Nonlinear Functional Analysis and Applications
Vol. 18, No. 1 (2013), pp. 85-102

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DOUBLE GAI SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULUS FUNCTIONS

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Abstract. In the present paper we introduce some double gai sequence spaces defined by a sequence of modulus functions $F = (f_{kl})$. We also study some topological properties and prove some inclusion relations between these spaces.

1. Introduction

The initial work on double sequences is found in Bromwich [6]. Later on, it was studied by Hardy [8], Moricz [12], Moricz and Rhoades [13], Tripathy [28, 29], Başarır and Sonalcan [4] and many others. Hardy [8] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [31] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [16] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Nextly, Mursaleen [15] and Mursaleen and Edely [17] have defined the almost strong regularity of matrices for double sequences and applied these

[0]Received October 11, 2012. Revised January 25, 2013.
[0]2000 Mathematics Subject Classification: 40A05, 40C05, 40D05.
[0]Keywords: Double sequence, difference sequence space, modulus function, solid, monotone.

matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{kl})$ into one whose core is a subset of the M-core of x. By the convergence of a double sequence we mean the convergence in the Pringsheim sense i.e. a double sequence $x = (x_{kl})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{kl} - L| < \epsilon$ whenever $k, l > n$ see [17]. We shall write more briefly as P-convergent. The four dimensional matrix transformation $(Ax)_{kl} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{kl}^{mn} x_{kl}$ was studied extensively by Robison [22]. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise. The double sequence $x = (x_{kl})$ is bounded if there exists a positive number M such that $|x_{kl}| < M$ for all k and l.

The notion of difference sequence spaces was introduced by Kızmaz [10], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et. and Çolak [7] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let r be non-negative integer, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta^r) = \{x = (x_k) \in w : (\Delta^r x_k) \in Z\},$$

where $\Delta^r x = (\Delta^r x_k) = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^r x_k = \sum_{w=0}^{r} (-1)^w \begin{pmatrix} r \\ w \end{pmatrix} x_{k+w}.$$

Taking $r = 1$, we get the spaces which were introduced and studied by Kızmaz [10].

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,
(2) $f(x + y) \le f(x) + f(y)$ for all $x \ge 0$, $y \ge 0$,
(3) f is increasing,
(4) f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. For more details about modulus function and sequence spaces we may refer to [2, 3, 11, 14, 19, 20, 21] and references therein.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$,
(2) $p(-x) = p(x)$, for all $x \in X$,
(3) $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [30] Theorem 10.4.2, P-183).

Definition 1.1. Let p, q be semi norms on a vector space X. Then p is said to be stronger than q if whenever (x_{mn}) is a sequence such that $p(x_{mn}) \to 0$, then $q(x_{mn}) \to 0$ also. If each is stronger than the others then p and q ae said to be equivalent.

Lemma 1.2. *Let p and q be semi norms on a linear space X. Then p is stronger than q if and only if there exists a constant M such that $q(x) \leq Mp(x)$ for all $x \in X$.*

Definition 1.3. A sequence E is said to be solid or normal if $\alpha_{mn} x_{mn} \in E$ whenever $x_{mn} \in E$ and for all sequence of scalars α_{mn} with $|\alpha_{mn}| \leq 1$, for all $m, n \in \mathbb{N}$.

Definition 1.4. A sequence space E is said to be monotone if it contains the cannonical pre-images of all its step spaces.

Remark 1.5. From the above two definitions it is clear that a sequence space E is solid implies that E is monotone.

By the double gai sequence we mean the gai on the Pringsheim sense that is, a double sequence $x = (x_{mn}) \in E$ has Pringsheim limit 0 (denoted by $P - \lim x = 0$) if $((m+n)!|x_{mn}|)^{\frac{1}{m+n}} \to 0$, whenever $m, n \to \infty$. We shall denote the space of all P-gai sequences by χ^2. The double sequence x is analytic if there exists a positive number M such that $|x_{mn}|^{\frac{1}{m+n}} < M$ for all m, n. We will denote the set of all analytic double sequences by Λ^2.

Definition 1.6. Let $A = (a_{kl}^{mn})$ denote a four dimensional summability method that maps the complex double sequences x into the double sequences

Ax where the (k,l) term of Ax is as follows:

$$(Ax)_{kl} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\big(a_{kl}^{mn}\big)x^{mn}$$

such transformation is said to be non negative if (a_{kl}^{mn}) is non-negative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman [26] and Toeplitz [27]. Follows Silverman and Toeplitz presented the following four dimensional analog of regularity for double sequences in which they both added an additional assumption of boundedness. This assumption was made because a double sequence which is $P-$convergent is not necessarily bounded.

Definition 1.7. The four dimensional matrix A is said to be $RH-$regular if it maps every bounded $P-$gai sequence into a $P-$gai sequence with the same $P-$limit.

In addition to this definition, Robison and Hamilton also presented the following Silverman-Toeplitz type multidimensional characterization of regularity in [21] and [9] respectively.

Theorem 1.8. *The four dimensional matrix A is $RH-$regular if and only if*

$RH_1 : P-\lim_{k,l} a_{kl}^{mn} = 0$ *for each m and n;*

$RH_2 : P-\lim_{k,l} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{kl}^{mn} = 1;$

$RH_3 : P-\lim_{k,l} \sum_{m=1}^{\infty} |a_{kl}^{mn}| = 0$ *for each n;*

$RH_4 : P-\lim_{k,l} \sum_{n=1}^{\infty} |a_{kl}^{mn}| = 0$ *for each m;*

$RH_5 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{kl}^{mn}$ *is P-convergent; and*

$RH_6 :$ *there exist positive numbers M and N such that* $\sum_{m,n>N} |(a_{kl}^{mn})_{k,l}| < M.$

Definition 1.9. A double sequence (x_{mn}) of complex numbers is said to be strongly A-summable to 0, if

$$P-\lim_{k,l}\sum_{mn}\big(a_{kl}^{mn}\big)\big((m+n)!|x_{mn}-0|\big)^{\frac{1}{m+n}} = 0.$$

Let σ be a one to one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n)), m = 1, 2, 3, \cdots$. A continuous linear functional ϕ is said to be an invariant mean or a $\sigma-$ mean if and only if

(1) $\phi(x) \geq 0$ when the sequence $x = (x_{mn})$ has $x_{mn} \geq 0$ for all m, n.

(2) $\phi(e) = 1$, where e= $\begin{pmatrix} 1 & 1 & . & . & . & 1 \\ 1 & 1 & . & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & . & . & . & 1 \end{pmatrix}$ and

(3) $\phi(x_{\sigma(m)}, \sigma(n)) = \phi(x_{\sigma(m)})$ for all $x \in \Lambda^2$.

For certain kinds of mapping σ, every invariant mean ϕ extends the limit functional on the space C of all real convergent sequences in the sense that $\phi(x) = \lim x$ for all $x \in C$ consequently $C \subset V_\sigma$, where V_σ is the set of double analytic sequences for which $\sigma-$ means are equal. If $x = (x_{mn})$, set

$$Tx = (Tx)^{\frac{1}{m+n}} = (x_{\sigma(m),\sigma(n)}).$$

It can be shown that

$V_\sigma = \left\{ x \in \Lambda^2 : \lim_{m\to\infty} t_{mn}(x_n)^{\frac{1}{n}} = Le \text{ uniformly in } n, L = \sigma - \lim(x_{mn})^{\frac{1}{m+n}} \right\}$

where

$$t_{mn}(x) = \frac{(x_n + Tx_n + \cdots + T^m x_n)^{\frac{1}{m+n}}}{m+1}, \tag{1.1}$$

we say that a double analytic sequence $x = (x_{mn})$ is σ-convergent if and only if $x \in V_\sigma$.

Definition 1.10. A double analytic sequence $x = (x_{mn})$ of real numbers is said to be $\sigma-$ convergent to zero provided that

$$P - \lim_{p,q} \frac{1}{pq} \sum_{m=1}^{p} \sum_{n=1}^{q} |x_{\sigma^m(k),\sigma^m(l)}|^{\frac{1}{\sigma^m(k)+\sigma^m(l)}} = 0,$$

uniformly in (k, l).

In this case we write $\sigma_2 - \lim x = 0$. We shall also denote the set of all double σ-convergent sequences by V_σ^2. Clearly $V_\sigma^2 \subset \Lambda^2$. One can see that in contrast to the case for single sequences, a $P-$ convergent double sequence need not be σ-convergent. But, it is easy to see that every bounded P-convergent double sequence is convergent. In addition, if we let $\sigma(m) = m+1$, and $\sigma(n) = n+1$, in then σ-convergence of double sequences reduces to the almost convergence of double sequences.

The following inequality will be used throughout the paper. Let $p = (p_{mn})$ be a sequence of positive real numbers with $0 < p = (p_{mn}) < \sup p_{mn} = G$ and $D = \max(1, 2^{G-1})$. Then for $a_{mn}, b_{mn} \in \mathbb{C}$, the set of complex numbers and for all $m, n \in \mathbb{N}$, we have

$$|a_{mn} + b_{mn}|^{\frac{1}{m+n}} \leq D\Big\{|a_{mn}|^{\frac{1}{m+n}} + |b_{mn}|^{\frac{1}{m+n}}\Big\}.$$

Let $F = (f_{kl})$ be a sequence of modulus functions and $A = (a_{kl}^{mn})$ be a non-negative RH-regular summability matrix method. Now, we define the following sequence spaces in this paper:

$$\chi^2(A, F, u, \Delta^r) = \Big\{x \in \chi^2 : P - \lim_{kl} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{kl}^{mn}) \Big[f_{kl}\Big(\big(\sigma^m(k) + \sigma^n(l)\big)!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big] = 0\Big\}$$

and

$$\Lambda^2(A, F, u, \Delta^r) = \Big\{x \in \Lambda^2 : \sup_{kl} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{kl}^{mn}) \Big[f_{kl}\big(|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big] < \infty\Big\}.$$

If $f_{kl}(x) = x$, for all k, l then the sequence spaces defined above reduced to the following spaces:

$$\chi^2(A, u, \Delta^r) = \Big\{x \in \chi^2 : P - \lim_{kl} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{kl}^{mn}) \Big(\big(\sigma^m(k) + \sigma^n(l)\big)!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} = 0\Big\}$$

and

$$\Lambda^2(A, u, \Delta^r) = \Big\{x \in \Lambda^2 : \sup_{kl} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{kl}^{mn}) \big(|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} < \infty\Big\}.$$

If $A = (C, 1, 1))$, the sequence space defined above reduced to following spaces:

$$\chi^2(F, u, \Delta^r) = \Big\{x \in \chi^2 : P - \lim_{kl} \frac{1}{kl} \sum_{m=0}^{k-1} \sum_{n=0}^{l-1} \Big[f_{kl}\Big(\big(\sigma^m(k) + \sigma^n(l)\big)!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big] = 0\Big\}$$

and

$$\Lambda^2(F,u,\Delta^r)$$
$$= \left\{x \in \Lambda 2 : \sup_{kl} \frac{1}{kl} \sum_{m=0}^{k-1} \sum_{n=0}^{l-1} \left[f_{kl}\left(|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} \right] < \infty \right\}.$$

For $A = (C,1,1)$ and $f_{kl}(x) = x$, for all k, l we obtain the following spaces:

$$\chi^2(u,\Delta^r) = \left\{x \in \chi^2 : P - \lim_{kl} \frac{1}{kl} \sum_{m=0}^{k-1} \sum_{n=0}^{l-1} \left((\sigma^m(k)+\sigma^n(l))! |u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} = 0 \right\}$$

and

$$\Lambda^2(u,\Delta^r) = \left\{x \in \Lambda 2 : \sup_{kl} \frac{1}{kl} \sum_{m=0}^{k-1} \sum_{n=0}^{l-1} \left(|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} < \infty \right\}.$$

The main purpose of this paper is to establish some new types of double P-gai sequence spaces defined by a sequence of modulus functions. We also make an efforts to study some topological properties and inclusion relations between $\chi^2(A,F,u,\Delta^r)$ and $\Lambda^2(A,F,u,\Delta^r)$ spaces in the second section of this paper.

2. Main Results

Theorem 2.1. *Let $A = (a_{kl}^{mn})$ be a non-negative matrix, $F = (f_{kl})$ be a sequence of modulus functions and $u = (u_{kl})$ be a sequence of strictly positive real numbers. Then the spaces $\chi^2(A,F,u,\Delta^r)$ and $\Lambda^2(A,F,u,\Delta^r)$ are linear spaces over the field of complex numbers $\mathbb{C}$.*

Proof. Let $x, y \in \chi^2(A,F,u,\Delta^r)$ and for $\alpha, \beta \in \mathbb{C}$ there exist integers M_α and N_β such that $|\alpha| < M_\alpha$ and $|\beta| < N_\beta$. Since $F = (f_{kl})$ is a sequence of modulus functions, so we have

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{kl}^{mn})$$

$$\left[f_{kl}\left(\left((\sigma^m(k)+\sigma^n(l))! |\alpha u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)} + \beta u_{kl}\Delta^r y_{\sigma^m(k),\sigma^n(l)}| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} \right) \right]$$

$$\leq M_\alpha \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big]$$
$$+ N_\beta \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r y_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big].$$

Thus $\alpha x + \beta y \in \chi^2(A, F, u, \Delta^r)$ for all k, l. Hence $\chi^2(A, F, u, \Delta^r)$ is a linear space. Similarly we can prove that $\Lambda^2(A, F, u, \Delta^r)$ is a linear space. □

Theorem 2.2. *Let $A = (a_{kl}^{mn})$ be a non-negative matrix, $F = (f_{kl})$ be a sequence of modulus functions and $u = (u_{kl})$ be a sequence of strictly positive real numbers. Then the space $\chi^2(A, F, u, \Delta^r)$ is a complete linear topological space with the paranorm defined by*

$$g(x) = \sup_{kl} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn}) f_{kl}\Big(|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}.$$

Proof. Let $x \in \chi^2(A, F, u, \Delta^r)$. Then $g(x)$ exists. Clearly, $g(\theta) = 0$, where $\theta = (0, 0, \cdots, 0)$, $g(-x) = g(x)$ and $g(x+y) \leq g(x) + g(y)$. Now we show that the scalar multiplication is continuous. We have

$$g(\lambda x) = \sup_{kl} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn}) f_{kl}\Big(|\lambda u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}$$
$$\leq (1 + [\lambda])g(x),$$

where $\big[|\lambda|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\big]$ denotes the integral part of $|\lambda|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}$. In addition observe that $g(x)$ and λ approaches to 0 implies $g(\lambda x)$ approaches to 0. For fixed λ, if x approaches to 0 then $g(\lambda x)$ approaches to 0. We now show that for a fixed $x, g(\lambda x)$ approaches to 0 whenever λ approaches to 0. Since $x \in \chi^2(A, F, u, \Delta^r)$, thus

$$P - \lim_{kl} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}(((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}})\Big]$$
$$= 0.$$

If $|\lambda|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} < 1$ and $M \in \mathbb{N}$. We have,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|\lambda||u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big]$$
$$\leq \sum_{m\leq M}\sum_{n\leq M}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|\lambda||u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big]$$
$$+\sum_{m\geq M}\sum_{n\geq M}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|\lambda||u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big].$$

Let $\epsilon > 0$ and choose N such that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})f_{kl}\Big(((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big) < \frac{\epsilon}{2} \tag{2.1}$$

for $k, l > N$. Also for each k, l with $1 \leq k, l \leq N$, we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big] < \infty,$$

so that there exist an integer $(M_{k,l})$ such that

$$\sum_{m>M_{k,l}}\sum_{n>M_{k,l}}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big] < \frac{\epsilon}{2}.$$

Taking

$$M = \inf_{1\leq k\leq N(\text{or})1\leq l\leq N}\{M_{k,l}\}.$$

We have for each (k, l) with $1 \leq k \leq N(\text{or })1 \leq l \leq N$

$$\sum_{m>M}\sum_{n>M}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big] < \frac{\epsilon}{2},$$

for $k, l > N$ we have

$$\sum_{m>M}\sum_{n>M}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big] < \frac{\epsilon}{2}.$$

Thus M is an integer independent of (k,l) such that

$$\sum_{m>M}\sum_{n>M}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big] < \frac{\epsilon}{2}. \tag{2.2}$$

Further for $|\lambda|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} < 1$ and for all (k,l)

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|\lambda u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big] \\ \leq \sum_{m>M}\sum_{n>M}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|\lambda u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big] \\ +\sum_{m\leq M}\sum_{n\leq M}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|\lambda u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big].\end{aligned}$$

For each (k,l) and $\lambda \to 0$, we have the following

$$\sum_{m\leq M}\sum_{n\leq M}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|\lambda u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big).$$

Now choose $\delta < 1$ such that $|\lambda|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} < \delta$ implies

$$\sum_{m\leq M}\sum_{n\leq M}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|\lambda u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big] < \frac{\epsilon}{2}. \tag{2.3}$$

It follows that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|\lambda u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big] < \epsilon$$

for all (k,l). Thus $g(\lambda x) \to 0$ as $\lambda \to 0$. Therefore $\chi^2(A,F,u,\Delta^r)$ is a paranormed linear topological space.
Now let us show that $\chi^2(A,F,u,\Delta^r)$ is complete with respect to its paranormed topologies. Let (x^i_{mn}) be a sequence in $\chi^2(A,F,u,\Delta^r)$. Then, we

write $g(x^i - x^j) \to 0$ as $i, j \to \infty$, for all (k, l)

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn}) \left[f_{kl}\left(\left((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x^i_{\sigma^m(k),\sigma^n(l)} - \Delta^r x^j_{\sigma^m(k),\sigma^n(l)}|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\right)\right] \to 0. \quad (2.4)$$

Thus for each fixed m and n as $i, j \to \infty$ we are granted

$$f_{kl}\left((m+n)!|x^i_{mn} - x^j_{mn}|\right) \to 0$$

and so (x^i_{mn}) is a Cauchy sequence in $\mathbb{C}$ for each fixed m and n. Since $\mathbb{C}$ is complete we have $x^i_{mn} \to x_{mn}$ as $i \to \infty$ for each (mn). Now we have for $\epsilon > 0$ there exist a natural number N such that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{}\sum_{i,j>N}^{\infty}(a_{kl}^{mn}) \left[f_{kl}\left(\left((\sigma^m(k)+\sigma^n(l))!|u_{kl}(\Delta^r x^i_{\sigma^m(k),\sigma^n(l)} - \Delta^r x^j_{\sigma^m(k),\sigma^n(l)})|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\right)\right] < \epsilon$$

for (k, l). Since for any fixed natural number M, we have from (2.1)

$$\sum_{m\leq M}^{\infty}\sum_{n\leq M}^{}\sum_{i,j>N}^{\infty}(a_{kl}^{mn}) \left[f_{kl}\left(\left((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x^i_{\sigma^m(k),\sigma^n(l)} - \Delta^r x^j_{\sigma^m(k),\sigma^n(l)}|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\right)\right] < \epsilon$$

for all (k, l), by letting $j \to \infty$ in the above expression we obtain

$$\sum_{m\leq M}^{\infty}\sum_{n\leq M}^{}\sum_{i>N}^{\infty}(a_{kl}^{mn}) \left[f_{kl}\left(\left((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x^i_{\sigma^m(k),\sigma^n(l)} - \Delta^r x_{\sigma^m(k),\sigma^n(l)}|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\right)\right] < \epsilon.$$

Since M is arbitrary. By letting $M \to \infty$ we obtain

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn}) \left[f_{kl}\left(\left((\sigma^m(k)+\sigma^n(l))!|u_{kl}(\Delta^r x^i_{\sigma^m(k),\sigma^n(l)} - \Delta^r x_{\sigma^m(k),\sigma^n(l)})|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\right)\right] < \epsilon$$

for all (k, l). Thus $g(x^i - x) \to 0$ as $i, j \to 0$. Also (x^i) being a sequence in $\chi^2(A, F, u, \Delta^r)$ by definition of $\chi^2(A, F, u, \Delta^r)$ for each i with

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})$$

$$\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x^i_{\sigma^m(k),\sigma^n(l)}-\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big]\to 0$$

as $(k,l)\to 0$. Thus $x\in\chi^2(A,F,u,\Delta^r)$. This completes the proof. □

Theorem 2.3. *Let $A=(a_{kl}^{mn})$ be a non-negative matrix such that*

$$\sup_{kl}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})<\infty$$

and let $F=(f_{kl})$ be a sequence of modulus functions, then $\chi^2(A,F,u,\Delta^r)\subset\Lambda^2(A,F,u,\Delta^r)$.

Proof. Let $x\in\chi^2(A,F,u,\Delta^r)$. Then, we have

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big]\\ &\le\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}-L|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big]\\ &\quad+f_{kl}(|L|)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn}).\end{aligned}$$

There exist an integer N_p such that $|L|\le N_p$. Thus we have

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big]\\ &\le\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(\Big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}-L|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big]\\ &\quad+N_p f_{kl}(1)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn}).\end{aligned}$$

Since $\sup_{kl}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})<\infty$ and $x\in\chi^2(A,F,u,\Delta^r)$ and this implies that $x\in\Lambda^2(A,F,u,\Delta^r)$. This completes the proof of the theorem. □

Theorem 2.4. *Let* $A = (a_{kl}^{mn})$ *be a non negative matrix such that*

$$\sup_{kl} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{kl}^{mn}) < \infty$$

and $F = (f_{kl})$ *be a sequence of modulus functions. Then* $\Lambda^2(A, u, \Delta^r) \subset \Lambda^2(A, F, u, \Delta^r)$.

Proof. Let $x \in \Lambda^2(A, u, \Delta^r)$, so that

$$\sup_{kl} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{kl}^{mn}) |u_{kl} \Delta^r x_{\sigma^m(k), \sigma^n(l)}|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} < \infty.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. Consider

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{kl}^{mn}) \left[f_{kl} \left(|u_{kl} \Delta^r x_{\sigma^m(k), \sigma^n(l)}| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} \right]$$

$$= \sum_{m=0, n=0}^{\infty} \sum_{|\Delta^r x_{\sigma^m(k), \sigma^n(l)}|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} \leq \delta}^{\infty} (a_{kl}^{mn}) \left[f_{kl} \left(|u_{kl} \Delta^r x_{\sigma^m(k), \sigma^n(l)}| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} \right]$$

$$+ \sum_{m=0, n=0}^{\infty} \sum_{|\Delta^r x_{\sigma^m(k), \sigma^n(l)}|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} > \delta}^{\infty} (a_{kl}^{mn}) \left[f_{kl} \left(|u_{kl} \Delta^r x_{\sigma^m(k), \sigma^n(l)}| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} \right].$$

Then

$$\sum_{m=0 n=0}^{\infty} \sum_{|\Delta^r x_{\sigma^m(k), \sigma^n(l)}|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} \leq \delta}^{\infty} (a_{kl}^{mn}) f_{kl} \left(|u_{kl} \Delta^r x_{\sigma^m(k), \sigma^n(l)}| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}$$

$$\leq \epsilon \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{kl}^{mn}). \tag{2.5}$$

For $|\Delta^r x_{\sigma^m(k), \sigma^n(l)}|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} > \delta$, we use the fact that

$$|\Delta^r x_{\sigma^m(k), \sigma^n(l)}|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}} < \frac{|\Delta^r x_{\sigma^m(k), \sigma^n(l)}|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}}{\delta}$$

$$< \left[1 + | \frac{|\Delta^r x_{\sigma^m(k), \sigma^n(l)}|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}}{\delta} \right],$$

where $[t]$ denoted the integer part of t and $F = (f_{kl})$ be a sequence of modulus functions we have

$$f_{kl}\Big(|\Delta^r x_{\sigma^m(k),\sigma^n(l)}|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big) \leq \Big[1+|\frac{|\Delta^r x_{\sigma^m(k),\sigma^n(l)}|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}}{\delta}\Big] f_{kl}(1)$$
$$\leq 2f_{kl}(1)\frac{|\Delta^r x_{\sigma^m(k),\sigma^n(l)}|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}}{\delta}.$$

Hence

$$\sum_{m=0,n=0}^{\infty}\ \sum_{|\Delta^r x_{mn}|^{\frac{1}{m+n}}\leq\delta}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big]$$
$$\leq \frac{2f_{kl}(1)}{\delta}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}.$$

Which together with inequality (2.5) yield the following

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big]$$
$$\leq \epsilon\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})+\frac{2f_{kl}(1)}{\delta}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|^{\frac{1}{\sigma^m(k)+\sigma^n(l)}},$$

since $\sup_{kl}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn}) < \infty$ and hence $x \in \Lambda^2(A,u,F,\Delta^r)$. This completes the proof of the theorem. □

3. Double gai sequence spaces defined by seminorm and a sequence of modulus functions

In this section, we shall introduced double P-sequence spaces by using seminorm function q and a sequence of modulus functions $F=(f_{kl})$. We shall also establish some topological properties and inclusion relations between the sequence spaces $\chi^2(A,F,p,q,\Delta^r,u)$ and $\Lambda^2(A,F,p,q,\Delta^r,u)$.

Let (X,q) be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q and $F=(f_{kl})$ be a sequence of modulus functions. We define the following sequence spaces in this section :

$$\chi^2(A,F,p,q,\Delta^r,u)=\Big\{x\in\chi^2 : P-\lim_{kl}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})$$
$$\Big[f_{kl}\Big(q((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big]^{p_{mn}}=0\Big\},$$

$$\Lambda^2(A,F,p,q,\Delta^r,u) = \Big\{x \in \Lambda^2 : \sup_{kl} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn}) \\ \Big[f_{kl}\Big(q(|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big]^{p_{mn}} < \infty\}.$$

Theorem 3.1. *Let $F' = f'_{kl}$ and $F'' = f''_{kl}$ be two sequences of modulus functions. Then $\chi^2(A,F',p,q,\Delta^r,u) \cap \chi^2(A,F'',p,q,\Delta^r,u) \subseteq \chi^2(A,F'+F'',p,q,\Delta^r,u)$.*

Proof. The proof is easy so omitted. □

Proposition 3.2. *Let $F = (f_{kl})$ be a sequence of modulus functions q_1 and q_1 be two seminorm on X, we have*
(i) $\chi^2(A,F,p,q_1,\Delta^r,u)\bigcap\chi^2(A,F,p,q_2,\Delta^r,u) \subseteq \chi^2(A,F,p,q_1+q_2,\Delta^r,u)$.
(ii) *If q_1 is stronger than q_2 then $\chi^2(A,F,p,q_1,\Delta^r,u) \subseteq \chi^2(A,F,p,q_2,\Delta^r,u)$.*
(iii) *If q_1 is equivalent to q_2 then $\chi^2(A,F,p,q_1,\Delta^r,u) = \chi^2(A,F,p,q_2,\Delta^r,u)$.*

Theorem 3.3. *Let $A = (a_{kl}^{mn})$ be a non-negative matrix, $F = (f_{kl})$ be a sequence of modulus functions, $u = (u_{kl})$ be a sequence of strictly positive real numbers, $0 \le p_{mn} \le w_{mn}$ for all $m,n \in \mathbb{N}$ and let $\left\{\frac{w_{mn}}{p_{mn}}\right\}$ be bounded. Then $\chi^2(A,F,w,q,\Delta^r,u) \subseteq \chi^2(A,F,p,q,\Delta^r,u)$.*

Proof. Suppose $x \in \chi^2(A,F,w,q,\Delta^r,u)$,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn}) \\ \Big[f_{kl}\Big(q\Big(\big(\sigma^m(k)+\sigma^n(l)\big)!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big]^{w_{mn}}. \tag{3.1}$$

Let

$$t_{mn} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn}) \\ \Big[f_{kl}\Big(q\Big(\big(\sigma^m(k)+\sigma^n(l)\big)!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\Big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big]^{w_{mn}}, \tag{3.2}$$

we have $\gamma_{mn} = p_{mn}/w_{mn}$. Since $p_{mn} \le w_{mn}$, we have $0 \le \gamma_{mn} \le 1$.
Let $0 < \gamma < \gamma_{mn}$. Then

$$u_{mn} = \begin{cases} t_{mn}, & \text{if } t_{mn} \ge 1 \\ 0, & \text{if } t_{mn} < 1, \end{cases}$$

$$v_{mn} = \begin{cases} 0, & \text{if } \ t_{mn} \geq 1 \\ t_{mn}, & \text{if } \ t_{mn} < 1, \end{cases} \tag{3.3}$$

$t_{mn} = u_{mn} + v_{mn}, t_{mn}^{\gamma_{mn}} = u_{mn}^{\gamma_{mn}} + v_{mn}^{\gamma_{mn}}$. Now, it follows that

$$u_{mn}^{\gamma_{mn}} \leq u_{mn} \leq t_{mn}, \quad v_{mn}^{\gamma_{mn}} \leq v_{mn}^{\gamma}. \tag{3.4}$$

Since $t_{mn}^{\gamma_{mn}} = u_{mn}^{\gamma_{mn}} + v_{mn}^{\gamma_{mn}}$, we have $t_{mn}^{\gamma_{mn}} = t_{mn} + v_{mn}^{\gamma}$. Thus,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(q\Big(\big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)^{w_{mn}}\Big]^{\gamma_{mn}}$$
$$\leq \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(q\Big(\big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big)\Big]^{w_{mn}}$$
$$\Longrightarrow$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(q\Big(\big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)^{w_{mn}}\Big]^{\frac{p_{mn}}{w_{mn}}}$$
$$\leq \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(q\Big(\big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big)\Big]^{w_{mn}}$$
$$\Longrightarrow$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(q\Big(\big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big)\Big]^{p_{mn}}$$
$$\leq \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(q\Big(\big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big)\Big]^{w_{mn}}.$$

But

$$P-\lim_{kl}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(q\Big(\big((\sigma^m(k)+\sigma^n(l))!|x_{\sigma^m(k),\sigma^n(l)}|\big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big)\Big]^{w_{mn}}$$
$$= 0.$$

Therefore we have

$$P-\lim_{kl}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(q\Big(\big((\sigma^m(k)+\sigma^n(l))!|x_{\sigma^m(k),\sigma^n(l)}|\big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big)\Big]^{p_{mn}}$$
$$= 0.$$

Hence $x \in \chi^2(A, F, p, q, \Delta^r, u)$. We get

$$\chi^2(A, F, w, q, \Delta^r, u) \subset \chi^2(A, F, p, q, \Delta^r, u).$$

This completes the proof of the theorem. □

Theorem 3.4. *The space $\chi^2(A, F, p, q, \Delta^r, u)$ is solid and such are monotones.*

Proof. Let $x = (x_{mn}) \in \chi^2(A, F, p, q, \Delta^r, u)$ and (α_{mn}) be a sequence of scalars such that $|\alpha_{mn}| \leq 1$ for all $m, n \in \mathbb{N}$. Then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(q\Big(\big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big)\Big]^{p_{mn}}$$
$$\leq \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(q\Big(\big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big)\Big]^{p_{mn}}$$
$$\Longrightarrow$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(q\Big(\big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big)\Big]^{p_{mn}}$$
$$\leq \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{kl}^{mn})\Big[f_{kl}\Big(q\Big(\big((\sigma^m(k)+\sigma^n(l))!|u_{kl}\Delta^r x_{\sigma^m(k),\sigma^n(l)}|\big)^{\frac{1}{\sigma^m(k)+\sigma^n(l)}}\Big)\Big)\Big]^{p_{mn}}$$

for all $m, n \in \mathbb{N}$. This completes the proof of the theorem. □

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