIATIONAL-LIKE INEQUALITIES FOR PSEUDO-MONOTONE TYPE II OPERATORS ON NON-COMPACT SETS

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Abstract. We obtained results on upper hemi-continuous and pseudo-monotone type two mappings for sets which are not compact. M.S.R. Chowdhury and K.-K. Tan's improved result on Ky Fan's minimax inequality will be used.

1. Introduction

We have derived an advanced form of variational-like inequalities for upper hemi-continuous and (β, g)-pseudo-monotone type II and strong (β, g)-pseudo-monotone type II operators on compact domains in topological vector spaces(TVS). These advanced form of variational-like inequalities are extensions of more general form of variational inequalities. In 1985, the generalized quasi-variational inequalities problem was first introduced in [24]. During the last three decades many authors obtained the results on generalized quasi-variational inequalities and generalized quasi-variational-like inequalities and biquasi-variational inequalities (see [4], [6]-[17], [19], [21], [24]-[25]). Generalized quasi-variational-like inequalities are advanced form of variational inequalities and extends the definitions of generalized variational inequalities and generalized quasi-variational inequalities in TVS.

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We shall use the definition of generalized quasi-variational inequalities given in [7]. In 1998, Chowdhury [6] obtained generalized quasi-vriational inequalities for upper hemi-continuous and pseudo-monotone type II and strong pseudo-monotone type II operators in non-comact domains. We will derive some results on an advanced form of quasi-variational-like inequalities for (β, g)-pseudo-monotone type II and strong (β, g)-pseudo-monotone type II operators in non-compact domains. We refer to [7] for the definition of generalized quasi-variational-like inequalities problem.

For more results on generalized quasi-variational-like inequalities, readers can look into [4], [19], [21] and references therein.

Readers can look into [7] for preliminary background. Moreover, we will use the definition of (β, g)-pseudo-monotone type II (resp., a strong (β, g)-pseudo-monotone type II) operator given in [7] extensively in obtaining the our results of this paper on non-compact domains. This definition was originally given in [16] and was derived from the demi-operator [6] and pseudo-monotone type II operators [5].

Chowdhury and Tan obtained a more general form of minimax inequality in [10] which generalized the inequality of minimax given by Ky Fan in [18]. This is a main tool for our research findings given below:

The definition of 0-diagonally concave function [22] and the 0-diagonally concave relation [17] will be required (in Section 3) in addition to the definition of upper hemi-continuous given in [11] and in [13].

2. Preliminaries

The Lemma 1 in [24], Lemma 3 in [23], and the Lemma 3 in [25] will be used in our research findings. Next, we are giving the proof of a lemma used in [7] which extended one lemma in [16] and Lemma 4.2 in [13]. Please note that we also used this Lemma in [7] but we did not give its proof there.

Lemma 2.1. *Let G be a TVS over Ψ and we consider a subset of G which is both convex and nonempty. Suppose that H is a vector space with the scalar filed Ψ. We then equip H with the $\sigma\langle H, G\rangle$ topology such that we have a continuous function $a \mapsto Re\langle w, a\rangle$ for all $w \in H$. Let $J : A \to 2^H$ be upper hemi-continuous in some subsets of A which are only line segments in A. Let $\beta : A \times A \to G$ be such that, for each fixed $b \in A$, $\beta(\cdot, b)$ is continuous and, for each fixed $a \in A$, $\beta(a, \cdot)$ is affine. Suppose that $g : A \times A \to \mathbb{R}$ is a mapping. We assume that for all $b \in A$, arbitrarily fixed, $g(\cdot, b)$ is lower semi-continuous and convex on $co(B)$ for all $B \in \mathcal{F}(A)$ and for all $a \in A$ arbitrarily chosen. Also we assume that $g(a, \cdot)$ is concave and $g(a, a) = 0$, $\beta(a, a) = 0$ and J, β have the 0-diagonally concave relation. Let $\hat{b} \in A$ be such* *that* $\inf_{u\in J(a)} Re\langle u, \beta(\hat{b}, a)\rangle \le g(a, \hat{b})$ *for all* $a \in A$. *Then we have*

$$\inf_{w\in J(\hat{b})} Re\langle w, \beta(\hat{b}, a)\rangle \le g(a, \hat{b})$$

for all $a \in A$.

Proof. Suppose that

$$\inf_{u\in J(a)} Re\langle u, \beta(\hat{b}, a)\rangle \le g(a, \hat{b})$$

for all $a \in A$. Let $a \in A$ be arbitrarily fixed and let $z_t = ta+(1-t)\hat{b} = \hat{b}-t(\hat{b}-a)$ for all $t \in [0, 1]$. Then $z_t \in A$ since A is convex. Let $M = \{z_t : t \in [0, 1]\}$. Thus, for any $t \in [0, 1]$,

$$\inf_{u\in J(z_t)} Re\langle u, \beta(\hat{b}, z_t)\rangle \le g(z_t, \hat{b}).$$

Since, for each $b \in A$, $g(\cdot, b)$ is convex and, for each $a \in A$, $g(a, \cdot)$ is affine, we have

$$\begin{aligned} \inf\nolimits_{u\in J(z_t)} Re\langle u, \beta(\hat{b}, ta + (1-t)\hat{b})\rangle \quad &\le g(tx + (1-t)\hat{b}, \hat{b}) \\ &\le t(g(a, \hat{b})) + (1-t)g(\hat{b}, \hat{b}) \end{aligned}$$

for all $t \in (0, 1]$ and so

$$\inf_{u\in J(z_t)} [Re\langle u, t\beta(\hat{b}, a) + (1-t)\beta(\hat{b}, \hat{b})\rangle] \le t(g(a, \hat{b})),$$

that is,

$$\inf_{u\in J(z_t)} t[Re\langle u, \beta(\hat{b}, a)\rangle] \le t(g(a, \hat{b})).$$

This implies that $\inf_{u\in J(z_t)} Re\langle u, \beta(\hat{b}, a)\rangle \le g(a, \hat{b})$ for all t in $(0, 1]$. Because J is upper hemi-continuous on M, $f_{\beta(\hat{a},a)} : M \to (-\infty, +\infty]$ as given below

$$f_{\beta(\hat{a},a)}(z_t) = \inf_{u\in J(z_t)} Re\langle u, \beta(\hat{a}, a)\rangle$$

for each $z_t \in L$ is lower semi-continuous on M. Thus the set

$$A = \{z_t \in M : f_{\beta(\hat{a},a)}(z_t) \le g(a, \hat{b})\}$$

is a subset of M which is closed in its topology. Then $z_t \to \hat{b}$ in M because t converges to 0^+. Because $z_t \in A$ for all $t \in (0, 1]$, clearly $\hat{b} \in A$. Consequently, $f_{\beta(\hat{a},a)}(\hat{b}) = \inf_{u\in J(\hat{b})} Re\langle u, \beta(\hat{a}, a)\rangle \le g(a, \hat{b})$. Since $a \in A$ is arbitrary, we have

$$\inf_{w\in J(\hat{b})} Re\langle w, \beta(\hat{b}, a)\rangle \le g(a, \hat{b})$$

for all $a \in A$. Hence the proof of this lemma is completed. □

Finally, for our research findings a minimax theorem of Kneser in [20] and of Aubin in [1] will be extensively used.

3. Generalized Quasi Variational-like inequalities for upper hemi-continuous and $(\beta - g)$-pseudo-monotone type II operators

We derive some new findings on a more advanced variational inequalities for upper hemi-continuous operators and (β, g)-pseudo-monotone type II (resp., strong (β, g)-pseudo-monotone type II) operators J with non-compact domain in a locally convex topological vector spaces which is also Hausdorff. Our findings will be extensions of similar findings in [24].

In the beginning we start with the following findings:

Theorem 3.1. *Suppose that G is a locally convex topological vector space over Ψ which is also Hausdorff and A is a nonempty para-compact subset of G which is also a convex subset and a bounded subset of G. Suppose also that H is a vector space over Ψ with the topology $\sigma\langle H, G\rangle$, where $\langle \cdot, \cdot \rangle : H \times G \to \Psi$ is a bilinear functional. This bilinear functional separates points on H such that for all $w \in H$, we get the continuous function $a \mapsto Re\langle w, a\rangle$. Let $L : A \to 2^A$, $J : A \to 2^H$, $\beta : A \times A \to G$ and $g : G \times G \to \mathbb{R}$ be the mappings such that*

(1) *L is upper semi-continuous and each $L(a)$ is a compact subset of A which is also convex;*
(2) *$g(A, A)$ is a bounded subset of $\mathbb{R}$;*
(3) *J is a (β, g)-pseudo-monotone type II (resp., a strong (β, g)-pseudo-monotone type II) operator which is also upper hemi-continuous on a subset of A which is also a line segment in A. We assume that H has the topology $\sigma\langle H, G\rangle$ so that $J(a)$ is a compact subset of H in the topology $\sigma\langle H, G\rangle$ and is also a convex subset of H. Further we assume that $J(A)$ is a bounded subset of H in the strong topology $\delta\langle H, G\rangle$;*
(4) *J and β keep the property of 0 diagonally concave relation, also we assume continuity of β;*
(5) *for all $B \in \mathcal{F}(A)$, $a \mapsto g(a, b)$, $g(\cdot, b)$ is lower semi-continuous on $co(B)$, for all $a \in A$ arbitrarily chosen, also $g(a, \cdot)$ and $\beta(a, \cdot)$ are concave and $\beta(a, \cdot)$ is affine and $g(a, a) = 0$, $\beta(a, a) = 0$ for all $a \in A$ arbitrarily chosen;*
(6) *the set $\Sigma = \{b \in A : \sup_{a \in L(b)}[\inf_{u \in J(a)} Re\langle u, \beta(b, a)\rangle + g(b, a)] > 0\}$ is a subset of A which is also open in its topology;*
(7) *for all $B \in \mathcal{F}(a)$ and $b \in co(B)$ there exist $\bar{a} \in A$ and $\bar{u} \in J(\bar{a})$ such that*
$$\delta_0(b)[Re\langle \bar{u}, \beta(b, \bar{a})\rangle + g(b, \bar{a})] + \Sigma_{h \in G^*} \delta_h(b) Re\langle h, \beta(b, \bar{a})\rangle \leq 0$$
for every family $\{\delta_0, \delta_h : h \in G^*\}$ of functions from A into $[0, 1]$ which are real-valued and non-negative;
(8) *the bilinear functional $\langle \cdot, \cdot \rangle$ defined on the compact subset $[\cup_{b \in co(B)} J(b)] \times \beta(co(B) \times co(B))$ of $H \times G$ is continuous for all $B \in \mathcal{F}(A)$;*

(9) *there exists a nonempty compact subset* K *of* A *and a point* $a_0 \in A$ *such that* $a_0 \in K \cap L(b)$ *and* $\min_{u \in J(a_0)} Re\langle u, \beta(b, a_0)\rangle + g(b, a_0) > 0$ *for all* $b \in A \setminus K$.

Then there exists a point $\hat{b} \in K$ *such that*

(a) $\hat{b} \in L(\hat{b})$;

(b) *there exists a point* $\hat{w} \in J(\hat{b})$ *such that*

$$Re\langle \hat{w}, \beta(\hat{b}, a)\rangle + g(\hat{b}, a) \leq 0$$

for all $a \in L(\hat{b})$.

Proof. We shall complete the proof in several steps.

Step 1. We first derive that there exist $\hat{b} \in A$ such that $\hat{b} \in L(\hat{b})$ with

$$\sup_{a \in L(\hat{b})} [\inf_{u \in J(a)} Re\langle u, \beta(\hat{b}, a)\rangle + g(\hat{b}, a)] \leq 0.$$

Suppose, we do not agree with the above outcome. Then, for each $b \in A$, either $b \notin L(b)$ or there exists $a \in L(b)$ such that $\inf_{u \in J(a)} Re\langle u, \beta(b, a)\rangle + g(b, a) > 0$, that is, for each $b \in A$, either $b \notin L(b)$ or $b \in \Sigma$. If $b \notin L(b)$, we can use a general form of Hahn Banach theorem to derive that there is a linear functional h which is continuous on G such that

$$Re\langle h, b\rangle - \sup_{a \in L(b)} Re\langle h, x\rangle > 0.$$

For each $b \in A$, we define

$$\eta(b) := \sup_{a \in L(b)} [\inf_{u \in J(a)} Re\langle u, \beta(b, a) + g(b, a)],$$

$$W_0 := \Sigma = \{b \in A : \eta(b) > 0\}$$

and

$$W_h := \{b \in A : Re\langle h, b\rangle - \sup_{a \in L(b)} Re\langle h, a\rangle > 0\}.$$

Then we have

$$A = W_0 \cup \bigcup_{h \in LF(G)} W_h.$$

Here, we denote by $LF(G)$ the set of all linear functionals on G which are continuous. Now, by our assumption, W_0 is an open set and each W_h is open in A by Lemma 1 in [24]. So, $\{W_0, W_h : h \in LF(G)\}$ is an open covering for A. But A is para-compact, hence there exists a continuous partition of the unity $\{\delta_0, \delta_h : h \in LF(G)\}$ for A subordinated to the open cover $\{W_0, W_h : h \in LF(G)\}$. We conclude that for all $b \in A$, $B \in \mathcal{F}(A)$ and $a \mapsto g(a, b)$,

$g(\cdot, b)$ is a continuous function on $co(B)$ (for proof we refer to Cor. 10.1.1 in [22]). Next, we construct the function below $\psi : A \times A \to \mathbb{R}$ by

$$\begin{aligned}\psi(a,b) \quad &= \delta_0(b)[\min_{u \in J(a)} Re\langle u, \beta(b,a)\rangle + g(b,a)] \\ &+ \textstyle\sum_{h \in LF(G)} \delta_h(b) Re\langle h, b-a\rangle\end{aligned}$$

for all $a, b \in A$. Consequently, the following conclusions are derived:

(I) Since G is a T_2 topological space, for all $B \in \mathcal{F}(A)$ and $a \in co(B)$ arbitrarily chosen, the below defined formula

$$b \longmapsto \inf_{u \in J(a)} Re\langle u, \beta(b,a)\rangle + g(b,a)$$

is definitely a function which becomes continuous on $co(B)$ by a lemma in [10] and using the property that g is a continuous function on $co(B)$, and consequently the below defined function

$$b \longmapsto \delta_0(b)[\min_{u \in J(a)} Re\langle u, \beta(b,a)\rangle + g(b,a)]$$

is lower semi-continuous on $co(B)$ by a lemma in [25]. Also, for each fixed $a \in A$,

$$b \longmapsto \sum_{h \in LF(G)} \delta_h(b) Re\langle h, b-a\rangle$$

is a continuous function on A. Consequently, $\forall B \in \mathcal{F}(A)$ and $\forall a \in co(B)$ arbitrarily chosen, the mapping $b \mapsto \psi(a,b)$ is lower semi-continuous on $co(B)$.

(II) According to our assumption, $\{\delta_0, \delta_h : h \in LF(G)\}$ is a class of functions from A into $[0,1]$ which are real-valued and non-negative, and so by our assumption:

for all $B \in \mathcal{F}(A)$ and $b \in co(B)$, there exist $\bar{a}$ in A and $\bar{u}$ in $J(\bar{a})$ such that

$$\delta_0(b)[Re\langle \bar{u}, \beta(b,\bar{a})\rangle + g(b,\bar{a})] + \Sigma_{h \in LF(G)} \delta_h(b) Re\langle h, \beta(b,\bar{a})\rangle \leq 0.$$

Thus we have

$$\min_{u \in J(a)} [\delta_0(b)(Re\langle u, \beta(b,\bar{a})\rangle + g(b,\bar{a}))] + \Sigma_{h \in LF(G)} \delta_h(b) Re\langle h, \beta(b,\bar{a})\rangle \leq 0,$$

which implies that

$$\delta_0(b)[\min_{u \in J(a)} (Re\langle u, \beta(b,\bar{a})\rangle + g(b,\bar{a}))] + \Sigma_{h \in LF(G)} \delta_h(b) Re\langle h, \beta(b,\bar{a})\rangle \leq 0.$$

Therefore, we have

$$\min_{a \in A} [\delta_0(b)(\min_{u \in J(a)} (Re\langle u, \beta(b,a)\rangle + g(b,a))) + \Sigma_{h \in LF(G)} \delta_h(b) Re\langle h, \beta(b,a)\rangle] \leq 0.$$

Thus we have $\min_{a \in A} \psi(a,b) \leq 0$ for each $B \in \mathcal{F}(A)$ and $b \in co(B)$.

(III) Suppose that $B \in \mathcal{F}(A)$, $a, b \in co(B)$ and $\{b_\beta\}_{\beta\in\Gamma}$ is a net in A converging to b (resp., converging to b in weak topology) with $\psi(ta + (1-t)b, b_\beta) \le 0$ for all $\beta \in \Gamma$ and all $t \in [0, 1]$.

Case 1. Let $\delta_0(b) = 0$. Since δ_0 is continuous and $b_\beta \to b$, we have $\delta_0(b_\beta) \to \delta_0(b) = 0$. Note that $\delta_0(b_\beta) \ge 0$ for each $\beta \in \Gamma$. Since $J(A)$ is bounded in the strong topology and $\{b_\beta\}_{\beta\in\Gamma}$ is a generalized sequence, that is,, a net which is bounded and therefore we obtain

$$\limsup_{\beta}[\delta_0(b_\beta)(\min_{u\in J(a)} Re\langle u, \beta(b_\beta, a)\rangle + g(b_\beta, a))] = 0. \tag{3.1}$$

Clearly,

$$\delta_0(b)[\min_{u\in J(a)} Re\langle u, \beta(b, a)\rangle + g(b, a)] = 0.$$

Thus it follows from (3.1) that

$$\begin{aligned} &\limsup_\beta[\delta_0(b_\beta)(\min_{u\in J(a)} Re\langle u, \beta(b_\beta, a)\rangle + g(b_\beta, a))] \\ &\quad + \textstyle\sum_{h\in LF(G)} \delta_h(b) Re\langle h, b - a\rangle \\ &= \textstyle\sum_{h\in LF(G)} \delta_h(b) Re\langle h, b - a\rangle \\ &= \delta_0(b)[\min_{u\in J(a)} Re\langle u, \beta(b, a)\rangle + g(b, a)] \\ &\quad + \textstyle\sum_{h\in LF(G)} \delta_h(b) Re\langle h, b - a\rangle. \end{aligned} \tag{3.2}$$

If we make $t = 1$, then we see that $\psi(a, b_\beta) \le 0$ for all $\beta \in \Gamma$, that is,

$$\begin{aligned} &\delta_0(b_\beta)[\min_{u\in J(a)} Re\langle u, \beta(b_\beta, a)\rangle + g(b_\beta, a)] \\ &\quad + \textstyle\sum_{h\in LF(G)} \delta_h(b_\beta) Re\langle h, b_\beta - a\rangle \\ &\le 0 \end{aligned} \tag{3.3}$$

Consequently, the equation (3.3) gives

$$\begin{aligned} &\limsup_\beta[\delta_0(b_\beta)(\min_{u\in J(a)} Re\langle u, \beta(b_\beta, a)\rangle + g(b_\beta, a))] \\ &\quad + \liminf_\beta[\textstyle\sum_{h\in LF(G)} \delta_h(b_\beta) Re\langle h, b_\beta - a\rangle] \\ &\le \limsup_\beta[\delta_0(b_\beta)(\min_{u\in J(a)} Re\langle u, \beta(b_\beta, a)\rangle + g(b_\beta, a)) \\ &\quad + \textstyle\sum_{h\in LF(G)} \delta_h(b_\beta) Re\langle h, b_\beta - a\rangle] \\ &\le 0, \end{aligned}$$

and so

$$\begin{aligned} &\limsup_\beta[\delta_0(b_\beta)(\min_{u\in J(a)} Re\langle u, \beta(b_\beta, a)\rangle + g(b_\beta, a))] \\ &\quad + \textstyle\sum_{h\in LF(G)} \delta_h(b) Re\langle h, b - a\rangle \\ &\le 0. \end{aligned} \tag{3.4}$$

Hence, by (3.2) and (3.4), we have $\psi(a, b) \le 0$.

Case 2. Let $\delta_0(b) > 0$. Since δ_0 is continuous, $\delta_0(b_\beta) \to \delta_0(b)$. Again, since $\delta_0(b) > 0$, *exists* $\lambda \in \Gamma$ such that $\delta_0(b_\beta) > 0$ for all $\beta \ge \lambda$. If we make $t = 0$, then we see that $\psi(b, b_\beta) \le 0$ for all $\beta \in \Gamma$, that is,

$$\delta_0(b_\beta)[\min_{u\in J(y)} Re\langle u, \beta(b_\beta, b)\rangle + g(b_\beta, b)] + \sum_{h\in LF(G)} \delta_h(b_\beta) Re\langle h, b_\beta - b\rangle \le 0$$

for all $\beta \in \Gamma$, and so

$$\begin{aligned}&\limsup_\beta[\delta_0(b_\beta)(\min_{u\in T(y)} Re\langle u, \beta(b_\beta, y)\rangle + g(b_\beta, y)) \\ &\quad + \textstyle\sum_{h\in LF(G)} \delta_h(b_\beta) Re\langle h, b_\beta - y\rangle] \\ &\le 0.\end{aligned} \tag{3.5}$$

Hence, by (3.5), we have

$$\begin{aligned}&\limsup_\beta[\delta_0(b_\beta)(\min_{u\in J(y)} Re\langle u, \beta(b_\beta, b)\rangle + g(b_\beta, b))] \\ &\quad + \liminf_\beta[\textstyle\sum_{h\in LF(G)} \delta_h(b_\beta) Re\langle h, b_\beta - b\rangle] \\ &\le \limsup_\beta[\delta_0(b_\beta)(\min_{u\in J(y)} Re\langle u, \beta(b_\beta, b)\rangle + g(b_\beta, b)) \\ &\quad + \textstyle\sum_{h\in LF(G)} \delta_h(b_\beta) Re\langle h, b_\beta - b\rangle] \\ &\le 0.\end{aligned}$$

Since $\liminf_\beta[\sum_{h\in LF(G)} \delta_h(b_\beta) Re\langle h, b_\beta - b\rangle] = 0$, we have

$$\limsup_\beta[\delta_0(b_\beta)(\min_{u\in J(y)} Re\langle u, \beta(b_\beta, b)\rangle + g(b_\beta, b))] \le 0. \tag{3.6}$$

Since $\delta_0(b_\beta) > 0\ \forall \beta \ge \lambda$, we conclude that

$$\begin{aligned}&\delta_0(b) \limsup_\beta[\min_{u\in J(y)} Re\langle u, \beta(b_\beta, y)\rangle + g(b_\beta, b)] \\ &= \limsup_\beta[\delta_0(b_\beta)(\min_{u\in J(y)} Re\langle u, \beta(b_\beta, b)\rangle + g(b_\beta, b))].\end{aligned} \tag{3.7}$$

Since $\delta_0(b) > 0$, by the equations (3.6) and (3.7), the following is obtained

$$\limsup_\beta[\min_{u\in J(y)} Re\langle u, \beta(b_\beta, b)\rangle + g(b_\beta, b)] \le 0.$$

Since J is an operator which is (β, g)-pseudo-monotone type II, the following is derived

$$\begin{aligned}&\limsup_\beta[\min_{u\in J(a)} Re\langle u, \beta(b_\beta, a)\rangle + g(b_\beta, a)] \\ &\ge \min_{u\in J(a)} Re\langle u, \beta(b, a)\rangle + g(b, a)\end{aligned}$$

for all $a \in A$. Since $\delta_0(b) > 0$, we have

$$\begin{aligned}&\delta_0(b)[\limsup_\beta(\min_{u\in J(a)} Re\langle u, \beta(b_\beta, a)\rangle + g(b_\beta, a))] \\ &\ge \delta_0(b)[\min_{u\in J(a)} Re\langle u, \beta(b, a)\rangle + g(b, a)],\end{aligned}$$

and thus

$$\begin{aligned}&\delta_0(b)[\limsup_\beta(\min_{u\in J(a)} Re\langle u, \beta(b_\beta, a)\rangle + g(b_\beta, a))] \\ &\quad + \textstyle\sum_{h\in LF(G)} \delta_h(b) Re\langle h, b - a\rangle \\ &\ge \delta_0(b)[\min_{u\in J(a)} Re\langle u, \beta(b, a)\rangle + g(b, a)] \\ &\quad + \textstyle\sum_{h\in LF(G)} \delta_h(b) Re\langle h, b - a\rangle.\end{aligned} \tag{3.8}$$

If $t = 1$, then we can derive that $\psi(a, b_\beta) \leq 0$ for all $\beta \in \Gamma$, that is,

$$\delta_0(b_\beta)[\min_{u \in J(a)} Re\langle u, \beta(b_\beta, a)\rangle + g(b_\beta, a)] + \sum_{h \in LF(G)} \delta_h(b_\beta) Re\langle h, b_\beta - a\rangle \leq 0$$

for all $\beta \in \Gamma$ and so, by (3.8),

$$\begin{aligned} 0 &\geq \limsup_\beta[\delta_0(b_\beta)(\min_{u \in J(a)} Re\langle u, \beta(b_\beta, a)\rangle + g(b_\beta, a)) \\ &\quad + \textstyle\sum_{h \in LF(G)} \delta_h(b_\beta) Re\langle h, b_\beta - a\rangle] \\ &\geq \limsup_\beta[\delta_0(b_\beta)(\min_{u \in J(a)} Re\langle u, \beta(b_\beta, a)\rangle + g(b_\beta, a))] \\ &\quad + \liminf_\beta[\textstyle\sum_{h \in LF(G)} \delta_h(b_\beta) Re\langle h, b_\beta - a\rangle] \\ &= \delta_0(b)[\limsup_\beta(\min_{u \in J(a)} Re\langle u, \beta(b_\beta, a)\rangle + g(b_\beta, a))] \\ &\quad + \textstyle\sum_{h \in LF(G)} \delta_h(b) Re\langle h, b - a\rangle \\ &\geq \delta_0(b)[\min_{u \in J(a)} Re\langle u, \beta(b, a)\rangle + g(b, a)] \\ &\quad + \textstyle\sum_{h \in LF(G)} \delta_h(b) Re\langle h, b - a\rangle. \end{aligned} \tag{3.9}$$

Hence we have $\psi(a, b) \leq 0$.

(IV) Using our given assumptions in the statement of the theorem we see that there exists a nonempty subset K of A which is both compact and closed, and there exists $a_0 \in A$ such that $a_0 \in K \cap L(b)$ and

$$\min_{u \in J(a_0)} Re\langle u, \beta(b, a_0)\rangle + g(b, a_0) > 0$$

for all $b \in A \setminus K$. Thus, for all $b \in A \setminus K$, we have

$$\sup_{a \in L(b)} [\min_{u \in J(a)} Re\langle u, \beta(b, a)\rangle + g(b, a)] > 0.$$

Hence $b \in W_0$ and

$$\delta_0(b)[\min_{u \in J(a_0)} Re\langle u, \beta(b, a_0)\rangle + g(b, a_0)] > 0$$

for all $b \in A \backslash K$ whenever $\delta_0(b) > 0$ and $Re\langle h, \beta(b, a_0)\rangle > 0$ whenever $\delta_h(b) > 0$ for any $h \in LF(G)$. Consequently, we have

$$\begin{aligned} \psi(a_0, b) &= \delta_0(b)[\min_{u \in J(a_0)} Re\langle u, \beta(b, a_0)\rangle + g(b, a_0)] \\ &\quad + \textstyle\sum_{h \in LF(G)} \delta_h(b) Re\langle h, b - a_0\rangle \\ &> 0 \end{aligned}$$

for all $b \in A \setminus K$. (If J is an operator which is strong (β, g)-pseudo-monotone type II, we can consider a topology on G which is called weak.) Consequently, we have shown that all assumptions of Theorem 1.1 in [10] are fulfilled by the the function ψ. So, using the Theorem 1.1 in [10], we obtain $\hat{b} \in K$ such that

$\psi(a, \hat{b}) \leq 0$ for all $a \in A$, that is,

$$\begin{aligned} &\delta_0(\hat{b})[\min\nolimits_{u \in J(a)} Re\langle u, \beta(\hat{b}, a)\rangle + g(\hat{b}, a)] \\ &\quad + \textstyle\sum_{h \in LF(G)} \delta_h(\hat{b}) Re\langle h, \hat{b} - a\rangle \\ &\leq 0 \end{aligned} \tag{3.10}$$

for all $a \in A$.

If $\delta_0(\hat{b}) > 0$, then $\hat{b} \in W_0 = \Sigma$ so that $\gamma(\hat{b}) > 0$. Choose $\hat{a} \in L(\hat{b}) \subset A$ such that

$$\min_{u \in J(\hat{a})} Re\langle u, \beta(\hat{b}, \hat{a})\rangle + g(\hat{b}, \hat{a}) \geq \gamma(\hat{b})/2 > 0.$$

As a consequence, we obtain

$$\delta_0(\hat{b})[\min_{u \in J(\hat{a})} Re\langle u, \beta(\hat{b}, \hat{a})\rangle + g(\hat{b}, \hat{a})] > 0.$$

If $\delta_h(\hat{b}) > 0$ for some $h \in LF(G)$, then $\hat{b} \in W_h$ and hence

$$Re\langle h, \hat{b}\rangle > \sup_{a \in L(\hat{b})} Re\langle h, a\rangle \geq Re\langle h, \hat{a}\rangle,$$

which implies that $Re\langle h, \hat{b} - \hat{a}\rangle > 0$. Then we see that $\delta_h(\hat{b})[Re\langle h, \hat{b} - \hat{a}\rangle] > 0$ whenever $\delta_h(\hat{b}) > 0$ for all $h \in LF(G)$. Since $\delta_0(\hat{b}) > 0$ or $\delta_h(\hat{b}) > 0$ for some $h \in LF(G)$, we derive that

$$\psi(\hat{a}, \hat{b}) = \delta_0(\hat{b})[\min_{u \in J(\hat{a})} Re\langle u, \beta(\hat{b}, \hat{a})\rangle + g(\hat{b}, \hat{a})] + \Sigma_{h \in LF(G)} \delta_h(\hat{b}) Re\langle h, \hat{b} - \hat{a}\rangle > 0.$$

But this is contrary to our equation (3.10). So, we have proved our first step of this proof.

Consequently, we derived the conclusion that there exists $\hat{b} \in A$ such that $\hat{b} \in L(\hat{b})$ and

$$\sup_{a \in L(\hat{b})} [\inf_{u \in J(a)} Re\langle u, \beta(\hat{b}, a)\rangle + g(\hat{b}, a)] \leq 0.$$

Step 2. Now, we show that

$$\inf_{w \in J(\hat{b})} Re\langle w, \beta(\hat{b}, a)\rangle \leq g(a, \hat{b})$$

for all $a \in L(\hat{b})$. From Step 1, we know that $\hat{b} \in L(\hat{b})$, which is a convex subset of A, and

$$\inf_{u \in J(a)} Re\langle u, \beta(\hat{b}, a)\rangle \leq g(a, \hat{b})$$

for all $a \in L(\hat{b})$. Hence, by applying Lemma 2.1, we obtain

$$\inf_{w \in J(\hat{b})} Re\langle w, \beta(\hat{b}, a)\rangle \leq g(a, \hat{b})$$

for all $a \in L(\hat{b})$.

Step 3. There exists a point $\hat{w} \in J(\hat{b})$ with $Re\langle \hat{w}, \beta(\hat{b}, a)\rangle \leq g(a, \hat{b})$ for all $a \in L(\hat{b})$. From Step 2, we have

$$\sup_{a \in L(\hat{b})} [\inf_{w \in J(\hat{b})} Re\langle w, \beta(\hat{b}, a)\rangle + g(\hat{b}, a) \leq 0, \tag{3.11}$$

where $J(\hat{b})$ is a subset of the Hausdorff TVS $(H, \sigma\langle H, G\rangle)$ and is a convex subset of G which is also compact in the topology $\sigma\langle H, G\rangle$.

Now, we define a mapping $f : L(\hat{b}) \times J(\hat{b}) \to \mathbb{R}$ by

$$f(a, w) = Re\langle w, \beta(\hat{b}, a\rangle + g(\hat{b}, a)$$

for each $a \in L(\hat{b})$ and $w \in J(\hat{b})$. Then, for each fixed $a \in L(\hat{b})$, the mapping $w \mapsto f(a, w)$ is convex and continuous on $J(\hat{b})$ and, for each fixed $w \in J(\hat{b})$, the mapping $x \mapsto f(a, w)$ is concave on $L(\hat{b})$. Finally, using a theorem of minimax in [20] derived by Kneser, we conclude that

$$\min_{w \in J(\hat{b})} \sup_{a \in L(\hat{b})} [Re\langle w, \beta(\hat{b}, a)\rangle + g(\hat{b}, a)] = \sup_{a \in L(\hat{b})} [\min_{w \in J(\hat{b})} [Re\langle w, \beta(\hat{b}, a)\rangle + g(\hat{b}, a)].$$

Hence, by (3.11), we obtain

$$\min_{w \in J(\hat{b})} \sup_{a \in L(\hat{b})} [Re\langle w, \beta(\hat{b}, a)\rangle + g(\hat{b}, a)] \leq 0.$$

Since $J(\hat{b})$ is compact, there exists $\hat{w} \in J(\hat{b})$ such that

$$Re\langle \hat{w}, \beta(\hat{b}, a)\rangle + g(\hat{b}, a) \leq 0$$

for all $a \in L(\hat{b})$. Hence our proof is completed. □

In conclusion, we say that if every open subset U of A and for all $a, b \in U$, $\beta(a, b) = a - b$ and there exist $g' : A \to \mathbb{R}$ such that $g(a, b) = g'(a) - g'(b)$, and if the mapping $L : A \to 2^A$ is, in addition, lower semi-continuous and, for all $b \in \Sigma$, J is upper semi-continuous for some a in $L(b)$ with

$$\inf_{u \in J(a)} Re\langle u, \beta(b, a)\rangle + g(b, a) > 0,$$

then we can derive that Σ is an open subset of A in our last Theorem 3.1. This conclusion leads us to the result given below:

Theorem 3.2. *Let G be a locally convex topological vector spaces over Ψ which is also Hausdorff, A a nonempty para-compact and convex subset of G which is also bounded and H a vector space over Ψ with $\sigma\langle H, G\rangle$-topology, where $\langle \cdot, \cdot \rangle : H \times G \to \Psi$ is a bilinear functional separating points on H such that for each $w \in H$, the function $a \mapsto Re\langle w, a\rangle$ is continuous. Let $L : A \to 2^A$, $J : A \to 2^H$, $\beta : A \times A \to G$ and $g : G \times G \to \mathbb{R}$ be mappings such that*

(1) *L is continuous such that each $L(a)$ is compact and convex;*

(2) *$g(A, A)$ is bounded;*

(3) *J is a (β, g)-pseudo-monotone type II (resp., a strong (β, g)-pseudo-monotone type II) operator and is upper hemi-continuous on a subset of A which is also a line segment in A with the $\sigma\langle H, G\rangle$-topology on H such that each $J(a)$ is $\sigma\langle H, G\rangle$-compact and convex and $J(A)$ is $\delta\langle H, G\rangle$-bounded;*

(4) *J and β have the 0-diagonally concave relation and β is continuous;*

(5) *for each fixed $b \in A$, $a \mapsto g(a, b)$, $g(\cdot, b)$ is lower semi-continuous on $co(B)$ for each $B \in \mathcal{F}(A)$ and, for each fixed $a \in A$, $g(a, \cdot)$ and $\beta(a, \cdot)$ are concave, $\beta(a, \cdot)$ is affine, $g(a, a) = 0$ and $\beta(a, a) = 0$;*

(6) *for each open subset U of A and $a, b \in U$, $\beta(a, b) = a - b$ and there exists $g' : A \to \mathbb{R}$ such that $g(a, b) = g'(a) - g'(b)$;*

(7) *for each $b \in \Sigma = \{b \in A : \sup_{a\in L(b)}[\inf_{u\in J(a)} Re\langle u, \beta(b, a)\rangle + g(b, a)] > 0\}$, J is upper semi-continuous at some point a in $L(b)$ with*

$$\inf_{u\in J(a)} Re\langle u, \beta(b, a)\rangle + g(b, a) > 0;$$

(8) *for each $A \in \mathcal{F}(A)$ and $b \in co(B)$, there exist $\bar{a} \in A$ and $\bar{u} \in J(\bar{a})$ such that*

$$\delta_0(b)[Re\langle \bar{u}, \beta(b, \bar{a})\rangle + g(b, \bar{a})] + \Sigma_{h\in LF(G)}\delta_h(b)Re\langle h, y - \bar{a}\rangle \leq 0$$

for any family $\{\delta_0, \delta_h : h \in LF(G)\}$ of non-negative real-valued functions from A into $[0, 1]$;

(9) *for each $B \in \mathcal{F}(A)$, the bilinear functional $\langle\cdot, \cdot\rangle$ is continuous over the compact subset $[\cup_{b\in co(B)}J(b)] \times \beta(co(B) \times co(B))$ of $H \times G$.*

Further, suppose that there exist a nonempty compact subset K of A and a point $a_0 \in A$ such that

$$a_0 \in K \cap L(b), \qquad \min_{u\in J(a_0)} Re\langle u, \beta(b, a_0)\rangle + g(b, a_0) > 0$$

for all $b \in A \setminus K$. Then there exists a point $\hat{b} \in K$ such that

(a) *$\hat{b} \in L(\hat{b})$;*

(b) *there exists a point $\hat{w} \in J(\hat{b})$ with $Re\langle \hat{w}, \beta(\hat{b}, a)\rangle + g(\hat{b}, a) \leq 0$ for all $a \in L(\hat{b})$.*

Proof. The proof is similar to the proof of Theorem 3.2 in [15] and so omitted. □

Remark 3.3. (1) Theorems 3.1 and 3.2 of this paper are further generalizations of the results obtained in [15, Theorem 3.1] and in [15, Theorem 3.2], respectively, into generalized quasi-variational-like inequalities of (β, g)-pseudo-monotone type II operators and strong (β, g)-pseudo-monotone type II operators on non-compact sets;

(2) In 1985, Shih and Tan [24] obtained the results on generalized quasi-variational-like inequalities in locally convex topological vector spaces and their results were obtained on compact sets where the set-valued mappings were either lower semi-continuous or upper semi-continuous. Our present paper is another extension of the original work in [24] using (β, g)-pseudo-monotone type II and strong (β, g)-pseudo-monotone type II operators on non-compact sets;

(3) The results in [15] were obtained on non-compact sets where one of the set-valued mappings is a pseudo-monotone type II operators which were defined first in [6] and later renamed as pseudo-monotone type II operators in [5]. Our present results are extensions of the results in [15] using an extension of the operators defined in [5] (and originally in [6]).

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