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CONVERGENCE THEOREMS OF MIXED TYPE IMPLICIT ITERATION FOR NONLINEAR MAPPINGS IN CONVEX METRIC SPACES

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Abstract. In this paper, we propose and study an implicit iteration process for a finite family of total asymptotically quasi-nonexpansive mappings and a finite family of asymptotically quasi-nonexpansive mappings in the intermediate sense in convex metric spaces and establish some strong convergence results. Also, we give some applications of our result in the setting of convex metric spaces. The results of this paper are generalizations, extensions and improvements of several corresponding results.

1. Introduction and preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers and $J = \{1, 2, \cdots, N\}$, the set of first N natural numbers. Denote by $F(T)$ the set of all fixed points of T and by

$$\mathcal{F} := \Big(\bigcap_{j=1}^{N} F(S_j)\Big) \bigcap \Big(\bigcap_{j=1}^{N} F(T_j)\Big)$$

the set of common fixed points of two finite families of self-mappings $\{S_j : j \in J\}$ and $\{T_j : j \in J\}$.

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Lets remember some definitions and present simple results derived from the known theorem for leading the main results.

Definition 1.1. ([23]) Let (X,d) be a metric space. A mapping $W : X \times X \times [0,1] \to X$ is said to have a convex structure on X if for each $(x,y,\lambda) \in X \times X \times [0,1]$ and $u \in X$,

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y).$$

A metric space X together with the convex structure W is called a *convex metric space*.

Definition 1.2. ([23]) Let X be a convex metric space. A nonempty subset A of X is said to be convex if $W(x,y,\lambda) \in A$ whenever $(x,y,\lambda) \in A \times A \times [0,1]$.

In 1982, Kirk [15] used the term "hyperbolic type spaces" for convex metric spaces, and studied iteration processes for nonexpansive mappings in this abstract setting. Later on, many authors discussed the existence of the fixed point and the convergence of the iterative process for various mappings in convex metric spaces (see, for example, [3, 4, 5, 7, 10, 11, 13, 14, 17, 18, 19, 23]).

Recently, Yildirim and Khan [26] extend Definition 1.1 as follows:

Definition 1.3. A mapping $W : X^3 \times [0,1]^3 \to X$ is said to have a convex structure on X, if it satisfies the following condition: For any $(x,y,z;a,b,c) \in X^3 \times [0,1]^3$ with $a+b+c=1$ and $u \in X$,

$$d(W(x,y,z;a,b,c),u) \leq ad(x,u) + bd(y,u) + cd(z,u).$$

If (X,d) is a metric space with a convex structure W, then (X,d) is called a convex metric space.

Let (X,d) be a convex metric space. A nonempty subset A of X is said to be convex if $W(x,y,z;a,b,c) \in A$, for all $(x,y,z) \in A^3$, $(a,b,c) \in [0,1]^3$ with $a+b+c=1$.

Takahashi [23] has shown that open sphere $B(x,r) = \{y \in X : d(y,x) < r\}$ and closed sphere $B[x,r] = \{y \in X : d(y,x) \leq r\}$ are convex. All normed spaces and their convex subsets are convex metric spaces. But there are many examples of convex metric spaces which are not embedded in any normed space (see [23]).

Remark 1.4. Every normed space is a special convex metric space with a convex structure $W(x,y,z;\alpha,\beta,\gamma) = \alpha\, x + \beta\, y + \gamma\, z$ for all $x,y,z \in X$ and

$\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$. In fact,

$$\begin{aligned} d(u, W(x, y, z; \alpha, \beta, \gamma)) &= \|u - (\alpha\, x + \beta\, y + \gamma\, z)\| \\ &\leq \alpha\|u - x\| + \beta\|u - y\| + \gamma\|u - z\| \\ &= \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z),\ \forall\, u \in X. \end{aligned}$$

Definition 1.5. A mapping $T : X \to X$ is called:

(1) nonexpansive [6] if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$,

(2) quasi-nonexpansive [16] if $F(T) \neq \emptyset$ and $d(Tx, p) \leq d(x, p)$ for all $x \in X$ and $p \in F(T)$,

(3) asymptotically nonexpansive [6] if there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim\limits_{n\to\infty} u_n = 0$ such that

$$d(T^n x, T^n y) \leq (1 + u_n) d(x, y)$$

for all $x, y \in X$ and $n \in \mathbb{N}$,

(4) asymptotically quasi-nonexpansive [21] if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim\limits_{n\to\infty} u_n = 0$ such that

$$d(T^n x, p) \leq (1 + u_n) d(x, p)$$

for all $x \in X$, $p \in F(T)$ and $n \in \mathbb{N}$,

(5) total asymptotically nonexpansive [2] if there exist sequences $\{u_n\}$, $\{v_n\} \subset [0, \infty)$ with $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} v_n = 0$, and strictly increasing and continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + u_n \phi(d(x, y)) + v_n$$

for all $x, y \in X$ and $n \in \mathbb{N}$,

(6) total asymptotically quasi-nonexpansive [1] if $F(T) \neq \emptyset$ and there exist sequences $\{u_n\}, \{v_n\} \subset [0, \infty)$ with $\sum\limits_{n=1}^{\infty} u_n < \infty$, $\sum\limits_{n=1}^{\infty} v_n < \infty$, and strictly increasing and continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$d(T^n x, p) \leq d(x, p) + u_n \phi(d(x, p)) + v_n \tag{1.1}$$

for all $x \in X$, $p \in F(T)$ and $n \in \mathbb{N}$,

(7) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq L\, d(x, y)$$

for all $x, y \in X$ and $n \in \mathbb{N}$,

(8) asymptotically quasi-nonexpansive in the intermediate sense [26] if $F(T) \neq \emptyset$ and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{p\in F(T),\, y\in X} \Big(d(p, T^n y) - d(p, y)\Big) \leq 0. \tag{1.2}$$

If we define

$$\rho_n = \max\left\{0, \sup_{p\in F(T),\, y\in X} (d(p, T^n y) - d(p, y))\right\},$$

then $\rho_n \to 0$ as $n \to \infty$. It follows that (1.2) reduced to

$$d(p, T^n y) \leq d(p, y) + \rho_n \tag{1.3}$$

for all $p \in F(T)$, $y \in X$ and $n \in \mathbb{N}$.

Remark 1.6. From Definition 1.5, if $F(T) \neq \emptyset$, $\phi(\lambda) = \lambda$, then (1.1) takes the form

$$d(T^n x, p) \leq (1 + u_n) d(x, p) + v_n.$$

In addition, if $v_n = 0$ for all $n \geq 1$, then total asymptotically quasi-nonexpansive mappings coincide with asymptotically quasi-nonexpansive mappings. If $u_n = 0 = v_n$ for all $n \geq 1$, then we obtain from (1.1) the class of quasi-nonexpansive mappings. Thus the following statements are obvious.

(1) Every quasi-nonexpansive is asymptotically quasi-nonexpansive mapping.
(2) Every asymptotically quasi-nonexpansive is total asymptotically quasi-nonexpansive mapping.
(3) Every asymptotically quasi-nonexpansive mapping is asymptotically quasi-nonexpansive in the intermediate sense.
(4) The converse of these statements may not be true in general.

In 2003, Sun [22] introduced the following process for common fixed points of a finite family of asymptotically quasi-nonexpansive mappings $\{T_i : i \in I\}$, $I = \{1, 2, \cdots, N\}$ in uniformly convex Banach spaces:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_i^k x_n, \quad n \in \mathbb{N}, \tag{1.4}$$

where $n = (k-1)N + i$, $i \in I$ and $\{\alpha_n\}$ is a real sequence in $(0, 1)$.

Sun [22] studied the strong convergence of the process (1.4) for common fixed points of the finite family of mappings $\{T_i : i \in I\}$, requiring only one member of the family to be semicompact. The results of Sun [22] generalized and extended the corresponding results of Xu and Ori [25].

In 2008, Khan et al. [8] studied the following n-step iterative processes for a finite family of mappings $\{T_i : i = 1, 2, \cdots, k\}$. Let $x_1 \in K$. Then the iterative sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_{kn})x_n + \alpha_{kn}T_k^n y_{(k-1)n}, \\ y_{(k-1)n} = (1-\alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\ \qquad \vdots \\ y_{2n} = (1-\alpha_{2n})x_n + \alpha_{2n}T_2^n y_{1n}, \\ y_{1n} = (1-\alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n}, \quad n \geq 1, \end{cases} \tag{1.5}$$

where $y_{0n} = x_n$ for all $n \in \mathbb{N}\cup\{0\}$ and $\alpha_{in} \in [0, 1]$, $n \geq 1$ and $i \in \{1, 2, \cdots, k\}$.

In 2010, Khan and Ahmed [7] considered the iteration process (1.5) in convex metric space as follows:

$$\begin{cases} x_{n+1} = W(T_k^n y_{(k-1)n}, x_n; \alpha_{kn}), \\ y_{(k-1)n} = W(T_{k-1}^n y_{(k-2)n}, x_n; \alpha_{(k-1)n}), \\ \qquad \vdots \\ y_{2n} = W(T_2^n y_{1n}, x_n; \alpha_{2n}), \\ y_{1n} = W(T_1^n y_{0n}, x_n; \alpha_{1n}), \quad n \geq 1, \end{cases} \tag{1.6}$$

where $y_{0n} = x_n$ for all $n \in \mathbb{N}\cup\{0\}$ and $\alpha_{in} \in [0, 1]$, $n \geq 1$ and $i \in \{1, 2, \cdots, k\}$.

In 2019, Kim [12] studied the following double acting iterative process in $CAT(0)$ space:

Let $I = \{1, 2, \cdots, r\}$, where $r \geq 1$ and $\{T_i : i \in I\}$ be a family of generalized φ-weak contraction self-mappings on K. The scheme is

$$x_1 \in K, \quad x_{n+1} = U_{n(i)}x_n, \quad i \in I, \ \ n \geq 1,$$

where

$$\begin{cases} U_{n(0)} = I_d \ \ (\text{: the identity mapping}), \\ U_{n(1)}x = \alpha_{n(1)}x \oplus (1-\alpha_{n(1)})T_1^n U_{n(0)}x, \\ U_{n(2)}x = \alpha_{n(2)}x \oplus (1-\alpha_{n(2)})T_2^n U_{n(1)}x, \\ \qquad \vdots \\ U_{n(r-1)}x = \alpha_{n(r-1)}x \oplus (1-\alpha_{n(r-1)})T_{r-1}^n U_{n(r-2)}x, \\ U_{n(r)}x = \alpha_{n(r)}x \oplus (1-\alpha_{n(r)})T_r^n U_{n(r-1)}x, \end{cases}$$

where $\alpha_{n(i)} \in [0, 1]$ for each $i \in I$.

In 2010, Khan et al. [9] introduced an implicit iteration process for two finite families of nonexpansive mappings as follows:

Let $(E, \|\cdot\|)$ be a Banach space and $S_i, T_i : E \to E$ $(i \in I)$ be two families of nonexpansive mappings. For any given $x_0 \in E$, define an iteration process

$\{x_n\}$ as

$$x_n = \alpha_n x_{n-1} + \beta_n S_n x_n + \gamma_n T_n x_n, \;\; n \in \mathbb{N},$$

where $S_n = S_{n(mod\,N)}$, $T_n = T_{n(mod\,N)}$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$.

In 2017, Saluja and Hyun [20] introduced the following:

Let (X, d, W) be a convex metric space with convex structure W, $S_i : X \to X$ be finite families of asymptotically quasi-nonexpansive mappings and $T_i : X \to X$ be finite families of asymptotically quasi-nonexpansive mappings in the intermediate sense. For any given $x_0 \in X$, define iteration process $\{x_n\}$ as follows.

$$\begin{aligned}
x_1 &= W(x_0, S_1 x_1, T_1 x_1; \alpha_1, \beta_1, \gamma_1),\\
x_2 &= W(x_1, S_2 x_2, T_2 x_2; \alpha_2, \beta_2, \gamma_2),\\
&\;\;\vdots\\
x_N &= W(x_{N-1}, S_N x_N, T_N x_N; \alpha_N, \beta_N, \gamma_N),\\
x_{N+1} &= W(x_N, S_1^2 x_{N+1}, T_1^2 x_{N+1}; \alpha_{N+1}, \beta_{N+1}, \gamma_{N+1}),\\
&\;\;\vdots\\
x_{2N} &= W(x_{2N-1}, S_N^2 x_{2N}, T_N^2 x_{2N}; \alpha_{2N}, \beta_{2N}, \gamma_{2N}),\\
x_{2N+1} &= W(x_{2N}, S_1^3 x_{2N+1}, T_1^3 x_{2N+1}; \alpha_{2N+1}, \beta_{2N+1}, \gamma_{2N+1}),\\
&\;\;\vdots
\end{aligned}$$

This iteration process can be rewritten in the following form:

$$x_n = W\Big(x_{n-1}, S_i^k x_n, T_i^k x_n; \alpha_n, \beta_n, \gamma_n\Big), \; n \in \mathbb{N}, \tag{1.7}$$

where $n = (k-1)N + i$, $i \in I$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and they established some strong convergence results which generalized some recent results of [7, 9, 22, 24, 25, 26].

Notice that the iteration scheme (1.6) deals with one family and uses n-steps whereas (1.7) deals with two families and uses only one step.

Motivated and inspired by [20] and some others, we introduce and study the following iteration scheme:

Definition 1.7. Let (X, d, W) be a convex metric space with convex structure W, $S_j : X \to X$ be finite family of total asymptotically quasi-nonexpansive self-mappings and $T_j : X \to X$ be finite families of asymptotically quasi-nonexpansive self mappings in the intermediate sense. For any given $x_0 \in X$, we define iteration process $\{x_n\}$ as follows:

$$
\begin{aligned}
x_1 &= W(x_0, S_1x_1, T_1x_1; \alpha_1, \beta_1, \gamma_1),\\
x_2 &= W(x_1, S_2x_2, T_2x_2; \alpha_2, \beta_2, \gamma_2),\\
&\vdots\\
x_N &= W(x_{N-1}, S_Nx_N, T_Nx_N; \alpha_N, \beta_N, \gamma_N),
\end{aligned}
$$

$$
\begin{aligned}
x_{N+1} &= W(x_N, S_1^2x_{N+1}, T_1^2x_{N+1}; \alpha_{N+1}, \beta_{N+1}, \gamma_{N+1}),\\
&\vdots\\
x_{2N} &= W(x_{2N-1}, S_N^2x_{2N}, T_N^2x_{2N}; \alpha_{2N}, \beta_{2N}, \gamma_{2N}),\\
x_{2N+1} &= W(x_{2N}, S_1^3x_{2N+1}, T_1^3x_{2N+1}; \alpha_{2N+1}, \beta_{2N+1}, \gamma_{2N+1}),\\
&\vdots
\end{aligned}
$$

This iteration process can be rewritten in the following compact form:

$$x_n = W\Big(x_{n-1}, S_j^kx_n, T_j^kx_n; \alpha_n, \beta_n, \gamma_n\Big),\ n \in \mathbb{N}, \tag{1.8}$$

where $n = (k-1)N + j$, $j \in J$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and establish some strong convergence results in the setting of convex metric spaces.

Lemma 1.8. ([16]) *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences of nonnegative real numbers satisfying the following conditions:*

$$a_{n+1} \le (1 + b_n)a_n + c_n,\ n \ge 0, \quad \sum_{n=0}^{\infty} b_n < \infty, \quad \sum_{n=0}^{\infty} c_n < \infty.$$

Then

(i) $\lim_{n\to\infty} a_n$ *exists.*
(ii) *Moreover, if* $\liminf_{n\to\infty} a_n = 0$*, then* $\lim_{n\to\infty} a_n = 0$.

Remark 1.9. It is easy to verify that Lemma 1.8 (ii) holds under the hypothesis $\limsup_{n\to\infty} a_n = 0$ as well. Therefore, the condition of Lemma 1.8 (ii) can be modified and reformulated as follows.

(ii)' If either $\liminf_{n\to\infty} a_n = 0$ or $\limsup_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

2. Main results

In this section, we prove some strong convergence theorems for iteration scheme (1.8) in the framework of convex metric spaces. First, we shall need the following lemma.

Lemma 2.1. *Let (X, d, W) be a convex metric space with convex structure W and $\{S_j : X \to X : j \in J\}$ be a finite family of total asymptotically quasi-nonexpansive self-mappings with sequences $\{u_n^{(j)}\}, \{v_n^{(j)}\} \subset [0, \infty)$ such that $\sum_{i=1}^{\infty}\left(\max_{j\in J} u_i^{(j)}\right) < \infty$, $\sum_{i=1}^{\infty}\left(\max_{j\in J} v_i^{(j)}\right) < \infty$ and there exist constants M_0, $M > 0$ such that*

$$\phi(\lambda) \le M_0\lambda$$

for all $\lambda \ge M$. Let $\{T_j : X \to X : j \in J\}$ be a finite family of asymptotically quasi-nonexpansive self-mappings in the intermediate sense. Suppose that $\mathcal{F} \neq \emptyset$, $x_0 \in X$ and $\{x_n\}$ is as in (1.8) such that $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$. Put

$$\mathcal{A}_n = \max\left\{\max_{j\in J}\ \sup_{p\in\mathcal{F},\, y\in X}\Big(d(p, T_j^k y) - d(p, y)\Big),\ 0\right\}, \tag{2.1}$$

where $n = (k-1)N + j$ and $j \in J$ such that $\sum_{i=1}^{\infty}\mathcal{A}_i < \infty$. Then we have the following statements:

(i) $\lim_{n\to\infty} d(x_n, p)$ *exists for all* $p \in \mathcal{F}$,

(ii) $\lim_{n\to\infty} D(x_n, \mathcal{F})$ *exists, where* $D(x, \mathcal{F}) = \inf\{d(x, y) : y \in \mathcal{F}\}$,

(iii) *if* $\lim_{n\to\infty} D(x_n, \mathcal{F}) = 0$, *then* $\{x_n\}$ *is a Cauchy sequence.*

Proof. (i) Let $p \in \mathcal{F}$ and $n = (k-1)N + j$, $j \in J$. Then from (1.8) and (2.1), we have

$$\begin{aligned} d(x_n, p) &= d(W(x_{n-1}, S_j^k x_n, T_j^k x_n; \alpha_n, \beta_n, \gamma_n), p) \\ &\le \alpha_n d(x_{n-1}, p) + \beta_n d(S_j^k x_n, p) + \gamma_n d(T_j^k x_n, p) \\ &\le \alpha_n d(x_{n-1}, p) + \beta_n[d(x_n, p) + u_k^{(j)}\phi(d(x_n, p)) + v_k^{(j)}] \\ &\quad + \gamma_n[d(x_n, p) + \mathcal{A}_n] \\ &\le \alpha_n d(x_{n-1}, p) + (\beta_n + \gamma_n)d(x_n, p) \\ &\quad + \beta_n u_k^{(j)}\phi(d(x_n, p)) + \gamma_n\mathcal{A}_n + \beta_n v_k^{(j)}. \end{aligned} \tag{2.2}$$

Since ϕ is an increasing function, it follows that

$$\begin{cases} \phi(\lambda) \le \phi(M), & \text{if } \lambda \le M, \\ \phi(\lambda) \le M_0 \lambda, & \text{if } \lambda \ge M. \end{cases}$$

In either case we obtain

$$\phi(d(x_n, p)) \le \phi(M) + M_0\, d(x_n, p), \quad \forall\, n \ge 1.$$

Thus, from (2.2), we have

$$\begin{aligned}(1 - \beta_n - \gamma_n) d(x_n, p) &\le \alpha_n d(x_{n-1}, p) + \beta_n u_k^{(j)} (\phi(M) + M_0\, d(x_n, p)) \\ &\quad + \gamma_n \mathcal{A}_n + \beta_n v_k^{(j)} \\ &= \alpha_n d(x_{n-1}, p) + M_0\, \beta_n u_k^{(j)} d(x_n, p) \\ &\quad + \beta_n u_k^{(j)} \phi(M) + \gamma_n \mathcal{A}_n + \beta_n v_k^{(j)}. \end{aligned} \tag{2.3}$$

Since $\lim_{n\to\infty} \gamma_n = 0$, there exists an $n_1 \in \mathbb{N}$ such that $n \ge n_1$, $\gamma_n \le \frac{s}{2}$. Therefore

$$1 - \beta_n - \gamma_n \ge 1 - (1 - s) - \frac{s}{2} = \frac{s}{2}$$

for $n \ge n_1$. From (2.3) and $\gamma_n \le \frac{s}{2}$, we get

$$\begin{aligned} d(x_n, p) &\le \frac{\alpha_n}{1 - \beta_n - \gamma_n} d(x_{n-1}, p) + \frac{M_0\, \beta_n u_k^{(j)}}{1 - \beta_n - \gamma_n} d(x_n, p) \\ &\quad + \frac{1}{1 - \beta_n - \gamma_n} (\beta_n u_k^{(j)} \phi(M) + \gamma_n \mathcal{A}_n + \beta_n v_k^{(j)}) \\ &\le d(x_{n-1}, p) + \frac{2M_0}{s} u_k^{(j)} d(x_n, p) + \frac{2}{s} (\phi(M) u_k^{(j)} + v_k^{(j)}) + \mathcal{A}_n. \end{aligned}$$

Thus

$$\left(1 - \frac{2M_0}{s} u_k^{(j)}\right) d(x_n, p) \le d(x_{n-1}, p) + \frac{2}{s} \left(\phi(M) u_k^{(j)} + v_k^{(j)}\right) + \mathcal{A}_n. \tag{2.4}$$

Since $\lim_{k\to\infty} \max_{j\in J} u_k^{(j)} = 0$, there exists an $n_2 \in \mathbb{N}$ such that

$$u_k^{(j)} < \frac{s}{4M_0} \tag{2.5}$$

for $k \ge n_2$ and $j \in J$. From (2.4), we have

$$\begin{aligned} d(x_n, p) &\le \frac{s}{s - 2M_0 u_k^{(j)}} d(x_{n-1}, p) \\ &\quad + \frac{2}{s - 2M_0 u_k^{(j)}} \left(\phi(M) u_k^{(j)} + v_k^{(j)}\right) + \frac{s}{s - 2M_0 u_k^{(j)}} \mathcal{A}_n. \end{aligned} \tag{2.6}$$

Let

$$1+\varphi_k = \frac{s}{s-2M_0u_k^{(j)}} = 1 + \frac{2M_0u_k^{(j)}}{s-2M_0u_k^{(j)}}.$$

From (2.5), we have

$$2u_k^{(j)} \le \frac{s}{2M_0}, \quad s-2M_0u_k^{(j)} \ge s - \frac{s}{2} = \frac{s}{2},$$

for $k \ge n_2$ and $j \in J$. It means that

$$\frac{1}{s-2M_0u_k^{(j)}} \le \frac{2}{s} \tag{2.7}$$

and so

$$\varphi_k = \frac{2M_0u_k^{(j)}}{s-2M_0u_k^{(j)}} \le \frac{4M_0}{s}u_k^{(j)}.$$

Thus, we have

$$\sum_{k=1}^{\infty}\varphi_k \le \frac{4M_0}{s}\cdot\sum_{k=1}^{\infty}\max_{j\in J}u_k^{(j)} < \infty.$$

Now by (2.6) and (2.7), we have

$$d(x_n,p) \le (1+\varphi_k)d(x_{n-1},p) + \frac{4}{s}\left(\phi(M)u_k^{(j)} + v_k^{(j)}\right) + 2\mathcal{A}_n. \tag{2.8}$$

Since $\sum_{i=1}^{\infty}\varphi_i < \infty$, $\sum_{i=1}^{\infty}\max_{j\in J}u_i^{(j)} < \infty$, $\sum_{i=1}^{\infty}\max_{j\in J}v_i^{(j)} < \infty$ and $\sum_{i=1}^{\infty}\mathcal{A}_i < \infty$, it follows from Lemma 1.8-(i) that the sequence $\{d(x_n,p)\}$ is bounded and $\lim_{n\to\infty} d(x_n,p)$ exists.

(ii) Taking the infimum over all $p \in \mathcal{F}$ in (2.8), we have

$$D(x_n,\mathcal{F}) \le (1+\varphi_k)D(x_{n-1},\mathcal{F}) + \frac{4}{s}\left(\phi(M)u_k^{(j)} + v_k^{(j)}\right) + 2\mathcal{A}_n. \tag{2.9}$$

Since $\sum_{i=1}^{\infty}\varphi_i < \infty$, $\sum_{i=1}^{\infty}\max_{j\in J}u_i^{(j)} < \infty$, $\sum_{i=1}^{\infty}\max_{j\in J}v_i^{(j)} < \infty$ and $\sum_{i=1}^{\infty}\mathcal{A}_i < \infty$, it follows from (2.9) and Lemma 1.8-(i) that $\lim_{n\to\infty} D(x_n,\mathcal{F})$ exists.

(iii) First, we consider the following case. Let

$$n = (l-2)N + N, \quad n+m = (l-1)N + N$$

for some $l \in \mathbb{N}$. Since $1 + x \le e^x$, $x > 0$, from (2.8), we have

$$\begin{aligned}
& d(x_{n+m}, p) \\
& \le (1+\varphi_l) d(x_{n+m-1}, p) + \frac{4}{s}\left(\phi(M) u_l^{(N)} + v_l^{(N)}\right) + 2\mathcal{A}_{n+m} \\
& \le e^{\varphi_l}\left\{(1+\varphi_l) d(x_{n+m-2}, p) + \frac{4}{s}\left(\phi(M) u_l^{(N-1)} + v_l^{(N-1)}\right) + 2\mathcal{A}_{n+m-1}\right\} \\
& \quad + \frac{4}{s}\left(\phi(M) u_l^{(N)} + v_l^{(N)}\right) + 2\mathcal{A}_{n+m} \\
& \le e^{2\varphi_l} d(x_{n+m-2}, p) \\
& \quad + \frac{4}{s} \cdot e^{2\varphi_l}\left\{\left(\phi(M) u_l^{(N-1)} + v_l^{(N-1)}\right) + \left(\phi(M) u_l^{(N)} + v_l^{(N)}\right)\right\} \\
& \quad + 2 \cdot e^{2\varphi_l}\left(\mathcal{A}_{n+m-1} + \mathcal{A}_{n+m}\right) \\
& \quad \vdots \\
& \le e^{N\varphi_l} d(x_n, p) \\
& \quad + \frac{4}{s} \cdot e^{N\varphi_l}\left\{\left(\phi(M) u_l^{(1)} + v_l^{(1)}\right) + \cdots + \left(\phi(M) u_l^{(N)} + v_l^{(N)}\right)\right\} \\
& \quad + 2 \cdot e^{N\varphi_l}\left(\mathcal{A}_{n+1} + \mathcal{A}_{n+2} + \cdots + \mathcal{A}_{n+m}\right) \\
& \le e^{N\varphi_l} d(x_n, p) \\
& \quad + \frac{4}{s} \cdot e^{N\varphi_l} \cdot N \cdot \left(\phi(M) \cdot \max_{j \in J} u_l^{(j)} + \max_{j \in J} v_l^{(j)}\right) + 2 \cdot e^{N\varphi_l} \sum_{i=n+1}^{n+m} \mathcal{A}_i.
\end{aligned}$$

Next, we consider general case, there exist $k, l \in \mathbb{N}$ such that

$$n = (k-1)N + j, \quad n + m = (l-1)N + j'$$

for $j, j' \in J$, then

$$\begin{aligned}
& d(x_{n+m}, p) \\
& \le e^{N \sum_{i=k}^{l} \varphi_i} \cdot d(x_n, p) \\
& \quad + \frac{4}{s} \cdot e^{N \sum_{i=k}^{l} \varphi_i} \cdot N \cdot \sum_{i=k}^{l}\left(\phi(M) \cdot \max_{j \in J} u_i^{(j)} + \max_{j \in J} v_i^{(j)}\right) \\
& \quad + 2 \cdot e^{N \sum_{i=k}^{l} \varphi_i} \cdot \sum_{i=n+1}^{n+m} \mathcal{A}_i \\
& \le R\left(d(x_n, p) + \frac{4N}{s} \sum_{i=k}^{l}\left(\phi(M) \max_{j \in J} u_i^{(j)} + \max_{j \in J} v_i^{(j)}\right) + 2 \sum_{i=n+1}^{n+m} \mathcal{A}_i\right) \quad (2.10)
\end{aligned}$$

for all $p \in \mathcal{F}$, $n, m \in \mathbb{N}$ and $R = e^{N \sum_{i=k}^{\infty} \varphi_i}$.

Now, we use (2.10) to prove that $\{x_n\}$ is a Cauchy sequence. From the hypothesis $\lim_{n\to\infty} D(x_n, \mathcal{F}) = 0$, $\sum_{i=1}^{\infty} \max_{j\in J} u_i^{(j)} < \infty$, $\sum_{i=1}^{\infty} \max_{j\in J} v_i^{(j)} < \infty$ and $\sum_{i=1}^{\infty} \mathcal{A}_i < \infty$, for each $\varepsilon > 0$, there exists $n_3 \in \mathbb{N}$ such that

$$D(x_n, \mathcal{F}) < \frac{\varepsilon}{4(R+1)}, \quad \forall\, n \geq n_3, \tag{2.11}$$

$$\sum_{i=k}^{\infty} \max_{j\in J} u_i^{(j)} < \frac{s\varepsilon}{16\,\phi(M)NR} \quad , \quad \sum_{i=k}^{\infty} v_i^{(j)} < \frac{s\varepsilon}{16NR}, \quad \forall\, n \geq n_3 \tag{2.12}$$

and

$$\sum_{i=n+1}^{\infty} \mathcal{A}_i < \frac{\varepsilon}{8R}, \quad \forall\, n \geq n_3. \tag{2.13}$$

Thus, from (2.11), there exists $q \in \mathcal{F}$ such that

$$d(x_n, q) < \frac{\varepsilon}{4(R+1)}, \quad \forall\, n \geq n_3. \tag{2.14}$$

Using (2.12), (2.13) and (2.14) in (2.10), we obtain

$$\begin{aligned}
&d(x_{n+m}, x_n) \\
&\leq d(x_{n+m}, q) + d(x_n, q) \\
&\leq R\left(d(x_n, p) + \frac{4N}{s} \sum_{i=k}^{l} \left(\phi(M) \max_{j\in J} u_i^{(j)} + \max_{j\in J} v_i^{(j)}\right) + 2 \sum_{i=n+1}^{n+m} \mathcal{A}_i\right) + d(x_n, q) \\
&= (R+1) d(x_n, q) + \frac{4NR}{s} \sum_{i=k}^{l} \left(\phi(M) \max_{j\in J} u_i^{(j)} + \max_{j\in J} v_i^{(j)}\right) + 2R \sum_{i=n+1}^{n+m} \mathcal{A}_i \\
&< (R+1) \cdot \frac{\varepsilon}{4(R+1)} + \frac{4\,\phi(M)NR}{s} \cdot \frac{s\varepsilon}{16\,\phi(M)NR} + \frac{4NR}{s} \cdot \frac{s\varepsilon}{16NR} + 2R \cdot \frac{\varepsilon}{8R} \\
&= \varepsilon
\end{aligned}$$

for all $n, m \geq n_3$. Thus $\{x_n\}$ is a Cauchy sequence. This completes the proof. □

Theorem 2.2. *Let (X, d, W) be a convex metric space with convex structure W and $\{S_j : X \to X : j \in J\}$ be a finite family of total asymptotically quasi-nonexpansive self-mappings with sequences $\{u_n^{(j)}\}, \{v_n^{(j)}\} \subset [0, \infty)$ and $\{T_j : X \to X : j \in J\}$ be a finite family of asymptotically quasi-nonexpansive* *self mappings in the intermediate sense as in Lemma* 2.1. *Suppose that* $\mathcal{F} \neq \emptyset$, $x_0 \in X$ *and* $\{x_n\}$ *is as in* (1.8) *such that* $\{\beta_n\} \subset (s, 1-s)$ *for some* $s \in (0, \frac{1}{2})$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and let* $\mathcal{A}_n$ *as in* (2.1). *Then we have the following statements:*

(1) *if* $\{x_n\}$ *converges strongly to a point in* $\mathcal{F}$, *then*

$$\liminf_{n\to\infty} D(x_n, \mathcal{F}) = \limsup_{n\to\infty} D(x_n, \mathcal{F}) = 0,$$

(2) *if* X *is complete and either* $\liminf_{n\to\infty} D(x_n, \mathcal{F}) = 0$ *or* $\limsup_{n\to\infty} D(x_n, \mathcal{F}) = 0$, *then* $\{x_n\}$ *converges strongly to a point in* $\mathcal{F}$.

Proof. (1) Let $\{x_n\}$ converge to $q \in \mathcal{F}$. Then $\lim_{n\to\infty} d(x_n, q) = 0$. Thus, for a given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, q) < \varepsilon, \quad \forall\, n \geq n_0.$$

Taking the infimum with respect to $q \in \mathcal{F}$, we obtain

$$D(x_n, \mathcal{F}) < \varepsilon, \quad \forall\, n \geq n_0,$$

that is,

$$\lim_{n\to\infty} D(x_n, \mathcal{F}) = 0. \tag{2.15}$$

Therefore, we have

$$\liminf_{n\to\infty} D(x_n, \mathcal{F}) = \limsup_{n\to\infty} D(x_n, \mathcal{F}) = 0.$$

(2) Let X be complete and either $\liminf_{n\to\infty} D(x_n, \mathcal{F}) = 0$ or $\limsup_{n\to\infty} D(x_n, \mathcal{F}) = 0$. From Lemma 1.8-(ii) and Remark 1.9, we have $\lim_{n\to\infty} D(x_n, \mathcal{F}) = 0$. Moreover, from the completeness of X and Lemma 2.1-(iii), Cauchy sequence $\{x_n\}$ converges to a point $u \in X$, that is,

$$\lim_{n\to\infty} x_n = u. \tag{2.16}$$

From (2.15) and (2.16), we have

$$D(u, \mathcal{F}) = 0.$$

Since the set $\mathcal{F}$, which is the set of common fixed points of two finite families of $\{S_j : j \in J\}$ and $\{T_j : j \in J\}$, is closed, we get

$$u \in \mathcal{F}.$$

This shows that u is a common fixed point of $\{S_j : j \in J\}$ and $\{T_j : j \in J\}$. Hence $\{x_n\}$ converges to a point in $\mathcal{F}$. This completes the proof. □

If the strictly increasing and continuous function ϕ is defined by $\phi(\lambda) = \lambda$ and $v_n = 0$ for all $n \geq 1$ in (1.1), then we have following corollary which is a generalization of the result of [20].

Corollary 2.3. *Let (X, d, W) be a convex metric space with convex structure W and $\{S_j : X \to X : j \in J\}$ be a finite family of asymptotically quasi-nonexpansive self-mappings with sequence $\{u_n^{(j)}\} \subset [0, \infty)$ such that $\sum_{i=1}^{\infty} \max_{j \in J} u_i^{(j)} < \infty$ and $\{T_j : X \to X : j \in J\}$ be a finite family of asymptotically quasi-nonexpansive self-mappings in the intermediate sense. Suppose that $\mathcal{F} \neq \emptyset$, $x_0 \in X$ and $\{x_n\}$ is as in (1.8) such that $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$ and let $\mathcal{A}_n$ as in (2.1). Then we have the following statements:*

(1) *if $\{x_n\}$ converges strongly to a point in $\mathcal{F}$, then*
$$\liminf_{n\to\infty} D(x_n, \mathcal{F}) = \limsup_{n\to\infty} D(x_n, \mathcal{F}) = 0,$$
(2) *if X is complete and either $\liminf_{n\to\infty} D(x_n, \mathcal{F}) = 0$ or $\limsup_{n\to\infty} D(x_n, \mathcal{F}) = 0$, then $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.*

3. Applications

In this section, we establish some results for strong convergence as an application of Theorem 2.2.

Theorem 3.1. *Let (X, d, W) be a complete convex metric space with convex structure W and $\{S_j : X \to X : j \in J\}$ be a finite family of total asymptotically quasi-nonexpansive self-mappings with sequences $\{u_n^{(j)}\}, \{v_n^{(j)}\} \subset [0, \infty)$ and $\{T_j : X \to X : j \in J\}$ be a finite family of asymptotically quasi-nonexpansive self-mappings in the intermediate sense as in Theorem 2.2. Suppose that $\mathcal{F} \neq \emptyset$, $x_0 \in X$ and $\{x_n\}$ is as in (1.8) such that $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$ and let $\mathcal{A}_n$ as in (2.1). Assume that the following two conditions hold:*

(i) $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$,

(ii) *the sequence $\{z_n\}$ in X satisfying $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$ implies*
$$\text{either} \quad \liminf_{n\to\infty} D(z_n, \mathcal{F}) = 0 \quad \text{or} \quad \limsup_{n\to\infty} D(z_n, \mathcal{F}) = 0.$$

Then $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.

Proof. From hypothesis (i) and (ii), we have

$$\text{either} \quad \liminf_{n\to\infty} D(x_n, \mathcal{F}) = 0 \;\text{ or }\; \limsup_{n\to\infty} D(x_n, \mathcal{F}) = 0.$$

Therefore, we obtain from Theorem 2.2-(2) that the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$. This completes the proof. □

A mapping $T : X \to X$ is said to be semi-compact if every bounded sequence $\{x_n\}$ in X with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ has a convergent subsequence.

Theorem 3.2. *Let (X, d, W) be a complete convex metric space with convex structure W and $\{S_j : X \to X : j \in J\}$ be a finite family of total asymptotically quasi-nonexpansive self-mappings with sequences $\{u_n^{(j)}\}, \{v_n^{(j)}\} \subset [0, \infty)$ and $\{T_j : X \to X : j \in J\}$ be a finite family of asymptotically quasi-nonexpansive self-mappings in the intermediate sense as in Theorem 2.2. Suppose that $\mathcal{F} \neq \emptyset$, $x_0 \in X$ and $\{x_n\}$ is as in (1.8) such that $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$ and let $\mathcal{A}_n$ as in (2.1). Assume that $\lim_{n\to\infty} d(x_n, S_j x_n) = \lim_{n\to\infty} d(x_n, T_j x_n) = 0$ for all $j \in J$. If there exist one of the mapping in $\{S_j : X \to X : j \in J\}$ and $\{T_j : X \to X : j \in J\}$, which is semi-compact. Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.*

Proof. Without loss of generality, we can assume that $T_l \, (l \in J)$ is a semi-compact. From Lemma 2.1-(i), we know that the sequence $\{x_n\}$ is bounded and by hypothesis, we know that $\lim_{n\to\infty} d(x_n, T_l x_n) = 0$ for some $l \in J$. Since T_l is a semi-compact, there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ such that $x_{n_r} \to x^* \in X$. Thus

$$d(x^*, T_j x^*) = \lim_{r\to\infty} d(x_{n_r}, T_j x_{n_r}) = 0$$

and

$$d(x^*, S_j x^*) = \lim_{r\to\infty} d(x_{n_r}, S_j x_{n_r}) = 0$$

for all $j \in J$, which imply that

$$x^* \in \mathcal{F}.$$

Therefore, we have

$$\liminf_{n\to\infty} D(x_n, \mathcal{F}) \leq \liminf_{r\to\infty} D(x_{n_r}, \mathcal{F}) \leq \lim_{r\to\infty} d(x_{n_r}, x^*) = 0.$$

It follows from Theorem 2.2-(2) that $\{x_n\}$ converges strongly to a point in $\mathcal{F}$. This completes the proof. □

Remark 3.3. Theorem 3.2 is a generalization and modification of the result of [9, Theorem 13].

Theorem 3.4. *Let (X, d, W) be a complete convex metric space with convex structure W and $\{S_j : X \to X : j \in J\}$ be a finite family of total asymptotically quasi-nonexpansive self-mappings with sequences $\{u_n^{(j)}\}, \{v_n^{(j)}\} \subset [0, \infty)$ and $\{T_j : X \to X : j \in J\}$ be a finite family of asymptotically quasi-nonexpansive self-mappings in the intermediate sense as in Theorem 2.2. Suppose that $\mathcal{F} \neq \emptyset$, $x_0 \in X$ and $\{x_n\}$ is as in (1.8) such that $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$ and let $\mathcal{A}_n$ as in (2.1). Assume that $\lim_{n\to\infty} d(x_n, S_j x_n) = \lim_{n\to\infty} d(x_n, T_j x_n) = 0$ for all $j \in J$. If one of the following conditions holds, then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.*

(C_1) *If there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, $g(t) > 0$ for all $t \in (0, \infty)$ such that*

$$\text{either } d(x_n, S_j x_n) \geq g(D(x_n, \mathcal{F})) \text{ or } d(x_n, T_j x_n) \geq g(D(x_n, \mathcal{F}))$$

for all $n \in \mathbb{N}$ and $j \in J$.

(C_2) *There exists a function $h : [0, \infty) \to [0, \infty)$ which is right continuous at 0, $h(0) = 0$ and*

$$\text{either } h(d(x_n, S_j x_n)) \geq D(x_n, \mathcal{F}) \text{ or } h(d(x_n, T_j x_n)) \geq D(x_n, \mathcal{F})$$

for all $n \in \mathbb{N}$ and $j \in J$.

Proof. **Case I.** Assume that (C_1) holds. Then, we have either

$$\lim_{n\to\infty} g(D(x_n, \mathcal{F})) \leq \lim_{n\to\infty} d(x_n, S_j x_n) = 0$$

or

$$\lim_{n\to\infty} g(D(x_n, \mathcal{F})) \leq \lim_{n\to\infty} d(x_n, T_j x_n) = 0.$$

No matter what cases, we have

$$\lim_{n\to\infty} g(D(x_n, \mathcal{F})) = 0.$$

By the property of g, we obtain $\lim_{n\to\infty} D(x_n, \mathcal{F}) = 0$.

Case II. Assume that (C_2) holds. From hypothesis, we have either

$$\lim_{n\to\infty} D(x_n, \mathcal{F}) \leq \lim_{n\to\infty} h(d(x_n, S_j x_n)) = h\left(\lim_{n\to\infty} d(x_n, S_j x_n)\right) = h(0) = 0$$

or

$$\lim_{n\to\infty} D(x_n, \mathcal{F}) \leq \lim_{n\to\infty} h(d(x_n, T_j x_n)) = h\left(\lim_{n\to\infty} d(x_n, T_j x_n)\right) = h(0) = 0.$$

No matter what cases, we have $\lim_{n\to\infty} D(x_n, \mathcal{F}) = 0$.

Therefore, from above both cases, we obtain

$$\lim_{n\to\infty} D(x_n, \mathcal{F}) = 0.$$

Thus the condition of Theorem 2.2-(2) is satisfied, the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$. This completes the proof. □

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