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ITERATIVE ALGORITHMS FOR SOLUTIONS OF GENERALIZED REGULARIZED NONCONVEX VARIATIONAL INEQUALITIES

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Abstract. A new system of generalized nonlinear regularized nonconvex variational inequalities involving three different nonlinear operators is introduced and the equivalence between the aforesaid system and a fixed point problem is proved. Then by this equivalent formulation, the existence and uniqueness theorem for solution of the system of generalized nonlinear regularized nonconvex variational inequalities is established. Some new three-step projection iterative schemes for approximating the unique solution of the aforementioned system are constructed. The convergence analysis of the suggested iterative algorithms under some suitable conditions are also studied.

1. Introduction

The theory of variational inequalities, which was initially introduced by Stampacchia [30] in 1964, is a branch of the mathematical sciences dealing with general equilibrium problems. It has a wide range of applications in economics, operations research, industry, physical, and engineering sciences. Many research papers have been written lately, both on the theory and applications of this field. Important connections with main areas of pure and applied sciences have been made, see, for example, [4, 18, 20] and the references cited therein. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the

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one hand, it reveals the fundamental facts on the qualitative aspects of the solution to important classes of problems; on the other hand, it also enables us to develop highly efficient and powerful new numerical methods to solve, for example, obstacle, unilateral, free, moving and the complex equilibrium problems. One of the most interesting and important problems in variational inequality theory is the development of an efficient numerical method. There is a substantial number of numerical methods including projection method and its variant forms, Wiener-Holf (normal) equations, auxiliary principle, and descent framework for solving variational inequalities and complementarity problems. For the applications, physical formulations, numerical methods and other aspects of variational inequalities, see [1–18, 20, 21, 23–28, 30–34] and the references therein.

Projection method and its variant forms represent important tool for finding the approximate solution of various types of variational and quasi-variational inequalities, the origin of which can be traced back to Lions and Stampacchia [23]. The projection type methods were developed in 1970's and 1980's. The main idea in this technique is to establish the equivalence between the variational inequalities and the fixed point problems using the concept of projection. This alternate formulation enables us to suggest some iterative methods for computing the approximate solution.

It should be pointed that almost all the results regarding the existence and iterative schemes for solving variational inequalities and related optimizations problems are being considered in the convexity setting. Consequently, all the techniques are based on the properties of the projection operator over convex sets, which may not hold in general, when the sets are nonconvex. It is known that the uniformly prox-regular sets are nonconvex and include the convex sets as special cases, for more details, see, for example, [9, 16, 17, 27]. In recent years, Bounkhel et al. [9], Moudafi [24], Balooee et al. [3] and Pang et al. [26] have considered variational inequalities in the context of uniformly prox-regular sets.

On the other hand, related to the variational inequalities, we have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is natural to consider a unified approach to these two different problems. Motivated and inspired by the research going in this direction, Noor and Huang [25] considered the problem of finding the common element of the set of the solutions of variational inequalities and the set of the fixed points of the nonexpansive mappings. It is well known that every nonexpansive mapping is a Lipschitzian mapping. Lipschitzian mappings have been generalized by various authors. Sahu [29] introduced and investigated nearly uniformly Lipschitzian mappings as generalization of Lipschitzian mappings.

In the present paper, we introduce and consider a new system of generalized nonlinear regularized nonconvex variational inequalities involving three different nonlinear operators and establish the equivalence between the mentioned system and a fixed point problem. Then by this equivalent alternative formulation, we discuss the existence and uniqueness of solution of the system of generalized nonlinear regularized nonconvex variational inequalities. Applying three nearly uniformly Lipschitzian mappings S_i $(i = 1, 2, 3)$ and the aforementioned equivalent alternative formulation, we construct a new three-step projection iterative algorithm for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping $\mathcal{Q} = (S_1, S_2, S_3)$ which is the unique solution of the system of generalized nonlinear regularized nonconvex variational inequalities. The convergence analysis of the suggested iterative algorithms under some suitable conditions are also studied.

2. Preliminaries and basic results

Throughout this article, we will let $\mathcal{H}$ be a real Hilbert space which is equipped with an inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$ and let K be a nonempty and closed subset of $\mathcal{H}$. We denote by $d_K(\cdot)$ or $d(\cdot, K)$ the usual distance function to the subset K, i.e., $d_K(u) = \inf_{v \in K} \|u - v\|$. Let us recall the following well-known definitions and some auxiliary results of nonlinear convex analysis and nonsmooth analysis [15–17, 27].

Definition 2.1. Let $u \in \mathcal{H}$ is a point not lying in K. A point $v \in K$ is called *a closest point or a projection of u onto K* if, $d_K(u) = \|u - v\|$. The set of all such closest points is denoted by $P_K(u)$, i.e.,

$$P_K(u) := \{v \in K : d_K(u) = \|u - v\|\}.$$

Definition 2.2. The proximal normal cone of K at a point $u \in K$ is given by

$$N_K^P(u) := \{\xi \in \mathcal{H} : u \in P_K(u + \alpha\xi), \ for\ some\ \alpha > 0\}.$$

It can be easily seen $N_K^P(\cdot)$ is a closed set-valued map.

Clarke et al. [16], in Proposition 1.1.5, give a characterization of $N_K^P(u)$ as the following:

Lemma 2.3. *Let K be a nonempty closed subset in $\mathcal{H}$. Then $\xi \in N_K^P(u)$ if and only if there exists a constant $\alpha = \alpha(\xi, u) > 0$ such that $\langle \xi, v - u \rangle \leq \alpha \|v - u\|^2$ for all $v \in K$.*

The above inequality is called the *proximal normal inequality.* The special case in which K is closed and convex is an important one. In Proposition

1.1.10 of [16], the authors give the following characterization of the proximal normal cone for the closed and convex subset $K \subset \mathcal{H}$:

Lemma 2.4. *Let K be a nonempty, closed and convex subset in $\mathcal{H}$. Then $\xi \in N_K^P(u)$ if and only if $\langle \xi, v - u \rangle \leq 0$ for all $v \in K$.*

Definition 2.5. Let X is a real Banach space and $f : X \to \mathbb{R}$ be Lipschitz with constant τ near a given point $x \in X$; that is, for some $\varepsilon > 0$, we have $|f(y) - f(z)| \leq \tau \|y - z\|$ for all $y, z \in B(x; \varepsilon)$ where $B(x; \varepsilon)$ denotes the open ball of radius $\varepsilon > 0$ and centered at x. The generalized directional derivative of f at x in the direction v, denoted as $f^\circ(x; v)$, is defined as follows:

$$f^\circ(x; v) = \limsup_{y \to x, t \downarrow 0} \frac{f(y + tv) - f(y)}{t},$$

where y is a vector in X and t is a positive scalar.

The generalized directional derivative defined earlier can be used to develop a notion of tangency that does not require K to be smooth or convex.

Definition 2.6. The tangent cone $T_K(x)$ to K at a point x in K is defined as follows:

$$T_K(x) := \{v \in \mathcal{H} : d_K^\circ(x; v) = 0\}.$$

Having defined a tangent cone, the likely candidate for the normal cone is the one obtained from $T_K(x)$ by polarity. Accordingly, we define the normal cone of K at x by polarity with $T_K(x)$ as follows:

$$N_K(x) := \{\xi : \langle \xi, v \rangle \leq 0, \ \forall v \in T_K(x)\}.$$

Definition 2.7. The *Clarke normal cone*, denoted by $N_K^C(x)$, is given by $N_K^C(x) = \overline{co}[N_K^P(x)]$, where $\overline{co}[S]$ means the closure of the convex hull of S. It is clear that one always has $N_K^P(x) \subseteq N_K^C(x)$. The converse is not true in general. Note that $N_K^C(x)$ is always closed and convex cone, whereas $N_K^P(x)$ is always convex, but may not be closed (see [15, 16, 27]).

In 1995, Clarke et al. [17], introduced and studied a new class of nonconvex sets, called proximally smooth sets; subsequently Poliquin et. al in [27] investigated the aforementioned sets, under the name of uniformly prox-regular sets. These have been successfully used in many nonconvex applications in areas such as optimizations, economic models, dynamical systems, differential inclusions, etc. For such as applications see [6–8, 10]. This class seems particularly well suited to overcome the difficulties which arise due to the nonconvexity assumptions on K. We take the following characterization proved in [17] as a definition of this class. We point out that the original definition was given in terms of the differentiability of the distance function (see [17]).

Definition 2.8. For any $r \in (0, +\infty]$, a subset K_r of $\mathcal{H}$ is called *normalized uniformly prox-regular* (or *uniformly r-prox-regular* [17]) if every nonzero proximal normal to K_r can be realized by an r-ball. This means that for all $\bar{x} \in K_r$ and all $0 \neq \xi \in N^P_{K_r}(\bar{x})$,

$$\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2, \qquad \forall x \in K_r.$$

Obviously, the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of $\mathcal{H}$, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets, see [11, 17].

Lemma 2.9. ([17]) *A closed set $K \subseteq \mathcal{H}$ is convex if and only if it is proximally smooth of radius r for every $r > 0$.*

If $r = +\infty$, then in view of Definition 2.8 and Lemma 2.9, the uniform r-prox-regularity of K_r is equivalent to the convexity of K_r, which makes this class of great importance. For the case of that $r = +\infty$, we set $K_r = K$.

The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel. The proof of this results can be found in [17, 27].

Proposition 2.10. *Let $r > 0$ and K_r be a nonempty closed and uniformly r-prox-regular subset of $\mathcal{H}$. Set $U(r) = \{u \in \mathcal{H} : 0 < d_{K_r}(u) < r\}$. Then the following statements hold:*

(i) *For all $x \in U(r)$, one has $P_{K_r}(x) \neq \emptyset$;*
(ii) *For all $r' \in (0, r)$, P_{K_r} is Lipschitz continuous with constant $\frac{r}{r-r'}$ on $U(r') = \{u \in \mathcal{H} : 0 < d_{K_r}(u) < r'\}$.*

Since $N^P_K(\cdot)$ is a closed set-valued map, we have $N^C_{K_r}(x) = N^P_{K_r}(x)$. Therefore, we will define $N_{K_r}(x) := N^C_{K_r}(x) = N^P_{K_r}(x)$ for such a class of sets.

In order to make clear the concept of r-prox-regular sets, we state the following concrete example: The union of two disjoint intervals $[a, b]$ and $[c, d]$ is r-prox-regular with $r = \frac{c-b}{2}$. The finite union of disjoint intervals is also r-prox-regular and r depends on the distances between the intervals.

3. System of generalized nonlinear regularized nonconvex variational inequalities

This section is dedicated to introduce a new system of generalized nonlinear regularized nonconvex variational inequalities in Hilbert spaces and to prove the existence and uniqueness theorem for solution of the aforesaid system.

Let K_r be a uniformly r-prox-regular subset of $\mathcal{H}$. For given nonlinear operators $T_i : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ $(i = 1, 2, 3)$ and constants $\rho, \eta, \gamma > 0$, we consider the following *system of generalized nonlinear regularized nonconvex variational inequalities* (SGNRNVI): Find $(x^*, y^*, z^*) \in K_r \times K_r \times K_r$ such that for all $x \in K_r$

$$\begin{cases} \langle \rho T_1(y^*, z^*, x^*) + x^* - y^*, x - x^* \rangle + \frac{\lambda_1}{2r}\|x - x^*\|^2 \geq 0, \\ \langle \eta T_2(z^*, x^*, y^*) + y^* - z^*, x - y^* \rangle + \frac{\lambda_2}{2r}\|x - y^*\|^2 \geq 0, \\ \langle \gamma T_3(x^*, y^*, z^*) + z^* - x^*, x - z^* \rangle + \frac{\lambda_3}{2r}\|x - z^*\|^2 \geq 0, \end{cases} \tag{3.1}$$

where $\lambda_1 = \|\rho T_1(y^*, z^*, x^*) + x^* - y^*\|$, $\lambda_2 = \|\eta T_2(z^*, x^*, y^*) + y^* - z^*\|$ and $\lambda_3 = \|\gamma T_3(x^*, y^*, z^*) + z^* - x^*\|$.

If $r = \infty$, i.e., $K_r = K$, the convex set in $\mathcal{H}$, then the system (3.1) reduces to the following system:

Find $(x^*, y^*, z^*) \in K \times K \times K$ such that for all $x \in K$

$$\begin{cases} \langle \rho T_1(y^*, z^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \\ \langle \eta T_2(z^*, x^*, y^*) + y^* - z^*, x - y^* \rangle \geq 0, \\ \langle \gamma T_3(x^*, y^*, z^*) + z^* - x^*, x - z^* \rangle \geq 0, \end{cases}$$

which has been introduced and studied by Cho and Qin [14].

By taking different choices of the operators T_i $(i = 1, 2, 3)$ and constants ρ, η and γ in the above problems, one can easily obtain the systems and problems studied in [13, 30–33] and the references therein.

Proposition 3.1. *If K_r is a uniformly prox-regular set then the system* (3.1) *is equivalent to that of finding $(x^*, y^*, z^*) \in K_r \times K_r \times K_r$ such that*

$$\begin{cases} 0 \in \rho T_1(y^*, z^*, x^*) + x^* - y^* + N^P_{K_r}(x^*), \\ 0 \in \eta T_2(z^*, x^*, y^*) + y^* - z^* + N^P_{K_r}(y^*), \\ 0 \in \gamma T_3(x^*, y^*, z^*) + z^* - x^* + N^P_{K_r}(z^*), \end{cases} \tag{3.2}$$

where $N^P_{K_r}(s)$ denotes the P-normal cone of K_r at s in the sense of nonconvex analysis.

Proof. Let $(x^*, y^*, z^*) \in K_r \times K_r \times K_r$ be a solution of the system (3.1). If $\rho T_1(y^*, z^*, x^*) + x^* - y^* = 0$, because the vector zero always belongs to any normal cone, we have $0 \in \rho T_1(y^*, z^*, x^*) + x^* - y^* + N^P_{K_r}(x^*)$. If $\rho T_1(y^*, z^*, x^*) + x^* - y^* \neq 0$, then for all $x \in K_r$, one has

$$\langle -(\rho T_1(y^*, z^*, x^*) + x^* - y^*), x - x^* \rangle \leq \frac{\lambda_1}{2r}\|x - x^*\|^2,$$

where λ_1 is the same as in the system (3.1). Now, Lemma 2.3 implies

$$-(\rho T_1(y^*, z^*, x^*) + x^* - y^*) \in N^P_{K_r}(x^*)$$

and so

$$0 \in \rho T_1(y^*, z^*, x^*) + x^* - y^* + N^P_{K_r}(x^*).$$

Similarly we can get

$$0 \in \eta T_2(z^*, x^*, y^*) + y^* - z^* + N^P_{K_r}(y^*),$$
$$0 \in \gamma T_3(x^*, y^*, z^*) + z^* - x^* + N^P_{K_r}(z^*).$$

Conversely, if $(x^*, y^*, z^*) \in K_r \times K_r \times K_r$ is a solution of the system (3.2), then it follows from Definition 2.8 that $(x^*, y^*, z^*) \in K_r \times K_r \times K_r$ is a solution of the system (3.1). □

The problem (3.2) is called the *system of generalized nonlinear nonconvex variational inclusions* associated with SGNRNVI (3.1).

Now, we prove the existence and uniqueness theorem for solution of the SGNRNVI (3.1). For this end, we need the following lemma in which by using the projection operator technique, the equivalence between SGNRNVI (3.1) and a fixed point problem is established.

Lemma 3.2. *Let T_i $(i = 1, 2, 3)$, ρ, η and γ be the same as in the system (3.1). Then $(x^*, y^*, z^*) \in K_r \times K_r \times K_r$ is a solution of the system (3.1), if and only if*

$$\begin{cases} x^* = P_{K_r}(y^* - \rho T_1(y^*, z^*, x^*)), \\ y^* = P_{K_r}(z^* - \eta T_2(z^*, x^*, y^*)), \\ z^* = P_{K_r}(x^* - \gamma T_3(x^*, y^*, z^*)), \end{cases} \tag{3.3}$$

provided that $\rho < \frac{r'}{1+\|T_1(y^*,z^*,x^*)\|}$, $\eta < \frac{r'}{1+\|T_2(z^*,x^*,y^*)\|}$, $\gamma < \frac{r'}{1+\|T_3(x^*,y^*,z^*)\|}$, for some $r' \in (0, r)$, where P_{K_r} is the projection of $\mathcal{H}$ onto the uniformly prox-regular set K_r.

Proof. Let $(x^*, y^*, z^*) \in K_r \times K_r \times K_r$ be a solution of the system (3.1). Since $x^*, y^*, z^* \in K_r$, $\rho < \frac{r'}{1+\|T_1(y^*,z^*,x^*)\|}$, $\eta < \frac{r'}{1+\|T_2(z^*,x^*,y^*)\|}$ and $\gamma < \frac{r'}{1+\|T_3(x^*,y^*,z^*)\|}$, for some $r' \in (0, r)$, it follows that the points $y^* - \rho T_1(y^*, z^*, x^*)$,

$z^* - \eta T_2(z^*, x^*, y^*)$ and $x^* - \gamma T_3(x^*, y^*, z^*)$ belong to $U(r')$, for some $r' \in (0, r)$, that is, equations (3.3) are well-define. Then, by using Proposition 3.1, we have

$$\begin{cases} 0 \in \rho T_1(y^*, z^*, x^*) + x^* - y^* + N^P_{K_r}(x^*), \\ 0 \in \eta T_2(z^*, x^*, y^*) + y^* - z^* + N^P_{K_r}(y^*), \\ 0 \in \gamma T_3(x^*, y^*, z^*) + z^* - x^* + N^P_{K_r}(z^*), \end{cases}$$

$\Longleftrightarrow$

$$\begin{cases} y^* - \rho T_1(y^*, z^*, x^*) \in x^* + N^P_{K_r}(x^*), \\ z^* - \eta T_2(z^*, x^*, y^*) \in y^* + N^P_{K_r}(y^*), \\ x^* - \gamma T_3(x^*, y^*, z^*) \in z^* + N^P_{K_r}(z^*), \end{cases}$$

$\Longleftrightarrow$

$$\begin{cases} x^* = P_{K_r}(y^* - \rho T_1(y^*, z^*, x^*)), \\ y^* = P_{K_r}(z^* - \eta T_2(z^*, x^*, y^*)), \\ z^* = P_{K_r}(x^* - \gamma T_3(x^*, y^*, z^*)), \end{cases}$$

where I is identity operator and we have used the well-known fact that $P_{K_r} = (I + N^P_{K_r})^{-1}$. □

Definition 3.3. A three-variable operator $T : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is said to be

(i) *ϱ-strongly monotone* in the first variable if there exists a constant $\varrho > 0$ such that for all $x, x' \in \mathcal{H}$

$$\langle T(x, y, z) - T(x', y', z'), x - x' \rangle \geq \varrho \|x - x'\|^2, \quad \forall y, y', z, z' \in \mathcal{H};$$

(ii) *μ-Lipschitz continuous in the first variable* if there exists a constant $\mu > 0$ such that for all $x, x' \in \mathcal{H}$

$$\|T(x, y, z) - T(x', y', z')\| \leq \mu \|x - x'\|, \quad \forall y, y', z, z' \in \mathcal{H}.$$

Theorem 3.4. *Let T_i $(i = 1, 2, 3)$, ρ, η and γ be the same as in the system* (3.1) *and suppose further that for each $i = 1, 2, 3$, T_i is ϱ_i-strongly monotone and σ_i-Lipschitz continuous in the first variable. If constants ρ, η and γ satisfy the following conditions*

$$\rho < \frac{r'}{1 + \|T_1(y, z, x)\|}, \quad \eta < \frac{r'}{1 + \|T_2(z, x, y)\|}, \quad \gamma < \frac{r'}{1 + \|T_3(x, y, z)\|}, \tag{3.4}$$

for some $r' \in (0, r)$ and for all $x, y, z \in \mathcal{H}$, and

$$\begin{cases} |\rho - \frac{\varrho_1}{\sigma_1^2}| < \frac{\sqrt{r^2 \varrho_1^2 - \sigma_1^2 r'(2r - r')}}{r \sigma_1^2}, \\ |\eta - \frac{\varrho_2}{\sigma_2^2}| < \frac{\sqrt{r^2 \varrho_2^2 - \sigma_2^2 r'(2r - r')}}{r \sigma_2^2}, \\ |\gamma - \frac{\varrho_3}{\sigma_3^2}| < \frac{\sqrt{r^2 \varrho_3^2 - \sigma_3^2 r'(2r - r')}}{r \sigma_3^2}, \\ r \varrho_i > \sigma_i \sqrt{r'(2r - r')}, \ (i = 1, 2, 3), \end{cases} \tag{3.5}$$

then the system (3.1) *admits a unique solution.*

Proof. Define the mappings $\psi, \phi, \varphi : K_r \times K_r \times K_r \to K_r$ by

$$\begin{aligned} \psi(x,y,z) &= P_{K_r}(y - \rho T_1(y,z,x)), \\ \phi(x,y,z) &= P_{K_r}(z - \eta T_2(z,x,y)), \\ \varphi(x,y,z) &= P_{K_r}(x - \gamma T_3(x,y,z)), \end{aligned} \tag{3.6}$$

for all $(x,y,z) \in K_r \times K_r \times K_r$. By using the condition (3.4), one can easily check that the mappings ψ, ϕ, φ are well-defined. Define $\|.\|_*$ on $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$ by

$$\|(x,y,z)\|_* = \|x\| + \|y\| + \|z\|, \qquad \forall (x,y,z) \in \mathcal{H} \times \mathcal{H} \times \mathcal{H}.$$

Clearly $(\mathcal{H}\times\mathcal{H}\times\mathcal{H}, \|.\|_*)$ is a Banach space. Further, define $F : K_r\times K_r\times K_r \to K_r \times K_r \times K_r$ as follows:

$$F(x,y,z) = (\psi(x,y,z), \phi(x,y,z), \varphi(x,y,z)), \tag{3.7}$$

for all $(x,y,z) \in K_r \times K_r \times K_r$. Now, we establish that F is a contraction mapping. Let $(x,y,z), (\hat{x},\hat{y},\hat{z}) \in K_r \times K_r \times K_r$ be given. Since $y \in K_r$ and $\rho < \frac{r'}{1+\|T_1(y,z,x)\|}$, for some $r' \in (0,r)$, it follows that $y - \rho T_1(y,z,x) \in U(r')$, for some $r' \in (0,r)$. The r-prox-regularity of K_r implies that the set $P_{K_r}(y - \rho T_1(y,z,x))$ is nonempty and singleton. Similarly, one can deduce that the sets $P_{K_r}(\hat{y} - \rho T_1(\hat{y},\hat{z},\hat{x}))$, $P_{K_r}(z - \eta T_2(z,x,y))$, $P_{K_r}(\hat{z} - \eta T_2(\hat{z},\hat{x},\hat{y}))$, $P_{K_r}(x - \gamma T_3(x,y,z))$ and $P_{K_r}(\hat{x} - \gamma T_3(\hat{x},\hat{y},\hat{z}))$ are nonempty and singleton. By using Proposition 2.10, one has

$$\begin{aligned} &\|\psi(x,y,z) - \psi(\hat{x},\hat{y},\hat{z})\| \\ &= \|P_{K_r}(y - \rho T_1(y,z,x)) - P_{K_r}(\hat{y} - \rho T_1(\hat{y},\hat{z},\hat{x}))\| \\ &\leq \frac{r}{r-r'}\|y - \hat{y} - \rho(T_1(y,z,x) - T_1(\hat{y},\hat{z},\hat{x}))\|. \end{aligned} \tag{3.8}$$

Since T_1 is ϱ_1-strongly monotone and σ_1-Lipschitz continuous in the first variable, we conclude that

$$\begin{aligned} &\|y - \hat{y} - \rho(T_1(y,z,x) - T_1(\hat{y},\hat{z},\hat{x}))\|^2 \\ &= \|y-\hat{y}\|^2 - 2\rho\langle T_1(y,z,x) - T_1(\hat{y},\hat{z},\hat{x}), y - \hat{y}\rangle \\ &\quad + \rho^2\|T_1(y,z,x) - T_1(\hat{y},\hat{z},\hat{x})\|^2 \\ &\leq (1 - 2\rho\varrho_1 + \rho^2\sigma_1^2)\|y-\hat{y}\|^2. \end{aligned} \tag{3.9}$$

Substituting (3.9) in (3.8), we gain

$$\|\psi(x,y,z) - \psi(\hat{x},\hat{y},\hat{z})\| \leq \frac{r}{r-r'}\sqrt{1 - 2\rho\varrho_1 + \rho^2\sigma_1^2}\|y-\hat{y}\|. \tag{3.10}$$

In a similar way to that of proof of (3.10), we get

$$\|\phi(x,y,z)-\phi(\hat{x},\hat{y},\hat{z})\| \leq \frac{r}{r-r'}\sqrt{1-2\eta\varrho_2+\eta^2\sigma_2^2}\|z-\hat{z}\| \tag{3.11}$$

and

$$\|\varphi(x,y,z)-\varphi(\hat{x},\hat{y},\hat{z})\| \leq \frac{r}{r-r'}\sqrt{1-2\gamma\varrho_3+\gamma^2\sigma_3^2}\|x-\hat{x}\|. \tag{3.12}$$

It follows from (3.10)–(3.12) that

$$\begin{aligned}&\|\psi(x,y,z)-\psi(\hat{x},\hat{y},\hat{z})\|+\|\phi(x,y,z)-\phi(\hat{x},\hat{y},\hat{z})\|\\&\quad+\|\varphi(x,y,z)-\varphi(\hat{x},\hat{y},\hat{z})\|\\&\leq \upsilon\|x-\hat{x}\|+\vartheta\|y-\hat{y}\|+\varpi\|z-\hat{z}\|,\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}\vartheta&=\frac{r}{r-r'}\sqrt{1-2\rho\varrho_1+\rho^2\sigma_1^2}, \quad \varpi=\frac{r}{r-r'}\sqrt{1-2\eta\varrho_2+\eta^2\sigma_2^2},\\ \upsilon&=\frac{r}{r-r'}\sqrt{1-2\gamma\varrho_3+\gamma^2\sigma_3^2}.\end{aligned}$$

By (3.7) and (3.13), we conclude that

$$\|F(x,y,z)-F(\hat{x},\hat{y},\hat{z})\|_* \leq \theta\|(x,y,z)-(\hat{x},\hat{y},\hat{z})\|_*, \tag{3.14}$$

where $\theta=\max\{\upsilon,\vartheta,\varpi\}$. In view of the condition (3.5), we note that $0\leq\theta<1$, and so, from (3.14) we conclude that the mapping F is contraction. According to Banach fixed point theorem, there exists a unique point $(x^*,y^*,z^*)\in K_r\times K_r\times K_r$ such that $F(x^*,y^*,z^*)=(x^*,y^*,z^*)$. It follows from (3.6) and (3.7) that $x^*=P_{K_r}(y^*-\rho T_1(y^*,z^*,x^*))$, $y^*=P_{K_r}(z^*-\eta T_2(z^*,x^*,y^*))$ and $z^*=P_{K_r}(x^*-\gamma T_3(x^*,y^*,z^*))$. Now, Lemma 3.2 guarantees that $(x^*,y^*,z^*)\in K_r\times K_r\times K_r$ is a unique solution of the system (3.1) and this completes the proof. □

4. Projection iterative algorithms

We need to recall that a nonlinear mapping $T:\mathcal{H}\to\mathcal{H}$ is called *nonexpansive* if $\|Tx-Ty\|\leq\|x-y\|$, for all $x,y\in\mathcal{H}$. In recent years, the nonexpansive mappings have been generalized and investigated by various authors. One of these generalizations is the class of nearly uniformly Lipschitzian mappings. In this section, we first recall several generalizations of the nonexpansive mappings which have been introduced in recent years. Then, we use three nearly uniformly Lipschitzian mappings S_i $(i=1,2,3)$, and the equivalent alternative formulation (3.3) to suggest and analyze a new three-step projection iterative algorithm for finding an element of the set of the fixed points $\mathcal{Q}=(S_1,S_2,S_3)$ which is the unique solution of SGNRNVI (3.1).

In the next definitions, several generalizations of the nonexpansive mappings which have been introduced by various authors in recent years are stated.

Definition 4.1. A nonlinear mapping $T : \mathcal{H} \to \mathcal{H}$ is called

(a) *L-Lipschitzian* if there exists a constant $L > 0$ such that
$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in \mathcal{H};$$

(b) *generalized Lipschitzian* if there exists a constant $L > 0$ such that
$$\|Tx - Ty\| \leq L(\|x - y\| + 1), \quad \forall x, y \in \mathcal{H};$$

(c) *generalized (L, M)-Lipschitzian* [29] if there exist two constants $L, M > 0$ such that
$$\|Tx - Ty\| \leq L(\|x - y\| + M), \quad \forall x, y \in \mathcal{H};$$

(d) *asymptotically nonexpansive* [19] if there exists a sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for each $n \in \mathbb{N}$,
$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \quad \forall x, y \in \mathcal{H};$$

(e) *pointwise asymptotically nonexpansive* [22] if, for each integer $n \geq 1$,
$$\|T^n x - T^n y\| \leq \alpha_n(x)\|x - y\|, \quad x, y \in \mathcal{H},$$
where $\alpha_n \to 1$ pointwise on X;

(f) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that for each $n \in \mathbb{N}$,
$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

Definition 4.2. ([29]) A nonlinear mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be

(a) *nearly Lipschitzian* with respect to the sequence $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that
$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n), \quad \forall x, y \in \mathcal{H}, \tag{4.1}$$
where $\{a_n\}$ is a fix sequence in $[0, \infty)$ with $a_n \to 0$, as $n \to \infty$.

For an arbitrary, but fixed $n \in \mathbb{N}$, the infimum of constants k_n in (4.1) is called *nearly Lipschitz constant* and is denoted by $\eta(T^n)$. Notice that
$$\eta(T^n) = \sup\Big\{\frac{\|T^n x - T^n y\|}{\|x - y\| + a_n} : x, y \in \mathcal{H}, x \neq y\Big\}.$$

A nearly Lipschitzian mapping T with the sequence $\{(a_n, \eta(T^n))\}$ is said to be

(b) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$, that is,
$$\|T^n x - T^n y\| \leq \|x - y\| + a_n, \quad \forall x, y \in \mathcal{H};$$

(c) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) = 1$, in other words, $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} k_n = 1$;
(d) *nearly uniformly L-Lipschitzian* if $\eta(T^n) \leq L$ for all $n \in \mathbb{N}$, in other words, $k_n = L$ for all $n \in \mathbb{N}$.

Remark 4.3. It should be pointed that

(a) Every nonexpansive mapping is a asymptotically nonexpansive mapping and every asymptotically nonexpansive mapping is a pointwise asymptotically nonexpansive mapping. Also, the class of Lipschitzian mappings properly includes the class of pointwise asymptotically nonexpansive mappings.
(b) It is obvious that every Lipschitzian mapping is a generalized Lipschitzian mapping. Furthermore, every mapping with a bounded range is a generalized Lipschitzian mapping. It is easy to see that the class of generalized (L, M)-Lipschitzian mappings is more general than the class of generalized Lipschitzian mappings.
(c) Clearly, the class of nearly uniformly L-Lipschitzian mappings properly includes the class of generalized (L, M)-Lipschitzian mappings and that of uniformly L-Lipschitzian mappings.
Note that every nearly asymptotically nonexpansive mapping is nearly uniformly L-Lipschitzian.

Some interesting examples to investigate relations between these mappings, introduced in Definitions 4.1 and 4.2, can be found in [3].

Let $S_1 : K_r \to K_r$ be a nearly uniformly L_1-Lipschitzian mapping with the sequence $\{a_n\}_{n=1}^{\infty}$, $S_2 : K_r \to K_r$ be a nearly uniformly L_2-Lipschitzian mapping with the sequence $\{b_n\}_{n=1}^{\infty}$ and $S_3 : K_r \to K_r$ be a nearly uniformly L_3-Lipschitzian mapping with the sequence $\{c_n\}_{n=1}^{\infty}$. We define the mapping $\mathcal{Q}$ from $K_r \times K_r \times K_r$ into itself as follows:

$$\mathcal{Q}(x, y, z) = (S_1x, S_2y, S_3z), \qquad \forall x, y, z \in K_r. \tag{4.2}$$

Then $\mathcal{Q} = (S_1, S_2, S_3) : K_r \times K_r \times K_r \to K_r \times K_r \times K_r$ is a nearly uniformly $\max\{L_1, L_2, L_3\}$-Lipschitzian mapping with the sequence $\{a_n + b_n + c_n\}_{n=1}^{\infty}$ with respect to norm $\|.\|_*$ in $\mathcal{H}\times\mathcal{H}\times\mathcal{H}$. To see this fact, let $(x, y, z), (x', y', z') \in K_r \times K_r \times K_r$ be arbitrary. Then for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|\mathcal{Q}^n(x, y, z) - \mathcal{Q}^n(x', y', z')\|_* \\
&= \|(S_1^n x, S_2^n y, S_3^n z) - (S_1^n x', S_2^n y', S_3^n z')\|_* \\
&= \|(S_1^n x - S_1^n x', S_2^n y - S_2^n y', S_3^n z - S_3^n z')\|_*
\end{aligned}$$

$$
\begin{aligned}
&= \|S_1^n x - S_1^n x'\| + \|S_2^n y - S_2^n y'\| + \|S_3^n z - S_3^n z'\| \\
&\leq L_1(\|x - x'\| + a_n) + L_2(\|y - y'\| + b_n) + L_3(\|z - z'\| + c_n) \\
&\leq \max\{L_1, L_2, L_3\}(\|x - x'\| + \|y - y'\| + \|z - z'\| + a_n + b_n + c_n) \\
&= \max\{L_1, L_2, L_3\}(\|(x, y, z) - (x', y', z')\|_* + a_n + b_n + c_n).
\end{aligned}
$$

We denote the sets of all the fixed points of S_i $(i = 1, 2, 3)$ and $\mathcal{Q}$ by $\mathrm{Fix}(S_i)$ and $\mathrm{Fix}(\mathcal{Q})$, respectively, and the set of all the solutions of the system (3.1) by $\mathrm{SGNRNVI}(K_r, T_i, i = 1, 2, 3)$. It is clear that for any $(x, y, z) \in K_r \times K_r \times K_r$, $(x, y, z) \in \mathrm{Fix}(\mathcal{Q})$ if and only if $x \in \mathrm{Fix}(S_1)$, $y \in \mathrm{Fix}(S_2)$ and $z \in \mathrm{Fix}(S_3)$, that is, $\mathrm{Fix}(\mathcal{Q}) = \mathrm{Fix}(S_1, S_2, S_3) = \mathrm{Fix}(S_1) \times \mathrm{Fix}(S_2) \times \mathrm{Fix}(S_3)$. We now characterize the problem. Let the operators T_i $(i = 1, 2, 3)$ and constants ρ, η and γ be the same as in SGNRNVI (3.1). If $(x^*, y^*, z^*) \in \mathrm{Fix}(\mathcal{Q}) \cap \mathrm{SGNRNVI}(K_r, T_i, i = 1, 2, 3)$, $\rho < \frac{r'}{1+\|T_1(y^*, z^*, x^*)\|}$, $\eta < \frac{r'}{1+\|T_2(z^*, x^*, y^*)\|}$ and $\gamma < \frac{r'}{1+\|T_3(x^*, y^*, z^*)\|}$, for some $r' \in (0, r)$, then from Lemma 3.2 it follows that for each $n \in \mathbb{N}$,

$$
\begin{cases}
x^* = S_1^n x^* = P_{K_r}(y^* - \rho T_1(y^*, z^*, x^*)) = S_1^n P_{K_r}(y^* - \rho T_1(y^*, z^*, x^*)), \\
y^* = S_2^n y^* = P_{K_r}(z^* - \eta T_2(z^*, x^*, y^*)) = S_2^n P_{K_r}(z^* - \eta T_2(z^*, x^*, y^*)), \\
z^* = S_3^n z^* = P_{K_r}(x^* - \gamma T_3(x^*, y^*, z^*)) = S_3^n P_{K_r}(x^* - \gamma T_3(x^*, y^*, z^*)).
\end{cases}
$$

Remark 4.4. The above equalities can be written as follows:

$$
\begin{cases}
x^* = S_1^n P_{K_r}(u), \\
y^* = S_2^n P_{K_r}(v), \\
z^* = S_3^n P_{K_r}(w), \\
u = y^* - \rho T_1(y^*, z^*, x^*), \\
v = z^* - \eta T_2(z^*, x^*, y^*), \\
w = x^* - \gamma T_3(x^*, y^*, z^*).
\end{cases}
\tag{4.3}
$$

The fixed point formulation (4.3) enables us to suggest the following iterative algorithms.

Algorithm 4.5. *Let T_i $(i = 1, 2, 3)$, ρ, η and γ be the same as in the system* (3.1) *and suppose further that constants ρ, η and γ satisfy the condition* (3.4) *and $U(r')$ is a convex subset of $\mathcal{H}$. For an arbitrary chosen initial point $(u_1, v_1, w_1) \in U(r') \times U(r') \times U(r')$, compute the iterative sequence $\{(x_n, y_n, z_n)\}_{n=1}^{\infty}$ in $K_r \times K_r \times K_r$ in the following way:*

$$
\begin{cases}
x_n = S_1^n P_{K_r}(u_n), \; y_n = S_2^n P_{K_r}(v_n), \; z_n = S_3^n P_{K_r}(w_n), \\
u_{n+1} = (1 - \alpha_n) u_n + \alpha_n (y_n - \rho T_1(y_n, z_n, x_n)), \\
v_{n+1} = (1 - \alpha_n) v_n + \alpha_n (z_n - \eta T_2(z_n, x_n, y_n)), \\
w_{n+1} = (1 - \alpha_n) w_n + \alpha_n (x_n - \gamma T_3(x_n, y_n, z_n)),
\end{cases}
\tag{4.4}
$$

where $S_i : K_r \to K_r$ $(i = 1,2,3)$ *are three nearly uniformly Lipschitzian mappings and* $\{\alpha_n\}_{n=1}^{\infty}$ *is a sequence in* $[0,1]$ *satisfying* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If $S_i \equiv I$ $(i = 1,2,3)$, the identity operator, then Algorithm 4.5 collapses to the following algorithm:

Algorithm 4.6. *Suppose that* T_i $(i = 1,2,3)$, ρ, η *and* γ *are the same as in the system* (3.1) *and let constants* ρ, η *and* γ *satisfy the condition* (3.4) *and* $U(r')$ *be a convex subset of* $\mathcal{H}$. *For an arbitrary chosen initial point* $(u_1, v_1, w_1) \in U(r') \times U(r') \times U(r')$, *compute the iterative sequence* $\{(x_n, y_n, z_n)\}_{n=1}^{\infty}$ *in* $K_r \times K_r \times K_r$ *by the iterative processes*

$$\begin{cases} x_n = P_{K_r}(u_n), \ y_n = P_{K_r}(v_n), \ z_n = P_{K_r}(w_n), \\ u_{n+1} = (1-\alpha_n)u_n + \alpha_n(y_n - \rho T_1(y_n, z_n, x_n)), \\ v_{n+1} = (1-\alpha_n)v_n + \alpha_n(z_n - \eta T_2(z_n, x_n, y_n)), \\ w_{n+1} = (1-\alpha_n)w_n + \alpha_n(x_n - \gamma T_3(x_n, y_n, z_n)), \end{cases}$$

where the sequence $\{\alpha_n\}_{n=1}^{\infty}$ *is the same as in Algorithm* 4.5.

5. Convergence analysis

In the present section, we establish the strong convergence of the sequences generated by three-step projection iterative algorithms under some suitable conditions. For this end, we need the following lemma.

Lemma 5.1. ([34]) *Let* $\{a_n\}$ *be a nonnegative real sequence and* $\{b_n\}$ *be a real sequence in* $[0,1]$ *such that* $\sum_{n=0}^{\infty} b_n = \infty$. *If there exists a positive integer* n_0 *such that*

$$a_{n+1} \leq (1-b_n)a_n + b_n c_n, \quad \forall n \geq n_0,$$

where $c_n \geq 0$ *for all* $n \geq 0$ *and* $\lim_{n\to\infty} c_n = 0$, *then* $\lim_{n\to 0} a_n = 0$.

Theorem 5.2. *Let* T_i $(i = 1,2,3)$, ρ, η *and* γ *be the same as in Theorem* 3.4 *and let all the conditions of Theorem* 3.4 *hold. Suppose that* $S_1 : K_r \to K_r$ *is a nearly uniformly* L_1*-Lipschitzian mapping with the sequence* $\{b_n\}_{n=1}^{\infty}$, $S_2 : K_r \to K_r$ *is a nearly uniformly* L_2*-Lipschitzian mapping with the sequence* $\{c_n\}_{n=1}^{\infty}$, $S_3 : K_r \to K_r$ *is a nearly uniformly* L_3*-Lipschitzian mapping with the sequence* $\{d_n\}_{n=1}^{\infty}$, *and* $\mathcal{Q}$ *is a self-mapping of* $K_r \times K_r \times K_r$ *defined by* (4.2) *such that* $\mathrm{Fix}(\mathcal{Q}) \cap \mathrm{SGNRNVI}(K_r, T_i, i = 1,2,3) \neq \emptyset$. *Further, let for each* $i = 1,2,3$, $L_i\theta < 1$, *where* θ *is the same as in* (3.14). *Then the iterative* *sequence* $\{(x_n, y_n, z_n)\}_{n=1}^{\infty}$ *generated by Algorithm* 4.5*, converges strongly to the only element of* $\mathrm{Fix}(\mathcal{Q}) \cap \mathrm{SGNRNVI}(K_r, T_i, i = 1, 2, 3)$.

Proof. Theorem 3.4 guarantees that the system (3.1) has a unique solution $(x^*, y^*, z^*) \in K_r \times K_r \times K_r$. Since $\rho < \frac{r'}{1+\|T_1(y^*,z^*,x^*)\|}$, $\eta < \frac{r'}{1+\|T_2(z^*,x^*,y^*)\|}$ and $\gamma < \frac{r'}{1+\|T_3(x^*,y^*,z^*)\|}$, for some $r' \in (0, r)$, by using Lemma 3.2 conclude that (x^*, y^*, z^*) satisfies equations (3.3). Since $\mathrm{SGNRNVI}(K_r, T_i, i = 1, 2, 3)$ is a singleton set, it follows from $\mathrm{Fix}(\mathcal{Q}) \cap \mathrm{SGNRNVI}(K_r, T_i, i = 1, 2, 3) \neq \emptyset$ that $(x^*, y^*, z^*) \in \mathrm{Fix}(\mathcal{Q})$ and so $x^* \in \mathrm{Fix}(S_1)$, $y^* \in \mathrm{Fix}(S_2)$ and $z^* \in \mathrm{Fix}(S_3)$. Hence, in view of Remark 4.4, for each $n \in \mathbb{N}$, we can write

$$\begin{cases} x^* = S_1^n P_{K_r}(u),\ y^* = S_2^n P_{K_r}(v),\ z^* = S_3^n P_{K_r}(w), \\ u = (1-\alpha_n)u + \alpha_n(y^* - \rho T_1(y^*, z^*, x^*)), \\ v = (1-\alpha_n)v + \alpha_n(z^* - \eta T_2(z^*, x^*, y^*)), \\ w = (1-\alpha_n)w + \alpha_n(x^* - \gamma T_3(x^*, y^*, z^*)), \end{cases} \tag{5.1}$$

where the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is the same as in Algorithm 4.5. By using (4.4), (5.1) and the assumptions, we have

$$\begin{aligned} \|u_{n+1} - u\| \le{}& (1-\alpha_n)\|u_n - u\| + \alpha_n\|y_n - y^* \\ &- \rho(T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*))\| \\ \le{}& (1-\alpha_n)\|u_n - u\| + \alpha_n\sqrt{1 - 2\rho\varrho_1 + \rho^2\sigma_1^2}\|y_n - y^*\|. \end{aligned} \tag{5.2}$$

It follows from (4.4), (5.1) and Proposition 2.10 that

$$\begin{aligned} \|y_n - y^*\| &= \|S_2^n P_{K_r}(v_n) - S_2^n P_{K_r}(v)\| \\ &\le L_2(\|P_{K_r}(v_n) - P_{K_r}(v)\| + c_n) \\ &\le L_2(\frac{r}{r-r'}\|v_n - v\| + c_n). \end{aligned} \tag{5.3}$$

Substituting (5.3) in (5.2), conclude that

$$\begin{aligned} \|u_{n+1} - u\| \le{}& (1-\alpha_n)\|u_n - u\| \\ &+ \alpha_n L_2\sqrt{1 - 2\rho\varrho_1 + \rho^2\sigma_1^2}\Big(\frac{r}{r-r'}\|v_n - v\| + c_n\Big) \\ ={}& (1-\alpha_n)\|u_n - u\| + \alpha_n L_2\vartheta\|v_n - v\| \\ &+ \alpha_n L_2\sqrt{1 - 2\rho\varrho_1 + \rho^2\sigma_1^2}c_n. \end{aligned} \tag{5.4}$$

Like in the proofs of (5.2)–(5.4), one can prove that

$$\begin{aligned} \|v_{n+1} - v\| \le{}& (1-\alpha_n)\|v_n - v\| + \alpha_n L_3\varpi\|w_n - w\| \\ &+ \alpha_n L_3\sqrt{1 - 2\eta\varrho_2 + \eta^2\sigma_2^2}d_n \end{aligned} \tag{5.5}$$

and

$$\|w_{n+1} - w\| \leq (1-\alpha_n)\|w_n - w\| + \alpha_n L_1 v \|u_n - u\| + \alpha_n L_1 \sqrt{1 - 2\gamma\varrho_3 + \gamma^2\sigma_3^2}\, b_n. \tag{5.6}$$

Let $L = \max\{L_i : i = 1, 2, 3\}$. Then from (5.4)–(5.6), it follows that

$$\begin{aligned}
&\|(u_{n+1}, v_{n+1}, w_{n+1}) - (u, v, w)\|_* \\
&\leq (1-\alpha_n)\|(u_n, v_n, w_n) - (u, v, w)\|_* \\
&\quad + \alpha_n L\theta \|(u_n, v_n, w_n) - (u, v, w)\|_* \\
&\quad + \alpha_n L \sqrt{1 - 2\rho\varrho_1 + \rho^2\sigma_1^2}\, c_n \\
&\quad + \alpha_n L \sqrt{1 - 2\eta\varrho_2 + \eta^2\sigma_2^2}\, d_n \\
&\quad + \alpha_n L \sqrt{1 - 2\gamma\varrho_3 + \gamma^2\sigma_3^2}\, b_n \\
&\leq (1 - \alpha_n(1 - L\theta))\|(u_n, v_n, w_n) - (u, v, w)\|_* \\
&\quad + \alpha_n L\psi(b_n + c_n + d_n) \\
&= (1 - \alpha_n(1 - L\theta))\|(u_n, v_n, w_n) - (u, v, w)\|_* \\
&\quad + \alpha_n(1 - L\theta)\frac{L\psi(b_n + c_n + d_n)}{1 - L\theta},
\end{aligned} \tag{5.7}$$

where

$$\psi = \max\Big\{\sqrt{1 - 2\rho\varrho_1 + \rho^2\sigma_1^2}, \sqrt{1 - 2\eta\varrho_2 + \eta^2\sigma_2^2}, \sqrt{1 - 2\gamma\varrho_3 + \gamma^2\sigma_3^2}\Big\}.$$

Since $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$, $L\theta < 1$ and $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} d_n = 0$, we note that all the conditions Lemma 5.1 are satisfied. Hence, Lemma 5.1 and (5.7) guarantee that $(u_n, v_n, w_n) \to (u, v, w)$, as $n \to \infty$. It follows from (4.4) and (5.1) that

$$\begin{aligned}
\|x_n - x^*\| &= \|S_1^n P_{K_r}(u_n) - S_1^n P_{K_r}(u)\| \\
&\leq L_1\Big(\|P_{K_r}(u_n) - P_{K_r}(u)\| + b_n\Big) \\
&\leq L_1\Big(\frac{r}{r - r'}\|u_n - u\| + b_n\Big)
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
\|z_n - z^*\| &= \|S_3^n P_{K_r}(w_n) - S_3^n P_{K_r}(w)\| \\
&\leq L_3\Big(\|P_{K_r}(w_n) - P_{K_r}(w)\| + d_n\Big) \\
&\leq L_3\Big(\frac{r}{r - r'}\|w_n - w\| + d_n\Big).
\end{aligned} \tag{5.9}$$

Since $\lim_{n\to\infty} u_n = u$, $\lim_{n\to\infty} v_n = v$, $\lim_{n\to\infty} w_n = w$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 0$, inequalities (5.3), (5.8) and (5.9) imply that $y_n \to y^*$, $x_n \to x^*$ and $z_n \to z^*$, as $n \to \infty$. Thus the sequence $\{(x_n, y_n, z_n)\}_{n=1}^{\infty}$ generated by Algorithm 4.5, converges strongly to the unique solution (x^*, y^*, z^*) of the system (3.1), that is, the only element of $\mathrm{Fix}(\mathcal{Q}) \cap \mathrm{SGNRNVI}(K_r, T_i, i = 1, 2, 3)$. This completes the proof. $\square$

Corollary 5.3. *Let T_i $(i = 1, 2, 3)$, ρ, η and γ be the same as in Theorem* 3.4 *and let all the conditions of Theorem* 3.4 *hold. Then the iterative sequence $\{(x_n, y_n, z_n)\}_{n=1}^{\infty}$ generated by Algorithm* 4.6*, converges strongly to the unique solution (x^*, y^*, z^*) of* SGNRNVI (3.1).

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