Nonlinear Functional Analysis and Applications
Vol. 27, No. 4 (2022), pp. 757-771
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2022.27.04.05
http://nfaa.kyungnam.ac.kr/journal-nfaa

SOME FIXED POINT THEOREMS FOR RATIONAL (α, β, Z)-CONTRACTION MAPPINGS UNDER SIMULATION FUNCTIONS AND CYCLIC (α, β)-ADMISSIBILITY

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Abstract. In this paper, we present some fixed point theorems for rational type contractive conditions in the setting of a complete metric space via a cyclic (α, β)-admissible mapping imbedded in simulation function. Our results extend and generalize some previous works from the existing literature. We also give some examples to illustrate the obtained results.

[0]Received November 20, 2021. Revised May 17, 2022. Accepted June 21, 2022.
[0]2020 Mathematics Subject Classification: 47H10, 54H25.
[0]Keywords: Fixed point, metric space, simulation function, rational (α, β, Z)-contraction mapping, cyclic (α, β)-admissible mapping.
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1. Introduction

Recently, Samet et al. [18] proved a generalization of Banach contraction principle by introducing the notion of $\alpha - \psi$ contractive type mappings and α-admissible mappings. This concept is further generalized by many authors ([3, 5, 6, 13]) by introducing generalized $\alpha - \psi$ contractive type mapping and α-admissible mapping in different metric spaces.

The concept of cyclic (α, β)-admissible mapping was introduced by Alizadeh et al. [1] by generalizing the concept of α-admissible mapping of Samet et al. [18]. They proved various fixed point theorems in the setting of metric spaces. Also, Khojasteh et al. [14] introduced the notion of z-contraction by defining the concept of simulation function. The concept of Khojasteh et al. [14] is further modified by Argoubi et al. [4]. They proved the existence of common fixed point results of a pair of nonlinear operators satisfying a certain contractive condition involving simulation functions, in the setting of ordered metric spaces. Afterward, several authors discussed the existence of fixed point by using the simulation function, for instance see ([2, 7, 9, 10, 11, 12, 15, 16, 17]).

In this paper, we consider rational (α, β, Z) contraction mappings under simulation functions involving a cyclic (α, β)-admissibility in a metric space. For this kind of contractions, we establish some fixed point results. Our results are generalization and extension of the results [9] and [16]. For more results of rational type contractions and Z-contraction we refer the paper in ([7, 8, 9, 11, 12, 16, 17]) and references cited therein.

Now we will give some basic definitions and results in metric spaces before presenting our main results.

2. Preliminaries

Alizadeh et al. [1] introduced the notion of cyclic (α, β)-admissible mapping which is defined as follows:

Definition 2.1. ([1]) Let X be a nonempty set, f be a self-mapping on X and $\alpha, \beta : X \to [0, +\infty)$ be two mappings. We say that f is a cyclic (α, β)-admissible mapping if $x \in X$ with

$$\alpha(x) \geq 1 \Rightarrow \beta(f(x)) \geq 1$$

and

$$\beta(x) \geq 1 \Rightarrow \alpha(f(x)) \geq 1. \tag{2.1}$$

In 2015, Khojasteh et al. [14] introduced the class of simulation functions as given below and by using this definition they proved the following theorem:

Definition 2.2. Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping. Then ζ is called a simulation function if it satisfies the following conditions:

(ζ_1) $\zeta(0, 0) = 0$;
(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;
(ζ_3) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Theorem 2.3. ([14]) *Let (X, d) be a complete metric space and $T : X \to X$ be a Z-contraction mapping with respect to a simulation function ζ, that is,*

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0,$$

for all $x, y \in X$. Then T has a unique fixed point.

It is worth mentioning that the Banach contraction is an example of Z-contraction by defining $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ via $\zeta(t, s) = \gamma s - t$ for all $s, t \in [0, \infty)$, where $\gamma \in [0, 1)$.

Argoubi et al.[4] modified the definition of [14] as follows:

Definition 2.4. A simulation function is a function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ that satisfies the following conditions

(1) $\zeta(t, s) < s - t$ for all $t, s > 0$;
(2) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

It is clear that any simulation function in the sense of Khojasteh et al. [14] (Definition 2.2) is also a simulation function in the sense of Argoubi et al. [4] (Definition 2.4). The following example is a simulation function in the sense of Argoubi et al. [4].

Example 2.5. Define a function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\zeta(t, s) = \begin{cases} 1, & \text{if (s, t)= (0, 0);} \\ \lambda s - t, & \text{if otherwise,} \end{cases}$$

where $\lambda \in (0, 1)$. Then ζ is a simulation function.

3. Main results

Now, we are ready to prove our result with the following definitions.

Definition 3.1. Let (X, d) be a complete metric space, $T : X \to X$ be a mapping and $\alpha, \beta : X \to [0, \infty)$ be two functions. Then T is said to be a rational (α, β, Z)-contraction mapping if it satisfies the following conditions:

(1) T is cyclic (α, β)-admissible,

(2) there exists a simulation function $\zeta \in Z$ such that

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \zeta(d(Tx, Ty), M(x, y)) \geq 0, \tag{3.1}$$

holds for all $x, y \in X$, where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}, \frac{d(x, Tx)d(y, Ty)}{1 + d(Tx, Ty)}\right\}.$$

Theorem 3.2. *Let (X, d) be a complete metric space, $T : X \to X$ be a mapping and $\alpha, \beta : X \to [0, \infty)$ be two functions. Suppose that the following conditions hold:*

(1) *T is a rational (α, β, Z)-contraction mapping.*
(2) *There exists an element $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.*
(3) *T is continuous.*

Then T has a fixed point $u \in X$.

Proof. Assume that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$. We divide our proof into the following three steps:

Step 1. Define a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_n = x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, then T has a fixed point and the proof is finished. Hence, we assume that $x_n \neq x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, that is $d(x_n, x_{n+1}) \neq 0$ for $n \in \mathbb{N} \cup \{0\}$. Since T is a cyclic (α, β)-admissible mapping, $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,

$$\beta(x_1) = \beta(Tx_0) \geq 1.$$

It implies that

$$\alpha(x_2) = \alpha(Tx_1) \geq 1.$$

And also, we have

$$\alpha(x_1) = \alpha(Tx_0) \geq 1.$$

It implies that

$$\beta(x_2) = \beta(Tx_1) \geq 1.$$

By the continuing the above process, we have $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. Thus $\alpha(x_n)\beta(x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. Therefore, we get

$$\zeta(d(Tx_n, Tx_{n+1}), M(x_n, x_{n+1})) \geq 0 \tag{3.2}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\Big\{d(x_n, x_{n+1}), \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_{n+1})}{1 + d(x_n, x_{n+1})}, \\ &\qquad \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_{n+1})}{1 + d(Tx_n, Tx_{n+1})}\Big\} \\ &= \max\Big\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+1})}, \\ &\qquad \frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{1 + d(x_{n+1}, x_{n+2})}\Big\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned} \tag{3.3}$$

It follows that

$$\zeta(d(x_{n+1}, x_{n+2}), \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) \geq 0. \tag{3.4}$$

(ζ_2) of Definition 2.2 implies that

$$\begin{aligned} 0 &\leq \zeta(d(x_{n+1}, x_{n+2}), \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) \\ &< \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} - d(x_{n+1}, x_{n+2}). \end{aligned}$$

Thus, we conclude that

$$d(x_{n+1}, x_{n+2}) < \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} \tag{3.5}$$

for all $n \geq 1$. From (3.5), we have

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) \text{ for all } n \geq 1. \tag{3.6}$$

It follows that the sequence $\{d(x_n, x_{n+1})\}$ is nonincreasing. Therefore, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Note that if $r \neq 0$, that is $r > 0$, then by (ζ_2) of Definition 2.2, we have

$$0 \leq \limsup_{n\to\infty} \zeta(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})) < 0,$$

which is a contradiction. This implies that $r = 0$, that is

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.7}$$

Step 2. Now, we prove that $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary, that is, $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and two subsequences $\{x_{m_{(k)}}\}$ and $\{x_{n_{(k)}}\}$ of $\{x_n\}$ with $m_{(k)} > n_{(k)} > k$ and $m_{(k)}$ is the smallest index in $\mathbb{N}$ such that

$$d(x_{n_{(k)}}, x_{m_{(k)}}) \geq \epsilon.$$

So, $d(x_{n_{(k)}}, x_{m_{(k)-1}}) < \epsilon$. Triangular inequality implies that

$$\begin{aligned} \epsilon &\leq d(x_{n_{(k)}}, x_{m_{(k)}}) \\ &\leq d(x_{n_{(k)}}, x_{m_{(k)-1}}) + d(x_{m_{(k)-1}}, x_{m_{(k)}}) \\ &< \epsilon + d(x_{m_{(k)-1}}, x_{m_{(k)}}). \end{aligned}$$

Taking $k \to \infty$ in the above inequality and using (3.7), we get

$$\lim_{k\to\infty} d(x_{n_{(k)}}, x_{m_{(k)}}) = \epsilon. \tag{3.8}$$

Again, by triangular inequality, we have

$$\begin{aligned} d(x_{n_{(k)-1}}, x_{m_{(k)-1}}) &\leq d(x_{n_{(k)-1}}, x_{n_k}) + d(x_{n_{(k)}}, x_{m_{(k)}}) \\ &\quad + d(x_{m_{(k)}}, x_{m_{(k)-1}}) \\ &\leq d(x_{n_{(k)-1}}, x_{n_k}) + d(x_{n_{(k)}}, x_{n_{(k)-1}}) \\ &\quad + d(x_{n_{(k)-1}}, x_{m_{(k)}}) + d(x_{m_{(k)}}, x_{m_{(k)-1}}) \\ &\leq 2d(x_{n_{(k)}}, x_{n_{(k)-1}}) + d(x_{n_{(k)-1}}, x_{m_{(k)-1}}) \\ &\quad + d(x_{m_{(k)-1}}, x_{m_{(k)}}) + d(x_{m_{(k)}}, x_{m_{(k)-1}}) \\ &\leq 2d(x_{n_{(k)}}, x_{n_{(k)-1}}) + d(x_{m_{(k)-1}}, x_{n_{(k)-1}}) \\ &\quad + 2d(x_{m_{(k)-1}}, x_{m_{(k)}}). \end{aligned}$$

Taking $k \to \infty$ in the above inequality and using (3.7) and (3.8), we get

$$\begin{aligned} \lim_{k\to\infty} d(x_{n_{(k)}}, x_{m_{(k)}}) &= \lim_{k\to\infty} d(x_{n_{(k)}-1}, x_{m_{(k)}-1}) \\ &= \epsilon. \end{aligned} \tag{3.9}$$

Since $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all $n = 1, 2, 3, ...$, we conclude that

$$\alpha(x_{n_{(k)}-1})\beta(x_{m_{(k)}-1}) \geq 1.$$

Since T is a rational $(\alpha, \beta, Z)-$contraction, we have

$$\zeta(d(Tx_{n_{(k)}-1}, Tx_{m_{(k)}-1}), M(x_{n_{(k)}-1}, x_{m_{(k)}-1})) \geq 0 \tag{3.10}$$

for all $x, y \in X$, where

$$\begin{aligned}
M(x_{n_{(k)-1}}, x_{m_{(k)-1}}) &= \max\Big\{d(x_{n_{(k)-1}}, x_{m_{(k)-1}}),\\
&\quad \frac{d(x_{n_{(k)-1}}, Tx_{n_{(k)-1}})d(x_{m_{(k)-1}}, Tx_{m_{(k)-1}})}{1+d(x_{n_{(k)-1}}, x_{m_{(k)-1}})},\\
&\quad \frac{d(x_{n_{(k)-1}}, Tx_{n_{(k)-1}})d(x_{m_{(k)-1}}, Tx_{m_{(k)-1}})}{1+d(Tx_{n_{(k)-1}}, Tx_{m_{(k)-1}})}\Big\}\\
&= \max\Big\{d(x_{n_{(k)-1}}, x_{m_{(k)-1}}),\\
&\quad \frac{d(x_{n_{(k)-1}}, x_{n_{(k)}})d(x_{m_{(k)-1}}, x_{m_{(k)}})}{1+d(x_{n_{(k)-1}}, x_{m_{(k)-1}})},\\
&\quad \frac{d(x_{n_{(k)-1}}, x_{n_{(k)}})d(x_{m_{(k)-1}}, x_{m_{(k)}})}{1+d(x_{n_{(k)}}, x_{m_{(k)}})}\Big\}\\
&= \max\{d(x_{n_{(k)-1}}, x_{m_{(k)-1}}), d(x_{n_{(k)-1}}, x_{n_{(k)}})\}.
\end{aligned}$$

By (3.7) and (3.9), we conclude that

$$\lim_{n\to\infty} M(x_{n_{(k)-1}}, x_{m_{(k)-1}}) = \epsilon. \tag{3.11}$$

Note that by (ζ_2) and (ζ_3) of Definition 2.2, implies that

$$0 \le \limsup \zeta(d(Tx_{n_{(k)-1}}, Tx_{m_{(k)-1}}), M(x_{n_{(k)-1}}, x_{m_{(k)-1}})) < 0,$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence.

Step 3. Finally, we prove that T has a fixed point. Since $\{x_n\}$ is a Cauchy sequence in the complete metric space X, there exists a $x^* \in X$ such that $x_n \to x^*$. The continuity of T implies that $Tx_{2n} \to Tx^*$. Since $x_{2n+1} = Tx_{2n}$ and $x_{2n+1} \to x^*$, by uniqueness of limit, we get $Tx^* = x^*$. So x^* is a fixed point of T. This completes the proof. □

We begin our next result with the following definitions and notations.

Definition 3.3. We denote by Ψ the family of all nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that

(Ψ_1) ψ is a continuous;
(Ψ_2) $\psi^{-1}(\{0\}) = 0$.

Definition 3.4. Let (X, d) be a complete metric space, $T : X \to X$ be a mapping and $\alpha, \beta : X \to [0,\infty)$ be two functions. Then T is said to be a generalized rational (α, β, Z)-contraction mapping if T satisfies the following conditions:

(1) T is a cyclic (α, β)-admissible,
(2) there exists a simulation function $\zeta \in Z$ such that

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \zeta(\psi(d(Tx, Ty)), \psi(m(x, y))) \geq 0 \tag{3.12}$$

hold for all $x, y \in X$, where

$$m(x, y) = \max \left\{ d(y, Ty)\frac{1 + d(x, Tx)}{1 + d(x, y)}, \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)} \right\}.$$

From now on, let (X, d) be a metric space and let $\alpha, \beta : X \to [0, \infty)$ be functions, $\psi \in \Psi$ and $\zeta \in Z$.

Theorem 3.5. *Let (X, d) be a complete metric space, and let $T : X \to X$ be a generalized rational (α, β, Z)- contraction mapping with respect to ζ. Suppose that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$, where $x_0 \in X$. Assume that either*

(1) *T is continuous or*
(2) *if $\{x_n\} \subset X$ is a sequence such that $\lim_{n\to\infty} d(x_n, x) = 0$ and for all $n = 1, 2, 3, ...,$*

$$\beta(x_n) \geq 1. \tag{3.13}$$

If $T : X \to X$ is cyclic (α, β)-admissible, then T has a fixed point in X. Further if $\alpha(x)\beta(y) \geq 1$ for all fixed points x, y of T, then T has a unique fixed point.

Proof. Let $x_0 \in X$ be a point such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Define a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$ for all $n = 0, 1, 2, ...$ If $x_n = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T, and proof is completed. Assume that $x_n \neq x_{n+1}$ for all $n = 0, 1, 2, ...$ Since T is cyclic (α, β)-admissible and $\alpha(x_0) \geq 1$, $\beta(x_1) = \beta(Tx_0) \geq 1$, we have $\alpha(x_2) = \alpha(Tx_1) \geq 1$. By continuing this process, we have $\alpha(x_{2n}) \geq 1$ and $\beta(x_{2n+1}) \geq 1$ for all $n = 0, 1, 2, ...$ Again, since T is cyclic (α, β)-admissible and $\beta(x_0) \geq 1$, $\alpha(x_1) = \alpha(Tx_0) \geq 1$ and $\beta(x_2) = \beta(Tx_1) \geq 1$.

Recursively, we obtain that

$$\beta(x_{2n}) \geq 1 \quad \text{and} \quad \alpha(x_{2n+1}) \geq 1$$

for all $n = 0, 1, 2, ...$ Hence,

$$\alpha(x_n) \geq 1 \quad \text{and} \quad \beta(x_n) \geq 1$$

for all $n = 0, 1, 2, ...,$ and hence

$$\alpha(x_{n-1})\beta(x_n) \geq 1 \text{ for all } n = 0, 1, 2, ...$$

Now for all $n = 1, 2, 3, ...,$

$$\begin{aligned} m(x_{n-1}, x_n) &= \max\Big\{d(x_n, Tx_n)\frac{1+d(x_{n-1}, Tx_{n-1})}{1+d(x_{n-1}, x_n)}, \\ &\qquad \frac{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_n)+d(x_n, Tx_n)d(x_n, Tx_{n-1})}{d(x_{n-1}, Tx_n)+d(x_n, Tx_{n-1})}\Big\} \\ &= \max\Big\{d(x_n, x_{n+1})\frac{1+d(x_{n-1}, x_n)}{1+d(x_{n-1}, x_n)}, \\ &\qquad \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})+d(x_n, x_{n+1})d(x_n, x_n)}{d(x_{n-1}, x_{n+1})+d(x_n, x_n)}\Big\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}. \end{aligned} \tag{3.14}$$

It follows from (3.12) and (3.14), we have

$$\begin{aligned} 0 &\leq \zeta(\psi(d(Tx_{n-1}, Tx_n)), \psi(m(x_{n-1}, x_n))) \\ &= \zeta(\psi(d(x_n, x_{n+1})), \psi(\max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\})) \\ &< \psi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) - \psi(d(x_n, x_{n+1})). \end{aligned} \tag{3.15}$$

Consequently, we obtain that for all $n = 1, 2, 3, ...,$

$$\psi(d(x_n, x_{n+1})) < \psi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}).$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ for some n, then

$$\psi(d(x_n, x_{n+1})) < \psi(d(x_n, x_{n+1})),$$

which is a contradiction. Hence $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$ for all $n = 1, 2, 3...$ and hence from (3.15)

$$\begin{aligned} 0 &\leq \zeta(\psi(d(x_n, x_{n+1})), \psi(d(x_{n-1}, x_n))) \\ &< \psi(d(x_{n-1}, x_n)) - \psi(d(x_n, x_{n+1})), \end{aligned} \tag{3.16}$$

which implies

$$\psi(d(x_n, x_{n+1})) < \psi(d(x_{n-1}, x_n))$$

for all $n = 1, 2, 3, ...$ Since $\{\psi(d(x_{n-1}, x_n))\}$ is decreasing and bounded from below by 0, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \psi(d(x_n, x_{n-1})) = r.$$

Now, we show that $\lim_{n\to\infty} \psi(d(x_n, x_{n-1})) = 0$. On the contrary, assume that $r > 0$. Let $t_n = \psi(d(x_n, x_{n+1}))$ and $s_n = \psi(d(x_{n-1}, x_n))$, for all $n = 1, 2, 3, ...$ Then, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = r$. From condition (ζ_3) we have

$$0 \leq \limsup_{n\to\infty} \zeta(\psi(d(x_n, x_{n+1})), \psi(d(x_{n-1}, x_n))) < 0,$$

which is a contradiction. Hence, we have $r = 0$. Since $\psi \in \Psi$,

$$\lim_{n\to\infty} d(x_n, x_{n-1}) = 0. \tag{3.17}$$

We now show that $\{x_n\}$ is a Cauchy sequence. On contrary, let $\{x_n\}$ be not a Cauchy sequence. Then there exists $\epsilon > 0$ such that, for all $k > 0$ there exists $m(k) > n(k) > k$ with

$$d(x_{m_{(k)}}, x_{n_{(k)}}) \geq \epsilon \quad \text{and} \quad d(x_{m_{(k)}-1}, x_{n_{(k)}}) < \epsilon.$$

Then, we have

$$\begin{aligned} \epsilon &\leq d(x_{m_{(k)}}, x_{n_{(k)}}) \\ &\leq d(x_{m_{(k)}}, x_{m_{(k)}-1}) + d(x_{m_{(k)}-1}, x_{n_{(k)}}) \\ &< d(x_{m_{(k)}}, x_{m_{(k)}-1}) + \epsilon. \end{aligned}$$

Letting $k \to \infty$ in above inequality, we have

$$\lim_{k\to\infty} d(x_{m_{(k)}}, x_{n_{(k)}}) = \epsilon. \tag{3.18}$$

By using (3.17) and (3.18), we obtain

$$\lim_{k\to\infty} d(x_{m_{(k)}+1}, x_{n_{(k)}+1}) = \epsilon. \tag{3.19}$$

Since

$$\alpha(x_n) \geq 1 \quad \text{and} \quad \beta(x_n) \geq 1 \ \text{ for all } \ n = 1, 2, 3, ...,$$
$$\alpha(x_{m_{(k)}})\beta(x_{n_{(k)}}) \geq 1, \ \text{ for all } \ k = 1, 2, 3, ...$$

We deduce that

$$\begin{aligned} & m(x_{m_{(k)}}, x_{n_{(k)}}) \\ &= \max\Big\{ d(x_{n_{(k)}}, Tx_{n_{(k)}}) \frac{1 + d(x_{m_{(k)}}, Tx_{m_{(k)}})}{1 + d(x_{m_{(k)}}, x_{n_{(k)}})}, \\ &\qquad \frac{d(x_{m_{(k)}}, Tx_{m_{(k)}}) d(x_{m_{(k)}}, Tx_{n_{(k)}}) + d(x_{n_{(k)}}, Tx_{n_{(k)}}) d(x_{n_{(k)}}, Tx_{m_{(k)}})}{d(x_{m_{(k)}}, Tx_{n_{(k)}}) + d(x_{n_{(k)}}, Tx_{m_{(k)}})} \Big\} \\ &= \max\Big\{ d(x_{n_{(k)}}, x_{n_{(k)}+1}) \frac{1 + d(x_{m_{(k)}}, x_{m_{(k)}+1})}{1 + d(x_{m_{(k)}}, x_{n_{(k)}})}, \\ &\qquad \frac{d(x_{m_{(k)}}, x_{m_{(k)}+1}) d(x_{m_{(k)}}, x_{n_{(k)}+1}) + d(x_{n_{(k)}}, x_{n_{(k)}+1}) d(x_{n_{(k)}}, x_{m_{(k)}+1})}{d(x_{m_{(k)}}, x_{n_{(k)}+1}) + d(x_{n_{(k)}}, x_{m_{(k)}+1})} \Big\}. \end{aligned}$$

Let $s_k = \psi(m(x_{m_{(k)}}, x_{n_{(k)}}))$ and $t_k = \psi(d(x_{m_{(k)}+1}, x_{n_{(k)}+1}))$. Then it follows from (3.17), (3.18) and (3.19), we have

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} t_k = \psi(\epsilon). \tag{3.20}$$

Since $\psi(\epsilon) > 0$, it follows from condition (ζ_3) that

$$0 \leq \limsup_{n\to\infty} \zeta(\psi(d(x_{m_{(k)}+1}, x_{n_{(k)}+1})), \psi(m(x_{m_{(k)}}, x_{n_{(k)}}))) < 0,$$

which is a contradiction. Then $\{x_n\}$ is a Cauchy sequence. It follows from the completeness of X that there exists

$$x^* = \lim_{n\to\infty} x_n \in X. \tag{3.21}$$

If T is continuous, then $\lim_{n\to\infty} x_n = Tx^*$ and so $x^* = Tx^*$. Assume that (3.13) holds. Than $\alpha(x_n)\beta(x^*) \geq 1$ for all $n = 0, 1, 2, ...$ We have

$$\begin{aligned} m(x_n, x^*) &= \max\Big\{d(x^*, Tx^*)\frac{1 + d(x_n, Tx_n)}{1 + d(x_n, x^*)}, \\ &\qquad \frac{d(x_n, Tx_n)d(x_n, Tx^*) + d(x^*, Tx^*)d(x^*, Tx_n)}{d(x_n, Tx^*) + d(x^*, Tx_n)}\Big\} \\ &= \max\Big\{d(x^*, x_n)\frac{1 + d(x_n, x_{n+1})}{1 + d(x_n, x^*)}, d(x^*, Tx^*)\Big\}. \end{aligned}$$

Let $s_n := \psi(m(x_n, x^*))$ and $t_n := \psi(d(x_{n+1}, Tx^*))$. Then, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = \psi(d(x^*, Tx^*))$. Assume that $\psi(d(x^*, Tx^*)) > 0$. Then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n > 0,$$

it follows from (ζ_3) that

$$0 \leq \lim_{n\to\infty} \sup \zeta(\psi(d(x_{n+1}, Tx^*)), \psi(m(x_n, x^*))) < 0,$$

which is a contradiction.

Thus $\psi(d(x^*, Tx^*)) = 0$. From (ψ_2) we have $d(x^*, Tx^*) = 0$. Hence x^* is a fixed point of T.

We now show that the fixed point of T is unique under assumption that $\alpha(x)\beta(y) \geq 1$ for all fixed points x, y of T.

Let y^* be another fixed point of T. Then $\alpha(x^*)\beta(y^*) \geq 1$. Hence from (3.12), we have

$$\begin{aligned} 0 &\leq \zeta(\psi(d(Tx^*, Ty^*)), \psi(m(x^*, y^*))) \\ &= \zeta(\psi(d(x^*, y^*)), \psi(d(x^*, y^*))). \end{aligned} \tag{3.22}$$

If $d(x^*, y^*) > 0$, then $\psi(d(x^*, y^*)) > 0$. Hence it follows from (3.22) and (ζ_2) that

$$\begin{aligned} 0 &\leq \zeta(\psi(d(x^*, y^*)), \psi(d(x^*, y^*))) \\ &< \psi(d(x^*, y^*)) - \psi(d(x^*, y^*)) = 0, \end{aligned}$$

which is a contradiction. Hence $d(x^*, y^*) = 0$, and hence T has a unique fixed point. □

Corollary 3.6. *Let (X, d) be a complete metric space and let $T : X \to X$ be a generalized rational (α, β, Z)-contraction mapping with respect to ζ such that*

$$\zeta(d(Tx, Ty), m(x, y)) \geq 0$$

for all $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$. Suppose that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$, where $x_0 \in X$. Assume that either

(1) *T is continuous or*
(2) *if $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(x_n, x) = 0$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$.*

If $T : X \to X$ is cyclic (α, β)-admissible, then T has a fixed point in X. Further if $\alpha(x)\beta(y) \geq 1$ for all fixed points x, y of T, then T has a unique fixed point.

Note that the continuity of the mapping T in Theorem 3.2 can be dropped if we replace condition (3) by a suitable one as in the following result.

Corollary 3.7. *Let (X, d) be a complete metric space, $T : X \to X$ be a mapping and $\alpha, \beta : X \to [0, +\infty)$ be two functions. Suppose that the following conditions hold:*

(1) *T is a rational (α, β, Z)-contraction mapping.*
(2) *There exists an element $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.*
(3) *If $\{x_n\}$ is a sequence in X converges to $x \in X$ with $\alpha(x_n) \geq 1$ (or $\beta(x_n) \geq 1$) for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$ (or $\alpha(x) \geq 1$) for all $n \in \mathbb{N}$.*

Then T has a fixed point.

By taking the function $\beta : X \to [0, +\infty)$ to be α in Theorem 3.2, we get the following Corollary:

Corollary 3.8. *Let (X, d) be a complete metric space, $T : X \to X$ be a mapping and $\alpha : X \to [0, +\infty)$ be a function. Suppose that the following conditions hold:*

(1) *There exists $\zeta \in Z$ such that if $x, y \in X$ with $\alpha(x)\alpha(y) \geq 1$, then $\zeta(d(Tx, Ty), M(x, y)) \geq 0$, where*
$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}, \frac{d(x, Tx)d(y, Ty)}{1 + d(Tx, Ty)}\right\}.$$
(2) *If $x \in X$ with $\alpha(x) \geq 1$, then $\alpha(Tx) \geq 1$.*
(3) *There exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$.*
(4) *If $\{x_n\}$ is a sequence in X converges to $x \in X$ with $\alpha(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Example 3.9. Let $X = [-1, 1]$. Define $d : X \times X \to \mathbb{R}$ by $d(x, y) = |x - y|$. Also, define the mapping $T : X \to X$ the two functions $\alpha, \beta : X \to [0, \infty)$ and the function $\zeta : [0, +\infty) \times [0, \infty) \to \mathbb{R}$ as follows:

$$T(x) = \begin{cases} \frac{x}{4}, & \text{if} \quad x \in [0, 1], \\ 1/4, & \text{otherwise}, \end{cases}$$

$$\alpha(x) = \begin{cases} \frac{x+3}{2}, & \text{if} \quad x \in [0, 1], \\ 0, & \text{otherwise}, \end{cases}$$

$$\beta(x) = \begin{cases} \frac{x+5}{4}, & \text{if} \quad x \in [0, 1], \\ 0, & \text{otherwise}, \end{cases}$$

$$\zeta(t, s) = \frac{s}{s+1} - t.$$

Then, we have the following:

(1) (X, d) is a complete metric space.
(2) ζ is a simulation function.
(3) There exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.
(4) T is continuous.
(5) T is cyclic (α, β)-admissible mapping.
(6) For $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$, we have

$$\zeta(d(Tx, Ty), M(x, y)) \geq 0,$$

where

$$M(x, y) = \max\Big\{d(x, y), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}, \frac{d(x, Tx)d(y, Ty)}{1 + d(Tx, Ty)}\Big\}.$$

Indeed, the proof of $(1), (2), (3)$ and (4) are clear. To prove (5), let $x \in X$. If $\alpha(x) \geq 1$ then $x \in [0, 1]$. So,

$$\beta(Tx) = \beta(x/4) = \frac{(x/4) + 5}{4} = \frac{x + 20}{16} \geq 1.$$

If $\beta(x) \geq 1$, then $x \in [0, 1]$. So,

$$\alpha(Tx) = \alpha(x/4) = \frac{(x/4) + 3}{2} = \frac{x + 12}{8} \geq 1.$$

So, T is cyclic (α,β)-admissible. To prove (6), let $x, y \in X$ with $\alpha(x)\beta(x) \geq 1$. Then $x, y \in [0,1]$, therefore, we have

$$\begin{aligned}
\zeta(d(Tx,Ty),M(x,y)) &= \frac{M(x,y)}{1+M(x,y)} - d(Tx,Ty) \\
&\geq \frac{d(x,y)}{1+d(x,y)} - |T(x)-T(y)| \\
&= \frac{d(x,y)}{1+d(x,y)} - |x/4 - y/4| \\
&= \frac{|x-y|}{1+|x-y|} - |x/4-y/4| \\
&= \frac{3|x-y| - |x-y|^2}{4[1+|x-y|]} \geq 0.
\end{aligned}$$

So, T is a rational (α,β,Z)-contraction mapping. Hence this satisfies all the conditions of Theorem 3.2. So T has fixed point. Here 0 is the fixed point of T.

4. Conclusion

In this paper, we establish some unique fixed point results for rational (α,β,Z)-contraction mapping and generalized rational (α,β,Z)-contraction mapping in the setting of complete metric space via a cyclic (α,β)-admissible mapping imbedded in simulation function. Our results extend and generalize several results from the existing literature.

Acknowledgements: The authors are thankful to the learned referee for his/her deep observations and their suggestions, which greatly helped us to improve the paper significantly.

References

[1] S. Alizadeh, F. Moradlou and P. Salimi, *Some fixed point results for* (α,β)-(ψ,ϕ)-*contractive mappings,* Filomat, **28**(3) (2014), 635–647.

[2] H.H. Alsulami, E. Karapinar, F. Khojasteh and A.F. Roldān-Lōpez-de-Hierro, *A proposal to the study of contractions in quasi-metric spaces,* Discrete Dyna. Nature and Soc., (2014) Article ID 269286, 1–10.

[3] A.H. Ansari, J. Nantadilok and M.S. Khan, *Best proximity points of generalized cyclic weak* (F,ψ,φ)*-contractions in ordered metric spaces*, Nonlinear Funct. Anal. Appl., **25**(1) (2020), 55–67.

[4] H. Argoubi, B. Samet and C. Vetro, *Nonlinear contractions involving simulation functions in a metric space with a partial order,* J. Nonlinear Sci. Appl., **8** (2015), 1082–1094.

[5] H. Aydi and A. Felhi, *Fixed points in modular spaces via* α*-admissible mappings and simulation functions*, J. Nonlinear Sci. Appl., **9** (2016), 3686–3701.

[6] H. Aydi, A. Felhi and S. Sahmim, *On common fixed points for* (α, ψ)*-contractions and generalized cyclic contractions in b-metric like spaces and consequences*, J. Nonlinear Sci. Appl., **9** (2016), 2492–2510.

[7] H. Aydi, E. Karapinar and V. Rakoćević, *Non unique fixed point theorem on b-metric spaces via simulation functions*, Jordan J. Math. Stat., **12** (2019), 265–288.

[8] M. Bousselsal and Z. Mostefaoui, *Some common fixed point results in partial metric spaces for generalized rational type contraction mappings*, Nonlinear Funct. Anal. Appl., **20**(1) (2015), 43–54.

[9] Seong-Hoon Cho, *Fixed point theorem for* (α, β)*-z contractions in metric spaces*, Int. J. Math. Anal., **13**(4) (2019), 161–174.

[10] A. Das, B. Hazarika, H.K. Nashine and J.K. Kim, $\psi-$*coupled fixed point theorem via simulation functions in complete partially ordered metric space and its applications*, Nonlinear Funct. Anal. Appl., **26**(2) (2021), 273–288.

[11] E. Karapinar *Fixed points results via simulation functions,* Filomat, **30** (2016), 2343-2350.

[12] E. Karapinar and F. Khojasteh, *An approach to best proximity point results via simulation functions*, J. Fixed Point Theorey Appl., **19** (2017), 1983–1995.

[13] E. Karapinar and B. Samet, *Generalized* $\alpha - \psi$ *contractive type mappings and related fixed point theorems with applications*, Abst. Appl. Anal., **2012** (2012), 17 pages.

[14] F. Khojasteh, S. Shukla and S. Radenovic, *A new approach to the study of fixed point theory for simulation function*, Filomat, **29**(6) (2015), 1189–1194.

[15] H.K. Nashine, R.W. Ibrahim, Y.J. Cho and J.K. Kim, *Fixed point theorems for the modified simulation function and applications to fractional economics systems*, Nonlinear Funct. Anal. Appl., **26** (1) (2021), 137–155.

[16] H. Qawagneh, M.S. Noorani, W. Shatanawi, K. Abodayeh and H. Alsamir, *Fixed point for mappings under contractive condition based on simulation functions and cyclic* (α, β)*-admissibility*, J. Math. Anal., **9**(1) (2018), 38–51.

[17] A.F. Roldān-Lōpez-de-Hierro, E. Karapinar, C. Roldān-Lōpez-de-Hierro and J. Martinez-Moreno, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math., **275** (2015), 345–355.

[18] B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for* $(\alpha - \psi)$*-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165.