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REMARKS ON THE AUTOMORPHISM GROUP OF THE BOUNDED KOHN-NIRENBERG DOMAIN

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Abstract. In this paper, we completely describe the automorphism group of the bounded Kohn-Nirenberg domain in [5].

1. Introduction

The *automorphism group* $\mathrm{Aut}(D)$ of a domain D in $\mathbb{C}^n$ is the set of all biholomorphic self-mappings defined on D. The automorphism group is a topological group with compact-open topology. The bounded symmetric domain are well-known examples of domains with noncompact automorphism groups. If a domain has compact automorphism group, then the explicit description of automorphisms of a domain is difficult. There are few known examples of domains with explicit description of automorphisms. In general, a domain has the compact automorphism group. For example, Kohn and Nirenberg [7] constructed the domain D_{KN} in $\mathbb{C}^2$ defined by the following defining function

$$\psi(z, w) := \mathrm{Re}w + |zw|^2 + |z|^8 + \frac{15}{7}|z|^2\mathrm{Re}(z^6) < 0.$$

In [2], Byun proved that the automorphism group $\mathrm{Aut(D_{KN})}$ is generated by the map $\Pi : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$\Pi(z, w) = (e^{i\pi/3}z, w).$$

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It means that $\mathrm{Aut}(\mathrm{D}_{\mathrm{KN}})$ is equal to $\{\Pi^n \mid n = 1, \ldots, 6\}$, where Π^n is denoted by n-times composition function $\Pi \circ \cdots \circ \Pi$. This group is a cyclic group of order 6. This domain is an unbounded domain in $\mathbb{C}^2$.

In [5], Calami constructed an example of domain Ω called the bounded Kohn-Nirenberg domain defined by the following defining function

$$\rho(z, w) := \mathrm{Re}w + \frac{1}{5}|w|^2 + |zw|^2 + |z|^8 + \frac{15}{7}|z|^2\mathrm{Re}(z^6) < 0. \tag{1.1}$$

This domain Ω is bounded and real-analytic. The origin is a weakly pseudoconvex boundary point, while all the other boundary points are strongly pseudoconvex.

In this paper, we will show that all automorphism of Ω are generated by Π. The paper is organized as follows. Section 2 is devoted to the computation of the automorphism group. In Section 3, we will introduce main theorem and prove it base upon the result in Section 2.

2. Lemma for main theorem

Let $O = (0, 0)$ be the origin of $\mathbb{C}^2$ and let f be an automorphism of Ω. The notation $f(O) = O$ means that there is a sequence of points $q_j \in \Omega$ converging to the origin such that

$$\lim_{j\to\infty} f(q_j) = O.$$

In this case, we say that the automorphism f preserves the origin.

In this section, we will prove that all automorphisms $f \in \mathrm{Aut}(\Omega)$ should preserve the origin.

Lemma 2.1. *Let Ω is the bounded Kohn-Nirenberg domain defined by the equation* (1.1) *in $\mathbb{C}^2$. If f be an automorphism of Ω, then $f(O) = O$.*

Proof. Expecting a contradiction, we suppose that there is a sequence of points $q_j \in \Omega$ converging to the origin such that

$$\lim_{j\to\infty} f(q_j) \neq O.$$

Since Ω is bounded, we can suppose that $f(q_j)$ converges to a boundary point $p \in \partial\Omega$ not the origin. Since the result of [1, 6], f can be extended to the boundary. This implies that

$$8 = \tau(O) = \tau(p) = 2,$$

where $\tau(p)$ is the D'Angelo type at the boundary point p. This is contradict to the facts that the D'Angelo type should be preserved under the action of biholomorphisms. □

3. Main result

In this section, we will prove that $\mathrm{Aut}(\Omega)$ is a cyclic group of order 6.

Theorem 3.1. *The automorphism group of the bounded Kohn–Nirenberg domain Ω is equal to the set $\{\Pi^n : n = 1,2,3,4,5,6\}$. Therefore, it is compact and a cyclic group of order* 6*.*

Proof. Let f be an automorphism of Ω. By Lemma 2.1, f should preserves the origin O. It means that $f(O) = O$.

Let $\mathbf{H}_O$ be the holomorphic tangent space at O to the boundary of Ω. To follow the proof of the Kohn-Nirenberg domain and the Fornæss domain cases in [3, 4], it is needed the fact that the automorphism f with $f(O) = O$ preserves the subset $\mathbf{H}_O \cap \Omega$. It means that

$$f(\mathbf{H}_O \cap \Omega) \subset \mathbf{H}_O \cap \Omega.$$

Let

$$\rho(z,w) = \mathrm{Re} w + \frac{1}{5}|w|^2 + |zw|^2 + |z|^8 + \frac{15}{7}|z|^2\mathrm{Re}(z^6)$$

and $f = (f_1, f_2)$. Then ρ is a defining function for Ω. By assumption, $\rho \circ f$ is also a local defining function near the origin. Thus there is a positive real-analytic function μ such that

$$\rho \circ f \equiv \mu \cdot \rho. \tag{3.1}$$

We restrict this to the holomorphic tangent space $\mathbf{H}_O$ at the origin. Note that $\mathbf{H}_O = \{(z,0) : z \in \mathbb{C}\}$. The right-hand side of (3.1) is as follows :

$$\mu(z,0)\rho(z,0) = \mu(z,0)\big(|z|^8 + \frac{15}{7}|z|^2\mathrm{Re}(z^6)\big)$$

has no harmonic terms on $\mathbf{H}_O \cap \Omega$.

On the other hand, the left-hand side of (3.1)

$$\rho \circ f(z,0) = \mathrm{Re} f_2 + \frac{1}{5}|f_2|^2 + |f_1 f_2|^2 + |f_1|^8 + \frac{15}{7}|f_1|^2\mathrm{Re}(f_1)^6$$

the harmonic term $\mathrm{Re} f_2(z,0)$. Thus the harmonic term should vanish $\mathrm{Re} f_2(z,0) \equiv 0$. It means that $f(\mathbf{H}_O \cap \Omega) \subset \mathbf{H}_O \cap \Omega$.

The restriction map f on $\mathbf{H}_O \cap \Omega$ is determined by one-variable holomorphic function $h(z) = f_1(z,0)$ for all z satisfying $(z,0) \in \mathbf{H}_O \cap \Omega$.

Note that $\mathbf{H}_O \cap \Omega = \big\{(z,0) : |z|^6 + \frac{15}{7}\mathrm{Re}(z^6) < 0\big\}$. The holomorphic function $h : D \to D$ is a bijective map, where

$$D = \left\{ z \in \mathbb{C} : |z|^6 + \frac{15}{7}\mathrm{Re}(z^6) < 0 \right\}.$$

Note that h can be holomorphically extended near the origin by the result of [1, 6]. Since $f(O) = O$, we get $h(0) = 0$. If the first derivate $|h'(0)| < 1$, then the origin is attracting fixed point of h at the origin. It means that there is an automorphism orbit accumulating at the origin. This is a contradiction of the result [2]. If $|h'(0)| > 1$, then the inverse of h has attracting fixed point at the origin. Also, this is impossible by the same reason. Hence we get $|h'(0)| = 1$. So there is a θ such that $h'(0) = e^{i\theta}$. Since the boundary of D consists of six lines through the origin and h preserves the lines, we get

$$\theta = k\frac{\pi}{3}$$

for some $k \in \mathbb{Z}$. By Cartan Identity Theorem,

$$h(z) = e^{ik\frac{\pi}{3}} z = f_1(z,0).$$

Let $g(z,w) = \Pi^{-k} \circ f(z,w)$. Then $g \in \mathrm{Aut}(\Omega)$ and $g(z,0) = (z,0)$ for all $(z,0) \in \Omega$. For all $(z,0) \in \Omega$,

$$dg(z,0) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}.$$

Since $g(z,0) = (z,0)$, $a(z) = 1$ and $c(z) = 0$. Since all points $(z,0) \in \Omega$ are fixed points of the automorphism g and Ω is complete hyperbolic, the automorphism sequence $g^n = g \circ \cdots \circ g$ has a convergent subsequence converging to some automorphism. we get $d(z)$ has modulus 1 for all z with $(z,0) \in \Omega$. By Maximum modulus theorem, $d(z) \equiv 1$. Since

$$dg^n(z,0) = \begin{pmatrix} 1 & nb(z) \\ 0 & 1 \end{pmatrix},$$

we get $b(z) = 0$. Note that for all $(z,0) \in \Omega$, $g(z,0) = (z,0)$ and

$$dg(z,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By Cartan Identity Theorem, $g \equiv Id \equiv \Pi^{-k} \circ f$ is the identity map. Therefore

$$f \equiv \Pi^k.$$

This completes the proof. □

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