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## TRIGONOMETRIC JACKSON INTEGRALS APPROXIMATION BY A $k$ -GENERALIZED MODULUS OF SMOOTHNESS

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**Abstract.** The need for smoothness measures emerged by mathematicians working in the fields of approximation theory, functional analysis and real analysis. In the present paper, a new version of generalized modulus of smoothness is studied. The aim of defining that modulus, is to find the degree of best  $L_p$  functions approximation via trigonometric polynomials. We benefit from Jackson integrals to arrive to the essential approximation theorems.

### 1. INTRODUCTION

Many papers have been introduced about approximation theorems in terms of moduli of smoothness (see [2, 3, 4, 6]). In addition, numerous authors have investigated the problem of approximation by several types of functions, see [5].

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In [6], the authors proved direct and inverse inequalities for the best algebraic approximation with modulus of smoothness with order 2.

Here, we define  $k$ th order modulus of smoothness to serve as a generalization of their modulus.

## 2. PRELIMINARIES

Now let us list some definitions, which are used throughout our present paper. Let the following periodic function space is given by

$$L_p[-\pi, \pi] = \left\{ f : \|f\|_p = \left( \int_{-\pi}^{\pi} |f(x)|^p dx \right)^{1/p} < \infty \right\}.$$

We define here an operator that yields a version of trigonometric Jackson integrals as follow,

**Definition 2.1.** For any  $m, k \in N$ ,  $f \in L_p[-\pi, \pi]$ , define

$$G_{m,k}(f(x)) = \sum_{i=1}^k f \left( \cos \left( \arccos x + \left( \frac{k}{2} - i \right) h \right) \right) J_m(x),$$

where  $J_m(x) = \frac{1}{2} + \sum_{i=1}^k \rho_{i,m} \cos ix$ , is the Fejer Kernel from [1], that satisfies

$$\int_{-\pi}^{\pi} J_m(x) dx = 1.$$

Now, define

$$\Omega_k(f, t)_p = \sup_{|h| \leq t} \|\Delta_h^k f(\cdot)\|_p,$$

to be the  $k$ -genalized modulus of smoothness in the interval  $[-\pi, \pi]$ , where

$$\Delta_h^k f(x) = \sum_{i=1}^k \binom{k}{i} (-1)^k f \left( \cos \left( \arccos x + \left( \frac{k}{2} - i \right) h \right) \right).$$

Define  $\Xi$  to be the set of all operators of the above form of  $G_{m,k}$ .

Define,

$$E_n(f)_p = \inf_{G \in \Xi} \|f - G\|_p$$

to be the degree of best approximation of  $L_p[-\pi, \pi]$  out of  $\Xi$ .

3. MAIN RESULTS

We introduce the results that we need for the proof of our approximation theorems.

**Lemma 3.1.** *For any  $G_{m,k} \in \Xi$ , we have*

$$\Omega_{k,r}^\varphi(G_{m,k}, t)_p \leq C \|G_{m,k}\|_p,$$

where  $C$  is a constant of  $p, m$  and  $k$ .

*Proof.* It is direct from quasi-triangle inequality that

$$\begin{aligned} & \left\| \Delta_h^k G_{m,k}(f) \right\|_p^p \\ &= \left\| \sum_{i=1}^k \binom{k}{i} (-1)^k f \left( \cos \left( \arccos x + \left( \frac{k}{2} - i \right) h \right) \right) J_m(x) \right\|_p^p \\ &\leq C \sum_{i=1}^k \binom{k}{i} (-1)^k \left\| f \left( \cos \left( \arccos x + \left( \frac{k}{2} - i \right) h \right) \right) J_m(x) \right\|_p^p \\ &\leq C \left\| f \left( \cos \left( \arccos x + \left( \frac{k}{2} - i \right) h \right) \right) J_m(x) \right\|_p^p \\ &\leq C(m, k, p) \|G_{m,k}(f)\|_p^p. \end{aligned}$$

□

Also, we need the following lemma (see [2]).

**Lemma 3.2.** *Let  $I = \cup I_m, m = 1, \dots, n$ , where  $|I_m| \sim h \sim \frac{1}{n}$  be any partition of  $I$ . Then for any  $m < n$ , and  $f \in L_p$ ,*

$$\sum_{m=1}^n E_m(f)_{L_p(I_m)} \leq C E_n(f)_{L_p(I)}.$$

The generalized modulus of smoothness that defined in Definition 2.1 has the following properties.

**Theorem 3.3.** *For any  $f \in L_p[-\pi, \pi]$ , we have*

- (1)  $\lim_{t \rightarrow 0} \Omega_k(f, t) = 0,$
- (2)  $\Omega_k(f + g, t)_p \leq \Omega_k(f, t)_p + \Omega_k(g, t)_p,$
- (3)  $\Omega_k(f, t)_p \leq \Omega_k(f, t')_p,$  for any  $t \leq t',$
- (4)  $\Omega_k(f, nt)_p \leq n^k \Omega_k(f, t)_p,$  for any  $n \in N,$
- (5)  $\Omega_k(f, \lambda t)_p \leq (1 + \lambda)^k \Omega_k(f, t)_p,$   $\lambda \in R,$
- (6)  $\Omega_k(f, t)_p \leq C \|f\|_p.$

*Proof.* Proofs of properties (1), (2) and (3) follow directly from definition of generalized moduli of smoothness. To prove property (4), we use the difference identity from [2]:

$$\Delta_{nh}^k = \sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} \Delta_{nh}^k f \left( x + \left( i_1 + i_2 + \cdots + i_k - \frac{nk}{2} \right) h \right).$$

We get the result by induction on  $k$ . From definition of  $k$ th difference and operator  $G$ , we have

$$\Delta_{nh} f(x) = \sum_{i=0}^{n-1} f \left( \cos \left( \arccos x + \left( \frac{k}{2} - i \right) h \right) \right).$$

Suppose that the relation is true for  $k$ . Then

$$\begin{aligned} \Delta_{nh}^{k+1} f(x) &= \Delta_{nh}^k [\Delta_{nh} f(x)] \\ &= \sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} \Delta_{nh}^k f \left( \cos(\arccos x + (\frac{k}{2} - i_1)h + \cdots \right. \\ &\quad \left. + (\frac{k}{2} - i_k)h + (\frac{k+1}{2} - i_{k+1})h) \right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|\Delta_{nh}^{k+1} f(x)\|_p^p &\leq \sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} \left\| l \Delta_{nh}^k f \left( \cos(\arccos x + (\frac{k}{2} - i_1)h + \cdots \right. \right. \\ &\quad \left. \left. + (\frac{k}{2} - i_k)h + (\frac{k+1}{2} - i_{k+1})h) \right) \right\|_p^p \\ &= n^k \Omega_k(f, t)_p^p. \end{aligned}$$

Moreover, we let  $n-1 < \lambda \leq n$ , to prove (5). Then

$$\Omega_k(f, \lambda t)_p \leq \Omega_k(f, nt)_p \leq n^k \Omega_k(f, nt)_p \leq (\lambda + 1)^k \Omega_k(f, t)_p.$$

For (6), it is easily to be proved by using inequalities of quasi-normed space, as follow:

$$\begin{aligned} \Omega_k(f, t)_p^p &\leq 2^p \sum_{i=0}^k \binom{k}{i}^p \int_{-\pi}^{\pi} \sup_{h \leq t} \left| f \left( \cos \left( \arccos x + \left( \frac{k}{2} - i \right) h \right) \right) \right|^p dx \\ &\leq C(k, p) \|f\|_p^p. \end{aligned}$$

□

It is time to find and estimate the degree of best approximation in the following theorems.

**Theorem 3.4.** For any  $f \in L_p[-\pi, \pi]$ ,  $0 < p < 1$ , we have

$$E_n(f)_p \leq C\Omega_k\left(f, \frac{1}{n}\right)_p.$$

*Proof.* Let  $f \in L_p[-\pi, \pi]$ ,  $G_{m,k} \in \Xi$ . By using Lemma 3.1, we have

$$\begin{aligned} E_n(f)_p &\leq \|f - G_{n,k}(f)\|_p^p \\ &= \int_{-\pi}^{\pi} \left| \sum_{i=1}^k \binom{k}{i} (-1)^k f\left(\cos\left(\arccos x + \left(\frac{k}{2} - i\right)h\right)\right) J_m(x) - f(x) \right|^p dx \\ &\leq C \sum_{i=1}^k \binom{k}{i} (-1)^k \int_{-\pi}^{\pi} \left| f\left(\cos\left(\arccos x + \left(\frac{k}{2} - i\right)h\right)\right) - f(x) \right|^p \\ &\quad \times \int_{-\pi}^{\pi} |J_m(x)|^p dx \\ &\leq C\Omega_k(f, h)_p^p. \end{aligned}$$

□

To make the degree of approximation more accurate, we prove the following lower estimate, so that the degree of approximation is essentially estimated.

**Theorem 3.5.** For any  $f \in L_p[-\pi, \pi]$ ,  $0 < p < 1$ , we have

$$\Omega_k\left(f, \frac{1}{n}\right)_p \leq C(p)E_n(f)_p.$$

*Proof.* For  $m < n$ , suppose that

$$\|G_n - G_m\|_p \leq 2C(p)E_m(f)_p.$$

Now, by letting  $b = \max\{i : 2^{-i} < n\}$ , so that by Lemma 3.1, Lemma 3.2 and Theorem 3.4 we get

$$\begin{aligned} \Omega_k(f, t)_p^p &\leq C \left[ \Omega_k(f - G_n, t)_p^p + \Omega_k(G_n, t)_p^p \right] \\ &\leq C \left[ \|f - G_n\|_p^p + \Omega_k(G_{2^b} - G_{2^{b-1}}, 2^{-b})_p^p \right] \\ &\leq C \left[ \Omega_k(f, t)_p^p + \sum_{m=2}^n (m+1)^{p-1} E_m(f)_{L_p(I_m)}^p \right] \\ &\leq 2CE_n(f)_p^p. \end{aligned}$$

□

## 4. CONCLUSIONS

Studying inverse and direct theorems is very important in the field of function approximation theory. In  $L_p$  spaces, we estimated the degree of best approximation by using newly defined trigonometric operators via generalized modulus of smoothness. Thus more generalizations are done for the main theorems.

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