



DIFFERENTIAL INEQUALITIES ASSOCIATED WITH CARATHÉODORY FUNCTIONS

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Abstract. The purpose of the present paper is to estimate some real parts for certain analytic functions with some applications in connection with certain integral operators and geometric properties. Also we extend some known results as special cases of main results presented here.

1. INTRODUCTION

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We denote by \mathcal{P} the class of analytic functions $p : \mathbb{U} \rightarrow \mathbb{C}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

with $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{U}$. This class \mathcal{P} is known as the Carathéodory class or the class of functions with positive real part [3, 4], pioneered by Carathéodory. The theory of Carathéodory functions plays very important role on geometric function theory. For recent developments, the readers may refer to the works

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of Kim and Cho [5], Kwon and Sim [7], Nunokawa *et al.* [14], Sim *et al.* [18] and Wang [21]. Let \mathcal{A} denote the class of analytic functions f in the open unit disk \mathbb{U} with the usual normalization $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}^*(\alpha)$ denote the subclass of \mathcal{A} consisting of starlike functions of order α in \mathbb{U} . The class \mathcal{S}^* of starlike functions is identified by $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. A function $f \in \mathcal{A}$ is said to be in $\mathcal{B}(\alpha, \beta, \gamma)$ if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f^{1-\gamma}(z)g^\gamma(z)} \right\} > \beta \quad (z \in \mathbb{U})$$

for some $g \in \mathcal{S}^*(\alpha)$ and γ ($\gamma > 0$), β ($0 \leq \beta < 1$). Furthermore, we denote by $\mathcal{B}_1(0, \beta, \gamma)$ the subclass of $\mathcal{B}(\alpha, \beta, \gamma)$ for which $g(z) \equiv z$. We note that $\mathcal{B}_1(\beta, 0) = \mathcal{S}^*(\beta)$ and $\mathcal{B}_1(\beta, 1)$ is the subclass of \mathcal{A} consisting of functions such that $\operatorname{Re}f'(z) > \beta$ ($z \in \mathbb{U}$) (see, for details [1, 16, 19, 20]). Many authors (for example, see, [2, 6, 8, 9]) have studied the integral operators of the form:

$$\mathcal{J}_{c,\mu}(f) = \left(\frac{c+\mu}{z^c} \int_0^z t^{c-1} f^\mu(t) dt \right)^{\frac{1}{\mu}}, \quad (1.1)$$

where c and μ are suitably chosen real constants and f belongs to some favoured classes of univalent functions.

Motivated by the works mentioned above, in the present paper, we obtain some estimates of real parts for certain analytic functions. Also we give various applications for functions belonging to \mathcal{A} and integral operator given by (1.1).

2. MAIN RESULTS

In proving our results, we shall need the following lemmas due to Miller and Mocanu [11] and Nunokawa [12], respectively.

Lemma 2.1. *Let $\phi(u, v)$ be a complex-valued function,*

$$\phi : D \rightarrow \mathbb{C} \quad (D \subseteq \mathbb{C}^2, \mathbb{C} \text{ is the complex plane})$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u, v)$ satisfies the following conditions:

- (1) $\phi(u, v)$ is continuous ;
- (2) $(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} > 0$;
- (3) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -(1 + u_2^2)/2$, $\operatorname{Re}\{\phi(iu_2, v_1)\} \leq 0$.

Let $p(z) = 1 + p_1z + p_2z^2 + \dots$ be regular in \mathbb{U} such that $(p(z), zp'(z)) \in D$ for all $z \in \mathbb{U}$. If $\operatorname{Re}\{\phi(p(z), zp'(z))\} > 0$ ($z \in \mathbb{U}$), then $\operatorname{Re}\{p(z)\} > 0$ ($z \in \mathbb{U}$).

Lemma 2.2. *Let p be analytic in \mathbb{U} with $p(0) = 1$ and $p(z) \neq 0$ in \mathbb{U} . Suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg\{p(z)\}| < \frac{\pi}{2}\alpha \text{ for } |z| < |z_0| \tag{2.1}$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi}{2}\alpha \text{ (} 0 < \alpha \leq 1 \text{)}. \tag{2.2}$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha k, \tag{2.3}$$

where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \text{ when } \arg\{p(z_0)\} = \frac{\pi}{2}\alpha, \tag{2.4}$$

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \text{ when } \arg\{p(z_0)\} = -\frac{\pi}{2}\alpha \tag{2.5}$$

and

$$\{p(z_0)\}^{\frac{1}{\alpha}} = \pm ia \text{ (} a > 0 \text{)}. \tag{2.6}$$

With the help of Lemma 2.1, we now derive the following theorem.

Theorem 2.3. *Let p be analytic in \mathbb{U} with $p(0) = 1$. If*

$$\operatorname{Re}\{p(z) + r(z)zp'(z)\} > -\frac{1}{2}\delta \text{ (} 0 \leq \delta < 1; z \in \mathbb{U} \text{)},$$

where r is analytic in \mathbb{U} with $\operatorname{Re}\{r(z)\} \geq \delta$, then $\operatorname{Re}\{p(z)\} > 0$ ($z \in \mathbb{U}$).

Proof. Let us put

$$q(z) = \frac{2}{2+\delta} \left\{ p(z) + r(z)zp'(z) + \frac{1}{2}\delta \right\}.$$

Then q is analytic in \mathbb{U} with $q(0) = 1$ and $\operatorname{Re}\{q(z)\} > 0$ ($z \in \mathbb{U}$).

Let

$$\phi(u, v; z) = \frac{2}{2+\delta} \left\{ u + r(z)v + \frac{1}{2}\delta \right\}.$$

Then $\phi(u, v; z)$ satisfies

- (1) $\phi(u, v; z)$ is continuous in $D = \mathbb{C} \times \mathbb{C}$;
- (2) $(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} = 1 > 0$;

(3) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -(1 + u_2^2)/2$,

$$\begin{aligned} \operatorname{Re}\{\phi(iu_2, v_1; z)\} &= \frac{2}{2 + \delta} v_1 \operatorname{Re}\{r(z)\} + \frac{\delta}{2 + \delta} \\ &\leq -\frac{(1 + u_2^2)\delta}{2 + \delta} + \frac{\delta}{2 + \delta} \\ &\leq 0. \end{aligned}$$

Thus we see that $\phi(u, v; z)$ satisfies the conditions in Lemma 2.1. Therefore this shows that $\operatorname{Re}\{p(z)\} > 0$ ($z \in \mathbb{U}$). \square

Corollary 2.4. *Let $f, g \in \mathcal{A}$ and*

$$\operatorname{Re}\left\{\frac{g(z)}{zg'(z)}\right\} \geq \delta \quad (0 \leq \delta < 1; z \in \mathbb{U}).$$

If

$$\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > \beta - \frac{1}{2}\delta(1 - \beta) \quad (0 \leq \beta < 1; z \in \mathbb{U}),$$

then

$$\operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\} > \beta \quad (z \in \mathbb{U}).$$

Proof. Let

$$p(z) = \frac{1}{1 - \beta} \left(\frac{f(z)}{g(z)} - \beta \right).$$

Then p is analytic in \mathbb{U} with $p(0) = 1$. Hence we obtain

$$\frac{1}{1 - \beta} \left(\frac{f'(z)}{g'(z)} - \beta \right) = p(z) + \frac{g(z)}{zg'(z)} zp'(z).$$

Therefore, applying Theorem 2.3, we have the result. \square

Remark 2.5. Taking $\delta = 0$ in Corollary 2.4, we have the result obtained by Libera [8], MacGregor [10] and Sakaguchi [17].

Corollary 2.6. *Let c and μ be real numbers with $c \geq 0$, $\mu > 0$, respectively and let $f \in \mathcal{A}$. If*

$$\operatorname{Re}\left\{\frac{zf'(z)f^{\mu-1}(z)}{z^\mu}\right\} > \beta - \frac{1}{2}(c + \mu)(1 - \beta) \quad (0 \leq c + \mu < 1; \beta < 1; z \in \mathbb{U}),$$

then

$$\operatorname{Re}\left\{\frac{z(J_{c,\mu}(f))'J_{c,\mu}^{\mu-1}(f)}{z^\mu}\right\} > \beta \quad (z \in \mathbb{U}),$$

where $J_{c,\mu}$ is the integral operator defined by (1.1).

Proof. It follows from (1.1) that

$$\mu \frac{z(J_{\mu,c}(f))'}{J_{\mu,c}^{1-\mu}(f)} + cJ_{\mu,c}^\mu(f) = (\mu + c)f^\mu(z). \tag{2.7}$$

Let

$$p(z) = \frac{M(z)}{N(z)} \quad (z \in \mathbb{U}),$$

where

$$M(z) = \frac{1}{1-\beta} \left(\frac{(J_{\mu,c}(f))'}{\left(\frac{J_{\mu,c}f}{z}\right)^{1-\mu}} z^{\mu+c} - \beta z^{\mu+c} \right)$$

and

$$N(z) = z^{\mu+c}.$$

Then p is analytic in \mathbb{U} with $p(0) = 1$. From (2.7), we have

$$\frac{M'(z)}{N'(z)} = \frac{1}{1-\beta} \left(\frac{f'(z)}{\left(\frac{f(z)}{z}\right)^{1-\mu}} - \beta \right).$$

Applying Corollary 2.4, we have

$$\operatorname{Re} \left\{ \frac{z(J_{c,\mu}(f))' J_{c,\mu}^{\mu-1}}{z^\mu} \right\} > \beta.$$

□

Theorem 2.7. *Let p be analytic in \mathbb{U} with $p(0) = 1$. If*

$$\operatorname{Re} \left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \right\} > -\frac{\gamma}{2\beta^2} \quad (\beta > \gamma; \gamma \geq 0; z \in \mathbb{U}),$$

then $\operatorname{Re}\{p(z)\} > 0$ ($z \in \mathbb{U}$).

Proof. Let us put

$$g(z) = \frac{2\beta^2}{\gamma + 2\beta^2} \left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} + \frac{\gamma}{2\beta^2} \right\}.$$

Then g is analytic in \mathbb{U} with $g(0) = 1$ and $\operatorname{Re}\{g(z)\} > 0$. Let

$$\phi(u, v) = \frac{2\beta^2}{\gamma + 2\beta^2} \left\{ u + \frac{v}{\beta u + \gamma} + \frac{\gamma}{2\beta^2} \right\}.$$

Then $\phi(u, v)$ satisfies

- (1) $\phi(u, v)$ is continuous in $D = (\mathbb{C} \setminus \{-\gamma/\beta\}) \times \mathbb{C}$;

(2) $(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} = 1 > 0$;

(3) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -(1 + u_2^2)/2$,

$$\begin{aligned} \operatorname{Re}\{\phi(iu_2, v_1)\} &= \frac{2\beta^2}{\gamma + 2\beta^2} \left\{ \frac{\gamma v_1}{\gamma^2 + \beta^2 u_2^2} + \frac{\gamma}{2\beta^2} \right\} \\ &\leq \frac{2\beta^2}{\gamma + 2\beta^2} \left\{ -\frac{\gamma(1 + u_2^2)}{2(\gamma^2 + \beta^2 u_2^2)} + \frac{\gamma}{2\beta^2} \right\} \\ &\leq 0. \end{aligned}$$

Thus we see that $\phi(u, v)$ satisfies the conditions in Lemma 2.1. This completes the proof. \square

If we take $\beta = 1$ and $\gamma = 0$ in Theorem 2.7, we can get the following corollary.

Corollary 2.8. *Let p be analytic in \mathbb{U} with $p(0) = 1$ and satisfy the differential equation:*

$$B(z)p(z) + zp'(z) = 1 \quad (z \in \mathbb{U}),$$

where B is analytic with $B(0) = 1$. If $\operatorname{Re}\{B(z)\} > 0$ for $z \in \mathbb{U}$, then $\operatorname{Re}\{p(z)\} > 0$ for $z \in \mathbb{U}$.

Putting $p(z) = zf'(z)/f(z)$, $\beta = 1/\alpha$ and $\gamma = 0$ in Theorem 2.7 or $p(z) = f(z)/zf'(z)$ in Corollary 2.8, we have the following result.

Corollary 2.9. *Let $f \in \mathcal{A}$ and $\alpha > 0$. If*

$$\operatorname{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > 0 \quad (z \in \mathbb{U}),$$

then $f \in \mathcal{S}^*$.

Corollary 2.10. *Let $f \in \mathcal{A}$ and let $\mu > c$ ($c \geq 0$). If*

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > -\frac{c}{2\mu^2} \quad (z \in \mathbb{U}),$$

then $J_{c,\mu}(f) \in \mathcal{S}^*$, where $J_{c,\mu}(f)$ is the integral operator defined by (1.1).

Proof. Setting

$$p(z) = \frac{z(J_{c,\mu}(f))'}{J_{c,\mu}(f)} \quad (z \in \mathbb{U}),$$

then we obtain

$$p(z) + \frac{c}{\mu} = \frac{z^c f^\mu(z)}{\mu \int_0^z t^{c-1} f^\mu(t) dt}.$$

Taking logarithmic derivatives and multiply by z , after some simple calculations, we have

$$\frac{zp'(z)}{p(z) + c/\mu} = \frac{\mu zf'(z)}{f(z)} - \mu p(z).$$

and

$$\frac{zf'(z)}{f(z)} = p(z) + \frac{zp'(z)}{\mu p(z) + c}.$$

Therefore by using Theorem 2.7, we have desired result. □

Taking $\beta = 1$, $\gamma = 1/\alpha - 1$ ($\alpha > 0$) and $p(z) = zf'(z)/f(z)$ in Theorem 2.7, we have the following corollary.

Corollary 2.11. *Let $f \in \mathcal{A}$ and $0 < \alpha \leq 1$. If*

$$\operatorname{Re} \left\{ \frac{\alpha z(zf'(z))' + (1 - \alpha)zf'(z)}{\alpha zf'(z) + (1 - \alpha)f(z)} \right\} > \frac{\alpha - 1}{2\alpha} \quad (z \in \mathbb{U}),$$

then $f \in \mathcal{S}^$.*

Proof. Putting

$$p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathbb{U}),$$

then we have

$$\begin{aligned} &\alpha z(zf'(z))' + (1 - \alpha)zf'(z) \\ &= \alpha z f(z)p'(z) + \alpha zp(z)f'(z) + (1 - \alpha)p(z)f(z) \\ &= \{\alpha zp'(z) + p(z)(\alpha p(z) + (1 - \alpha))\}f(z) \end{aligned}$$

and

$$\alpha zf'(z) + (1 - \alpha)f(z) = (\alpha p(z) + (1 - \alpha))f(z).$$

Hence, we obtain

$$\begin{aligned} \frac{\alpha z(zf'(z))' + (1 - \alpha)zf'(z)}{\alpha zf'(z) + (1 - \alpha)f(z)} &= \frac{\alpha zp'(z) + p(z)(\alpha p(z) + 1 - \alpha)}{\alpha p(z) + 1 - \alpha} \\ &= p(z) + \frac{zp'(z)}{p(z) + (\frac{1}{\alpha} - 1)}. \end{aligned}$$

Therefore, applying Theorem 2.7, we have desired result. □

Next, by virtue of Lemma 2.2, we now prove the following theorem.

Theorem 2.12. *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in \mathbb{U} . Suppose that*

$$\operatorname{Re} \left\{ p(z) + \frac{zp'(z)}{p(z)} \right\} < \frac{\beta(2\beta - 1)}{2(\beta - 1)} \quad (\beta > 1; z \in \mathbb{U}),$$

and for arbitrary real number r ($0 < r < 1$), $p(z)$ satisfies the following condition:

$$\operatorname{Max}_{|z| \leq r} \operatorname{Re}\{p(z)\} = \operatorname{Re}\{p(z_0)\} \neq p(z_0).$$

Then

$$\operatorname{Re}\{p(z)\} < \beta \quad (z \in \mathbb{U}).$$

Proof. From the assumption of Theorem 2.12, we see that

$$p(z) \neq 0 \quad (z \in \mathbb{U}).$$

Let us put

$$q(z) = \frac{\beta - p(z)}{\beta - 1} \quad (\beta > 1; z \in \mathbb{U}).$$

Then $q(z)$ is analytic in \mathbb{U} with $q(0) = 1$. If there exists a point $z_0 \in \mathbb{U}$ such that

$$\operatorname{Re}\{q(z)\} > 0 \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re}\{q(z_0)\} = 0,$$

then from the assumption of Theorem 2.12, we obtain that $q(z_0) \neq 0$. Hence from Lemma 2.2 with $\alpha = 1$, we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik,$$

where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \geq 1 \quad \text{for } \arg\{q(z_0)\} = \frac{\pi}{2},$$

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \leq -1 \quad \text{for } \arg\{q(z_0)\} = -\frac{\pi}{2},$$

and

$$q(z_0) = \pm ia, \quad a > 0.$$

For the case $\arg q(z_0) = \pi/2$, it follows that

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{-z_0 p'(z_0)}{\beta - p(z_0)} = ik$$

and

$$-\frac{z_0 p'(z_0)}{p(z_0)} = \frac{-(\beta - 1)ak}{\beta - i(\beta - 1)a} = \frac{\beta + i(\beta - 1)a}{\beta^2 + (\beta - 1)^2 a^2} (-(\beta - 1)ak).$$

Therefore, we have

$$\operatorname{Re} \left\{ \frac{z_0 p'(z_0)}{p(z_0)} \right\} \geq \frac{\beta(\beta - 1)}{2} \frac{a^2 + 1}{(\beta - 1)^2 a^2 + \beta^2}.$$

While, the function

$$g(a) = \frac{a^2 + 1}{(\beta - 1)^2 a^2 + \beta^2}$$

is increasing for $a > 0$ and $\beta > 1$. Therefore, we obtain that

$$\operatorname{Re} \left\{ \frac{z_0 p'(z_0)}{p(z_0)} \right\} \geq \frac{\beta}{2(\beta - 1)},$$

and so

$$\operatorname{Re} \left\{ p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} \right\} \geq \beta + \frac{\beta}{2(\beta - 1)} = \frac{\beta(2\beta - 1)}{2(\beta - 1)}.$$

This is a contradiction to the assumption of Theorem 2.12. For the case $\arg q(z_0) = -\pi/2$, applying the same method as the above, we have

$$\operatorname{Re} \left\{ p(z) + \frac{z_0 p'(z_0)}{p(z_0)} \right\} \geq \frac{\beta(2\beta - 1)}{2(\beta - 1)}.$$

This is also a contradiction to the assumption of Theorem 2.12. Therefore we complete the proof of theorem. \square

Corollary 2.13. *Let $f \in \mathcal{A}$. Suppose that*

$$1 + \operatorname{Re} \left\{ \frac{z f''(z)}{f'(z)} \right\} < \alpha \quad \left(\alpha > \frac{3 + \sqrt{2}}{2}; z \in \mathbb{U} \right),$$

and for arbitrary real number r ($0 < r < 1$), $z f'(z)/f(z)$ satisfies the following property:

$$\operatorname{Max}_{|z| \leq r} \operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} = \operatorname{Re} \left\{ \frac{z_0 f'(z_0)}{f(z_0)} \right\} \neq \frac{z_0 f'(z_0)}{f(z_0)}.$$

Then

$$\operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} < \frac{1 + 2\alpha + \sqrt{4\alpha^2 - 12\alpha + 1}}{2} \quad (z \in \mathbb{U}).$$

Theorem 2.14. *Let $p(z)$ be analytic in \mathbb{U} with $p(0) = 1$, $p(z) \neq 0$ in \mathbb{U} and suppose that*

$$\left| \frac{p(z) + \beta \frac{z p'(z)}{p(z)} - \gamma}{p(z)} \right| < \sqrt{\left(1 + \frac{\beta}{2} + \frac{\beta}{2|p(z)|^2} \right)^2 + \frac{\gamma^2}{|p(z)|^2}} \quad (\beta > 0; \gamma \geq 0; z \in \mathbb{U}). \tag{2.8}$$

Then we have

$$\operatorname{Re}\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

Proof. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\operatorname{Re}\{p(z)\} > 0 \quad \text{for } |z| < |z_0|$$

and

$$p(z_0) = \pm ia \quad (a > 0).$$

Then by Lemma 2.2, we have

$$\begin{aligned} \left| \frac{p(z_0) + \beta \frac{z_0 p'(z_0)}{p(z_0)} - \gamma}{p(z_0)} \right| &= \left| 1 + \frac{\beta k}{a} + \frac{\gamma i}{a} \right| \\ &\geq \sqrt{\left(1 + \frac{\beta}{2a} \left(\frac{1+a^2}{a}\right)\right)^2 + \left(\frac{\gamma}{a}\right)^2} \\ &= \sqrt{\left(1 + \frac{\beta}{2} + \frac{\beta}{2|p(z_0)|^2}\right)^2 + \frac{\gamma^2}{|p(z_0)|^2}} \end{aligned}$$

which is a contradiction to (2.8). Therefore we obtain

$$\operatorname{Re}\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

□

Taking $p(z) = zf'(z)/f(z)$, $\beta = 1$ and $\gamma = 0$ in the Theorem 2.14, we have the corresponding result obtained by Nunokawa [13].

Corollary 2.15. *Let $f \in \mathcal{A}$ with $f(z)f'(z) \neq 0$ in $\mathbb{U} \setminus \{0\}$ and suppose that*

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < \frac{3}{2} \left| \frac{zf'(z)}{f(z)} \right| + \frac{1}{2} \left| \frac{f(z)}{zf'(z)} \right| \quad (z \in \mathbb{U}).$$

Then $f \in \mathcal{S}^$.*

Remark 2.16. Corollary 2.15 is also an improvement of the earlier result by Obradović and Owa [15].

Letting $p(z) = zf'(z)/f(z)$ and $\beta = \gamma = 1$ in Theorem 2.14, we have the following result.

Corollary 2.17. *Let $f \in \mathcal{A}$ with $f(z)f'(z) \neq 0$ in $\mathbb{U} \setminus \{0\}$ and suppose that*

$$\left| \frac{f(z)f''(z)}{(f'(z))^2} \right| < \sqrt{\left(\frac{3}{2} + \frac{1}{2} \left| \frac{f(z)}{zf'(z)} \right|^2\right)^2 + \left| \frac{f(z)}{zf'(z)} \right|^2} \quad (z \in \mathbb{U}).$$

Then $f \in \mathcal{S}^$.*

Proof. Set

$$p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathbb{U}).$$

Then p is analytic in \mathbb{U} and

$$\frac{f(z)f''(z)}{(f'(z))^2} = 1 - \frac{1}{p(z)} + \frac{zp'(z)}{(p(z))^2}.$$

Applying Theorem 2.14, we have Corollary 2.17. □

Making $p(z) = zf'(z)/f(z)$ and $\beta = \gamma$ in Theorem 2.14, we have:

Corollary 2.18. *Let $f \in \mathcal{A}$ with $f(z)f'(z) \neq 0$ in $\mathbb{U} \setminus \{0\}$ and suppose that*

$$\begin{aligned} & \left| \beta \frac{f(z)f''(z)}{(f'(z))^2} + 1 - \beta \right| \\ & < \sqrt{\left(1 + \frac{\beta}{2} + \frac{\beta}{2} \left| \frac{f(z)}{zf'(z)} \right|^2\right)^2 + \beta^2 \left| \frac{f(z)}{zf'(z)} \right|^2} \quad (\beta > 0; z \in \mathbb{U}). \end{aligned}$$

Then $f \in \mathcal{S}^$.*

Proof. Letting

$$p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathbb{U}),$$

we see that p is analytic in \mathbb{U} with $p(z) \neq 0$ in \mathbb{U} and

$$\beta \frac{zf''(z)}{f'(z)} + (1 - \beta) \frac{zf'(z)}{f(z)} = p(z) - \beta + \beta \frac{zp'(z)}{p(z)}.$$

Therefore, by Theorem 2.14, we have desired result. □

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