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CONVERGENCE OF COMPOSITE IMPLICIT ITERATIVE PROCESS WITH ERRORS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. The aim of this article is to study the weak and strong convergence of implicit iteration process with errors to a common fixed point for a finite family of asymptotically nonexpansive mappings in Banach spaces. The results presented in this paper extend and improve the corresponding results of [1]-[2], [4]-[11], [13]-[18].

1. Introduction

Let C be a nonempty subset of a real Banach space E. Let $T\colon C \to C$ be a mapping. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. Recall that a mapping $T\colon C \to C$ is said to be:

(1) asymptotically nonexpansive if there exists a sequence $\{a_n\}$ in $[1, \infty)$ with $a_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \quad \leq \quad a_n \|x - y\|, \tag{1.1}$$

for all $x, y \in C$ and $n \geq 1$.

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(2) A mapping T is said to be semi-compact if any bounded sequence $\{x_n\}$ in C with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some x^* in C.

(3) T is said to be demiclosed at the origin, if for each sequence $\{x_n\}$ in C, the condition $x_n \to x_0$ weakly and $Tx_n \to 0$ strongly imply $Tx_0 = 0$.

(4) Let $\{T_1, T_2, \dots, T_N\}\colon C \to C$ be N mappings. $\{T_1, T_2, \dots, T_N\}$ are said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T_i^n x - T_i^n y\| \quad \leq \quad L\,\|x - y\|\,, \tag{1.2}$$

for all $x, y \in C$, $i = 1, 2, \dots, N$ and $n \geq 1$.

Recall that E is said to satisfy the *Opial's condition* [9] if for each sequence $\{x_n\}$ in E weakly convergent to a point x and for all $y \neq x$

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|\,.$$

The examples of Banach spaces which satisfy the Opial's condition are Hilbert spaces and all $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition [9].

Proposition 1.1. *Let C be a nonempty subset of a real Banach space E, $\{T_i\}_{i=1}^N\colon C \to C$ be N asymptotically nonexpansive mappings. Then*

(i) *there exists a sequence $\{a_n\} \subset [1, \infty)$ with $a_n \to 1$ as $n \to \infty$ such that*

$$\|T_i^n x - T_i^n y\| \quad \leq \quad a_n\,\|x - y\|\,, \tag{1.3}$$

for all $x, y \in C$, $i = 1, 2, \dots, N$ and $n \geq 1$.

(ii) *$\{T_1, T_2, \dots, T_N\}$ is uniformly L-Lipschitzian, that is, if there exists a constant L such that*

$$\|T_i^n x - T_i^n y\| \quad \leq \quad L\,\|x - y\|\,, \tag{1.4}$$

where $L = \sup_{n\geq 1} a_n \geq 1$ and for all $x, y \in C$, $i = 1, 2, \dots, N$ and $n \geq 1$.

Proof. (i) Since for each $i = 1, 2, \dots, N$, $T_i\colon C \to C$ is an asymptotically nonexpansive mapping, there exists a sequence $\{a_n^{(i)}\} \subset [1, \infty)$ with $a_n^{(i)} \to 1$ as $n \to \infty$ such that

$$\|T_i^n x - T_i^n y\| \quad \leq \quad a_n^{(i)}\,\|x - y\|\,,$$

for all $x, y \in C$, $i = 1, 2, \dots, N$ and $n \geq 1$.

Letting

$$a_n = \max\{a_n^{(1)}, a_n^{(2)}, \dots, a_n^{(N)}\},$$

then we have that $\{a_n\} \subset [1, \infty)$ with $a_n \to 1$ as $n \to \infty$ and

$$\|T_i^n x - T_i^n y\| \leq a_n^{(i)} \|x - y\| \leq a_n \|x - y\|,$$

for all $x, y \in C$, $i = 1, 2, \ldots, N$ and $n \geq 1$.

(ii) Taking $L = \sup_{n\geq 1} a_n \geq 1$, then the conclusion of (ii) can be obtained from the conclusion of (i) immediately. □

Let E be a Hilbert space, let K be a nonempty closed convex subset of E, and let $\{T_1, T_2, \ldots, T_N\}\colon K \to K$ be N nonexpansive mappings. In 2001, Xu and Ori [17] introduced the following implicit iteration process $\{x_n\}$ defined by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{n(mod\ N)} x_n, \quad n \geq 1, \tag{1.5}$$

where $x_0 \in K$ is an initial point, $\{\alpha_n\}_{n\geq 1}$ is a real sequence in $(0, 1)$ and proved the weakly convergence of the sequence $\{x_n\}$ defined by (1.5) to a common fixed point $p \in F = \bigcap_{i=1}^N F(T_i)$.

In 2003, Sun [13] introduced the following implicit iterative sequence $\{x_n\}$

$$x_n = (1 - \alpha_n) x_{n-1} + \alpha_n T_{i(n)}^{k(n)} x_n, \quad n \geq 1, \tag{1.6}$$

for a finite family of asymptotically quasi-nonexpansive self-mappings on a bounded closed convex subset K of a Hilbert space E with $\{\alpha_n\}$ a sequence in $(0, 1)$ and an initial point $x_0 \in K$, where $n = (k(n) - 1)N + i(n)$, $i(n) \in \{1, 2, \ldots, N\}$, and proved the strong convergence of the sequence $\{x_n\}$ defined by (1.6) to a common fixed point $p \in F = \bigcap_{i=1}^N F(T_i)$.

Recently, in 2006, Gu [6] introduced the following implicit iterative sequence $\{x_n\}$ with errors

$$\begin{aligned} x_n &= (1 - \alpha_n) x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n + u_n, \\ y_n &= (1 - \beta_n) x_n + \beta_n T_{i(n)}^{k(n)} x_n + v_n, \quad n \geq 1, \end{aligned} \tag{1.7}$$

for a finite family of asymptotically nonexpansive mappings on a closed convex subset K of a Banach space X with $K + K \subset K$, $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in [0,1], $\{u_n\}$ and $\{v_n\}$ be two sequences in K, and an initial point $x_0 \in K$, where $n = (k(n) - 1)N + i(n)$, $i(n) \in \{1, 2, \ldots, N\}$, and proved the weak and strong convergence of the sequence $\{x_n\}$ defined by (1.7) to a common fixed point $p \in F = \bigcap_{i=1}^N F(T_i)$.

It should be pointed out that the sequence defined by (1.6) is a special case of the sequence defined by (1.7) with $u_n = v_n = 0$, $\beta_n = 0$, for all $n \geq 1$.

Recently concerning the convergence problems of an implicit (or nonimplicit) iterative process to a common fixed point for a finite family of asymptotically nonexpansive mappings (or nonexpansive mappings) in Hilbert spaces or uniformly convex Banach spaces have been considered by several authors (see, e.g., Bauschke [1], Chang and Cho [2], Goebel and Kirk [4], Gornicki [5], Gu [6], Halpern [7], Lions [8], Osilike [10], Reich [11], Schu [12], Sun [13], Tan and Xu [14], Wittmann [16], Xu and Ori [17] and Zhou and Chang [18]).

Inspired and motivated by [6, 13, 17] and many others, we introduce the following implicit iterative sequence $\{x_n\}$ with errors defined by

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + \beta_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n, \\ y_n &= \hat{\alpha_n} x_{n-1} + \hat{\beta_n} T_{i(n)}^{k(n)} x_n + \hat{\gamma_n} v_n, \quad n \geq 1, \end{aligned} \tag{1.8}$$

where $n = (k(n)-1)N + i(n)$, $i(n) \in \{1, 2, \ldots, N\}$, is called the implicit iterative sequence for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^N$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha_n}\}$, $\{\hat{\beta_n}\}$ and $\{\hat{\gamma_n}\}$ are six sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = \hat{\alpha_n} + \hat{\beta_n} + \hat{\gamma_n} = 1$ for all $n \geq 1$, x_0 is a given point in K, as well as $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K.

Especially, if $\{T_i\}_{i=1}^N \colon K \to K$ are N asymptotically nonexpansive mappings, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in [0,1], and x_0 is a given point in K, then the sequence $\{x_n\}$ defined by

$$x_n = \alpha_n x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_{n-1} + \gamma_n u_n, \quad n \geq 1, \tag{1.9}$$

is called the explicit iterative sequence for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^N$.

The purpose of this article is to study an iterative sequences defined by (1.8) and (1.9) for a finite family of asymptotically nonexpansive mappings in Banach spaces and also establish the weak and strong convergence theorems for said iteration schemes and mappings.

In the sequel we need the following lemmas to prove our main results:

Lemma 1.1. ([15]) *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2. ([12]) *Let E be a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \qquad \limsup_{n\to\infty} \|y_n\| \leq r,$$

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r,$$

for some $r \geq 0$. Then

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 1.3. ([3, 5, 14]) *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and $T\colon C \to C$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Then $I - T$ is semi-closed at zero, i.e., for each sequence $\{x_n\}$ in C, if $\{x_n\}$ converges weakly to $q \in C$ and $\{(I-T)x_n\}$ converges strongly to 0, then $(I-T)q = 0$.*

Lemma 1.4. *Let E be a real Banach space, C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N\colon C \to C$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in C, and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha_n}\}$, $\{\hat{\beta_n}\}$ and $\{\hat{\gamma_n}\}$ be six sequences in $[0,1]$ and $\{a_n\}$ be the sequence defined by (1.3) and $L = \sup_{n\geq 1} a_n \geq 1$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = \hat{\alpha_n} + \hat{\beta_n} + \hat{\gamma_n} = 1$;
(ii) $\sum_{n=1}^\infty (a_n - 1)\beta_n < \infty$;
(iii) $\tau = \sup\{\beta_n : n \geq 1\} < \frac{1}{L^2}$;
(iv) $\sum_{n=1}^\infty \gamma_n < \infty$, $\sum_{n=1}^\infty \hat{\gamma_n} < \infty$.

If $\{x_n\}$ is the implicit iterative sequence defined by (1.8), then for each $p \in F = \bigcap_{i=1}^N F(T_i)$ the limit $\lim_{n\to\infty} \|x_n - p\|$ exists.

Proof. Since $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, for any given $p \in F$, it follows from (1.8) and Proposition 1.1 that

$$\begin{aligned}
\|x_n - p\| &\leq (1-\beta_n-\gamma_n)\|x_{n-1}-p\| + \beta_n \left\|T_{i(n)}^{k(n)} y_n - p\right\| \\
&\quad + \gamma_n \|u_n - p\| \\
&= (1-\beta_n-\gamma_n)\|x_{n-1}-p\| + \beta_n \left\|T_{i(n)}^{k(n)} y_n - T_{i(n)}^{k(n)} p\right\| \\
&\quad + \gamma_n \|u_n - p\| \\
&\leq (1-\beta_n)\|x_{n-1}-p\| + \beta_n a_{k(n)} \|y_n - p\| \\
&\quad + \gamma_n \|u_n - p\|, \qquad (1.10)
\end{aligned}$$

Again it follows from (1.8) and Proposition 1.1 that

$$\begin{aligned} \|y_n - p\| &\leq (1-\hat{\beta_n}-\hat{\gamma_n})\,\|x_{n-1}-p\| + \hat{\beta_n}\left\|T_{i(n)}^{k(n)}x_n - p\right\| \\ &\quad +\hat{\gamma_n}\,\|v_n - p\| \\ &= (1-\hat{\beta_n}-\hat{\gamma_n})\,\|x_{n-1}-p\| + \hat{\beta_n}\left\|T_{i(n)}^{k(n)}x_n - T_{i(n)}^{k(n)}p\right\| \\ &\quad +\hat{\gamma_n}\,\|v_n - p\| \\ &\leq (1-\hat{\beta_n})\,\|x_{n-1}-p\| + \hat{\beta_n}a_{k(n)}\,\|x_n - p\| \\ &\quad +\hat{\gamma_n}\,\|v_n - p\|\,. \end{aligned} \tag{1.11}$$

Substituting (1.11) into (1.10), we obtain that

$$\begin{aligned} \|x_n - p\| &\leq (1-\beta_n\hat{\beta_n}a_{k(n)})\,\|x_{n-1}-p\| + \beta_n\hat{\beta_n}a_{k(n)}^2\,\|x_n - p\| \\ &\quad +\beta_n\hat{\gamma_n}a_{k(n)}\,\|v_n - p\| + \gamma_n\,\|u_n - p\|\,, \end{aligned}$$

which implies that

$$(1-\beta_n\hat{\beta_n}a_{k(n)}^2)\,\|x_n - p\| \leq (1-\beta_n\hat{\beta_n}a_{k(n)})\,\|x_{n-1}-p\| + \mu_n, \tag{1.12}$$

where $\mu_n = \beta_n\hat{\gamma_n}a_{k(n)}\,\|v_n - p\| + \gamma_n\,\|u_n - p\|$. By condition (iv) and boundedness of the sequences $\{\beta_n\}$, $\{a_{k(n)}\}$, $\{\|u_n - p\|\}$, and $\{\|v_n - p\|\}$, we have $\sum_{n=1}^{\infty}\mu_n < \infty$. From condition (iii) we know that

$$\beta_n\hat{\beta_n}a_{k(n)}^2 \leq \beta_n a_{k(n)}^2 \leq \tau < 1,$$

and so

$$1-\beta_n\hat{\beta_n}a_{k(n)}^2 \geq 1-\tau L^2 > 0, \tag{1.13}$$

hence from (1.12) we have

$$\begin{aligned} \|x_n - p\| &\leq \Big(\frac{1-\beta_n\hat{\beta_n}a_{k(n)}}{1-\beta_n\hat{\beta_n}a_{k(n)}^2}\Big)\,\|x_{n-1}-p\| + \frac{\mu_n}{1-\tau L^2} \\ &= \Big(1+\frac{(a_{k(n)}-1)\beta_n\hat{\beta_n}a_{k(n)}}{1-\beta_n\hat{\beta_n}a_{k(n)}^2}\Big)\,\|x_{n-1}-p\| + \frac{\mu_n}{1-\tau L^2} \\ &\leq \Big(1+\frac{(a_{k(n)}-1)\beta_n\hat{\beta_n}a_{k(n)}}{1-\tau L^2}\Big)\,\|x_{n-1}-p\| + \frac{\mu_n}{1-\tau L^2} \\ &= (1+A_n)\,\|x_{n-1}-p\| + B_n, \end{aligned} \tag{1.14}$$

where

$$A_n = \frac{(a_{k(n)}-1)\beta_n\hat{\beta_n}a_{k(n)}}{1-\tau L^2} \quad \text{and} \quad B_n = \frac{\mu_n}{1-\tau L^2}.$$

By conditions (ii) and (iii) we have that

$$\begin{aligned}\sum_{n=1}^{\infty} A_n &= \frac{1}{1-\tau L^2}\sum_{n=1}^{\infty}(a_{k(n)}-1)\beta_n\hat{\beta}_n a_{k(n)}\\ &\leq \frac{1}{1-\tau L^2}\sum_{n=1}^{\infty}(a_{k(n)}-1)\beta_n a_{k(n)}\\ &\leq \frac{L}{1-\tau L^2}\sum_{n=1}^{\infty}(a_{k(n)}-1)\beta_n < \infty\end{aligned}$$

and

$$B_n = \sum_{n=1}^{\infty}\frac{\mu_n}{1-\tau L^2} < \infty.$$

Taking $a_n = \|x_{n-1}-p\|$ in inequality (1.14), we have

$$a_{n+1} \leq (1+A_n)a_n + B_n, \quad \forall n \geq 1,$$

and satisfied all conditions in Lemma 1.1. Therefore the limit $\lim_{n\to\infty}\|x_n - p\|$ exists. Without loss of generality we may assume that

$$\lim_{n\to\infty}\|x_n - p\| = d, \quad p \in F.$$

This completes the proof of Lemma 1.4. □

2. Main Results

We are now in a position to prove our main results in this paper.

Theorem 2.1. *Let E be a real Banach space, C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N : C \to C$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in C, and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha_n}\}$, $\{\hat{\beta_n}\}$ and $\{\hat{\gamma_n}\}$ be six sequences in $[0,1]$ and $\{a_n\}$ be the sequence defined by (1.3) and $L = \sup_{n\geq 1} a_n \geq 1$ satisfying the following conditions:*

- (i) $\alpha_n + \beta_n + \gamma_n = \hat{\alpha_n} + \hat{\beta_n} + \hat{\gamma_n} = 1$;
- (ii) $\sum_{n=1}^{\infty}(a_n - 1)\beta_n < \infty$;
- (iii) $\tau = \sup\{\beta_n : n \geq 1\} < \frac{1}{L^2}$;
- (iv) $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\hat{\gamma_n} < \infty$.

Then the implicit iterative sequence $\{x_n\}$ *defined by* (1.8) *converges strongly to a common fixed point* $p \in F = \bigcap_{i=1}^{N} F(T_i)$ *if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0. \tag{2.1}$$

Proof. The necessity of condition (2.1) is obvious.

Next we prove the sufficiency of Theorem 2.1. For any given $p \in F$, it follows from (1.14) in Lemma 1.4 that

$$\|x_n - p\| \leq (1 + A_n)\|x_{n-1} - p\| + B_n \quad \forall n \geq 1, \tag{2.2}$$

where

$$A_n = \frac{(a_n - 1)\beta_n \hat{\beta}_n a_n}{1 - \tau L^2} \quad \text{and} \quad B_n = \frac{\mu_n}{1 - \tau L^2}$$

with $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$. Hence, we have

$$d(x_n, F) \leq (1 + A_n) d(x_{n-1}, p) + B_n \quad \forall n \geq 1. \tag{2.3}$$

It follows from (2.3) and Lemma 1.1 that the limit $\lim_{n\to\infty} d(x_n, F)$ exists. By the condition (2.1), we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Next, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in C. In fact, since $\sum_{n=1}^{\infty} A_n < \infty$, $1 + x \leq e^x$ for all $x > 0$, and (2.2), therefore we have

$$\|x_n - p\| \leq e^{A_n}\|x_{n-1} - p\| + B_n \quad \forall n \geq 1. \tag{2.4}$$

Hence, for any positive integers n, m, from (2.4) it follows that

$$\begin{aligned}
\|x_{n+m} - p\| &\leq e^{A_{n+m}}\|x_{n+m-1} - p\| + B_{n+m} \\
&\leq e^{A_{n+m}}\left[e^{A_{n+m-1}}\|x_{n+m-2} - p\| + B_{n+m-1}\right] + B_{n+m} \\
&\leq \cdots \\
&\leq e^{\left\{\sum_{i=n+1}^{n+m} A_i\right\}}\|x_n - p\| + e^{\left\{\sum_{i=n+2}^{n+m} A_i\right\}}\sum_{i=n+1}^{n+m} B_i \\
&\leq W\|x_n - p\| + W\sum_{i=n+1}^{n+m} B_i, \qquad (2.5)
\end{aligned}$$

where $W = e^{\left\{\sum_{n=1}^{\infty} A_n\right\}} < \infty$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} B_n < \infty$, for any given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$d(x_n, F) < \frac{\varepsilon}{4(W+1)}, \quad \sum_{i=n+1}^{\infty} B_i < \frac{\varepsilon}{2W}, \quad \forall n \geq n_0.$$

Therefore there exists $p_1 \in F$ such that

$$d(x_n, p_1) < \frac{\varepsilon}{2(W+1)}, \quad \forall n \geq n_0.$$

Consequently, for any $n \geq n_0$ and for all $m \geq 1$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\
&\leq W\|x_n - p_1\| + W\sum_{i=n+1}^{n+m} B_i + \|x_n - p_1\| \\
&= (W+1)\|x_n - p_1\| + W\sum_{i=n+1}^{n+m} B_i \\
&< (W+1).\frac{\varepsilon}{2(W+1)} + W.\frac{\varepsilon}{2W} \\
&= \varepsilon.
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in C. By the completeness of C, we can assume that $\lim_{n\to\infty} x_n = p^*$. Since the set of fixed points of an asymptotically nonexpansive mapping is closed, hence F is closed. This implies that $p^* \in F$, and so p^* is a common fixed point of $T_1, T_2, \ldots, T_N$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let E be a real Banach space, C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N : C \to C$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in C, and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three sequences in $[0,1]$ and $\{a_n\}$ be the sequence defined by* (1.3) *and $L = \sup_{n\geq 1} a_n \geq 1$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $\sum_{n=1}^{\infty}(a_n - 1)\beta_n < \infty$;
(iii) $0 < \tau = \sup\{\beta_n : n \geq 1\} < \frac{1}{L^2}$;
(iv) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then the implicit iterative sequence $\{x_n\}$ defined by (1.8) *converges strongly to a common fixed point $p \in F = \bigcap_{i=1}^N F(T_i)$ if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. Taking $\hat{\beta_n} = \hat{\gamma_n} = 0$, for all $n \geq 1$ in Theorem 2.1, then the conclusion of Theorem 2.2 can be obtained from Theorem 2.1 immediately. This completes the proof of Theorem 2.2. □

Theorem 2.3. *Let E be a real uniformly convex Banach space satisfying Opial condition, C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N : C \to C$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$*

and $\{v_n\}$ *be two bounded sequences in* C*, and let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha_n}\}$, $\{\hat{\beta_n}\}$ *and* $\{\hat{\gamma_n}\}$ *be six sequences in* $[0,1]$ *and* $\{a_n\}$ *be the sequence defined by* (1.3) *and* $L = \sup_{n\geq 1} a_n \geq 1$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = \hat{\alpha_n} + \hat{\beta_n} + \hat{\gamma_n} = 1$;
(ii) $\sum_{n=1}^{\infty}(a_n - 1)\beta_n < \infty$;
(iii) $0 < \tau_1 = \inf\{\beta_n : n \geq 1\} \leq \sup\{\beta_n : n \geq 1\} = \tau_2 < \frac{1}{L^2}$;
(iv) $\hat{\beta_n} \to 0$, $(n \to \infty)$;
(v) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \hat{\gamma_n} < \infty$;
(vi) $0 \leq \delta = \sup\{\hat{\beta_n} : n \geq 1\} < \frac{1}{L}$.

Then the implicit iterative sequence $\{x_n\}$ *defined by* (1.7) *converges weakly to a common fixed point of the mappings* $\{T_1, T_2, \ldots, T_N\}$.

Proof. First, we prove that

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l = 1, 2, \ldots, N. \tag{2.6}$$

Let $p \in F$. Put $d = \|x_n - p\|$. It follows from (1.7) that

$$\begin{aligned} \|x_n - p\| &= \|(1-\beta_n)[x_{n-1} - p + \gamma_n(u_n - x_{n-1})] \\ &\quad + \beta_n[T_{i(n)}^{k(n)} y_n - p + \gamma_n(u_n - x_{n-1})]\| \\ &\to d, \quad n \to \infty. \end{aligned} \tag{2.7}$$

Again since $\lim_{n\to\infty} \|x_n - p\|$ exists, so $\{x_n\}$ is a bounded sequence in C. By virtue of condition (v) and the boundedness of sequences $\{x_n\}$ and $\{u_n\}$ we have

$$\begin{aligned} \limsup_{n\to\infty} \|x_{n-1} - p + \gamma_n(u_n - x_{n-1})\| &\leq \limsup_{n\to\infty} \|x_{n-1} - p\| \\ &\quad + \limsup_{n\to\infty} \gamma_n \|u_n - x_{n-1}\| \\ &= d, \quad p \in F. \end{aligned} \tag{2.8}$$

It follows from (1.11) and condition (iv) that

$$\begin{aligned} \limsup_{n\to\infty} \left\|T_{i(n)}^{k(n)} y_n - p + \gamma_n(u_n - x_{n-1})\right\| &\leq \limsup_{n\to\infty} \left\|T_{i(n)}^{k(n)} y_n - p\right\| \\ &\quad + \limsup_{n\to\infty} \gamma_n \|u_n - x_{n-1}\| \\ &= \limsup_{n\to\infty} \|y_n - p\| \\ &\leq \limsup_{n\to\infty} [(1-\hat{\beta_n}) \|x_{n-1} - p\| \\ &\quad + \hat{\beta_n} a_{k(n)} \|x_n - p\| \\ &\quad + \hat{\gamma_n} \|v_n - p\|] \\ &\leq d, \quad p \in F. \end{aligned} \tag{2.9}$$

Therefore from condition (iii), (2.7) - (2.9), and Lemma 1.2 we know that

$$\lim_{n\to\infty}\left\|T_{i(n)}^{k(n)}y_n - x_{n-1}\right\| = 0. \tag{2.10}$$

From (1.7), (2.10) and condition (v), we have

$$\begin{aligned}\|x_n - x_{n-1}\| &= \left\|\beta_n[T_{i(n)}^{k(n)}y_n - x_{n-1}] + \gamma_n(u_n - x_{n-1})\right\| \\ &\le \beta_n\left\|T_{i(n)}^{k(n)}y_n - x_{n-1}\right\| + \gamma_n\|u_n - x_{n-1}\| \\ &\to 0, \text{ as } n\to\infty, \end{aligned}\tag{2.11}$$

which implies that

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0, \tag{2.12}$$

and so

$$\lim_{n\to\infty}\|x_n - x_{n+j}\| = 0 \ \ \forall j = 1,2,\ldots,N. \tag{2.13}$$

On the other hand, we have

$$\begin{aligned}\left\|x_n - T_{i(n)}^{k(n)}x_n\right\| &\le \|x_n - x_{n-1}\| + \left\|x_{n-1} - T_{i(n)}^{k(n)}y_n\right\| \\ &\quad + \left\|T_{i(n)}^{k(n)}y_n - T_{i(n)}^{k(n)}x_n\right\|. \end{aligned}\tag{2.14}$$

Now, we consider the third term of the right hand side of (2.14). From the Proposition 1.1, (1.7) and the condition (vi) we have

$$\begin{aligned}\left\|T_{i(n)}^{k(n)}y_n - T_{i(n)}^{k(n)}x_n\right\| &\le L\|y_n - x_n\| \\ &\le L\left\|\hat{\alpha_n}x_{n-1} + \hat{\beta_n}T_{i(n)}^{k(n)}x_n + \hat{\gamma_n}v_n - x_n\right\| \\ &\le L\Big[\hat{\alpha_n}\|x_{n-1} - x_n\| + \hat{\beta_n}\left\|T_{i(n)}^{k(n)}x_n - x_n\right\| \\ &\quad + \hat{\gamma_n}\|v_n - x_n\|\Big] \\ &\le L\hat{\alpha_n}\|x_{n-1} - x_n\| + L\delta\left\|T_{i(n)}^{k(n)}x_n - x_n\right\| \\ &\quad + L\hat{\gamma_n}\|v_n - x_n\|. \end{aligned}\tag{2.15}$$

Substituting (2.15) into (2.14), we obtain that

$$\begin{aligned}(1 - L\delta).\left\|x_n - T_{i(n)}^{k(n)}x_n\right\| &\le (1 + L\hat{\alpha_n})\|x_n - x_{n-1}\| + \left\|x_{n-1} - T_{i(n)}^{k(n)}y_n\right\| \\ &\quad + L\hat{\gamma_n}\|v_n - x_n\|. \end{aligned}\tag{2.16}$$

Hence, by virtue of the condition (v), (2.10) and (2.12), we have

$$(1 - L\delta).\limsup_{n\to\infty}\left\|x_n - T_{i(n)}^{k(n)}x_n\right\| \le 0. \tag{2.17}$$

From the condition (vi), $0 \le L\delta < 1$, hence from (2.17) we have

$$\lim_{n\to\infty} \left\| x_n - T_{i(n)}^{k(n)} x_n \right\| = 0. \tag{2.18}$$

Now, we prove that (2.6) holds. In fact, since for each $n > N$, $n = (n - N)(mod\ N)$ and $n = (k(n)-1)N + i(n)$, hence $n - N = ((k(n)-1)-1)N + i(n) = (k(n-N)-1)N + i(n-N)$, that is,

$$k(n-N) = k(n) - 1 \ \text{ and } \ i(n-N) = i(n).$$

From (2.13), (2.18) and Proposition 1.1 that

$$\begin{aligned}
\|x_n - T_n x_n\| &\le \left\| x_n - T_{i(n)}^{k(n)} x_n \right\| + \left\| T_{i(n)}^{k(n)} x_n - T_n x_n \right\| \\
&\le \left\| x_n - T_{i(n)}^{k(n)} x_n \right\| + L \left\| T_{i(n)}^{k(n)-1} x_n - x_n \right\| \\
&\le \left\| x_n - T_{i(n)}^{k(n)} x_n \right\| + L \Big\{ \left\| T_{i(n)}^{k(n)-1} x_n - T_{i(n-N)}^{k(n)-1} x_{n-N} \right\| \\
&\quad + \left\| T_{i(n-N)}^{k(n)-1} x_{n-N} - x_{n-N} \right\| + \| x_{n-N} - x_n \| \Big\} \\
&\le \left\| x_n - T_{i(n)}^{k(n)} x_n \right\| + L^2 \| x_n - x_{n-N} \| \\
&\quad + L \left\| T_{i(n-N)}^{k(n)-1} x_{n-N} - x_{n-N} \right\| + L \| x_{n-N} - x_n \| \\
&= \left\| x_n - T_{i(n)}^{k(n)} x_n \right\| + L(L+1) \| x_n - x_{n-N} \| \\
&\quad + L \left\| T_{i(n-N)}^{k(n)-1} x_{n-N} - x_{n-N} \right\| \\
&\to 0 \ \text{ as } \ n \to \infty,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0, \tag{2.19}$$

and so, from (2.10) and (2.19), it follows that, for any $j = 1, 2, \ldots, N$,

$$\begin{aligned}
\|x_n - T_{n+j} x_n\| &\le \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\| \\
&\quad + \|T_{n+j} x_{n+j} - T_{n+j} x_n\| \\
&\le (1+L) \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\| \\
&\to 0 \ \text{ as } \ n \to \infty,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_n - T_{n+j} x_n\| = 0, \tag{2.20}$$

for all $j = 1, 2, \ldots, N$.

Without loss of generality, we can assume that $n_k = i(mod\ N)$ for all k and some $i \in \{1, 2, \ldots, N\}$. For any fixed $l \in \{1, 2, \ldots, N\}$, we can find a

$j \in \{1, 2, \ldots, N\}$, independent of k, such that $i + j = l(mod\ N)$, and so $n_k + j = l(mod\ N)$ for all k. Hence, from (2.20), we have

$$\lim_{n_k \to \infty} \|x_{n_k} - T_l x_{n_k}\| = 0, \tag{2.21}$$

for all $l = 1, 2, \ldots, N$. Thus

$$\lim_{n \to \infty} \|x_n - T_l x_n\| = 0, \tag{2.22}$$

for all $l = 1, 2, \ldots, N$. That is, (2.6) holds. Since E is uniformly convex, every bounded subset of E is weakly compact. Again since $\{x_n\}$ is a bounded sequence in C, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q_1 \in C$. Hence from (2.6), it follows that

$$\lim_{n_k \to \infty} \|x_{n_k} - T_l x_{n_k}\| = 0, \tag{2.23}$$

for all $l = 1, 2, \ldots, N$.

By Lemma 1.3, we have that $(I - T_j)q_1 = 0$, that is, $q_1 \in F(T_j)$. Further, by the arbitrariness of $j \in \{1, 2, \ldots, N\}$, we know that $q_1 \in F = \bigcap_{j=1}^N F(T_j)$.

Finally, we prove that the sequence $\{x_n\}$ converges weakly to q_1. In fact, suppose the contrary, then there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $q_2 \in C$ and $q_1 \neq q_2$. Then by the same method as given above, we can also prove that $q_2 \in F = \bigcap_{j=1}^N F(T_j)$.

Taking $p = q_1$ and $p = q_2$ and by using the same method given in the proof of Lemma 1.4, we can prove that the limits $\lim_{n\to\infty} \|x_n - q_1\|$ and $\lim_{n\to\infty} \|x_n - q_2\|$ exist, and we have

$$\lim_{n \to \infty} \|x_n - q_1\| = d_1, \quad \lim_{n \to \infty} \|x_n - q_2\| = d_2$$

where d_1 and d_2 are two nonnegative numbers. By virtue of the Opial condition of E, we have

$$\begin{aligned} d_1 &= \limsup_{n_k \to \infty} \|x_{n_k} - q_1\| < \limsup_{n_k \to \infty} \|x_{n_k} - q_2\| = d_2 \\ &= \limsup_{n_j \to \infty} \left\|x_{n_j} - q_2\right\| < \limsup_{n_j \to \infty} \left\|x_{n_j} - q_1\right\| = d_1. \end{aligned}$$

This is a contradiction. Hence $q_1 = q_2$. This implies that $\{x_n\}$ converges weakly to q_1. This completes the proof of Theorem 2.3. □

Theorem 2.4. *Let E be a real uniformly convex Banach space satisfying Opial condition, C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N : C \to C$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in C, and let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $[0, 1]$ and $\{a_n\}$ be the sequence defined by (1.3) and $L = \sup_{n\geq 1} a_n \geq 1$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $\sum_{n=1}^{\infty}(a_n - 1)\beta_n < \infty$;
(iii) $0 < \tau_1 = \inf\{\beta_n : n \geq 1\} \leq \sup\{\beta_n : n \geq 1\} = \tau_2 < \frac{1}{L^2}$;
(iv) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then the explicit iterative sequence $\{x_n\}$ *defined by* (1.8) *converges weakly to a common fixed point of the mappings* $\{T_1, T_2, \ldots, T_N\}$.

Proof. Taking $\hat{\beta_n} = \hat{\gamma_n} = 0$, for all $n \geq 1$ in Theorem 2.3, then the conclusion of the Theorem 2.4 can be obtained from Theorem 2.3 immediately. This completes the proof of Theorem 2.4. □

Theorem 2.5. *Let* E *be a real uniformly convex Banach space satisfying Opial condition,* C *be a nonempty closed convex subset of* E*. Let* $\{T_i\}_{i=1}^{N} : C \to C$ *be* N *nonexpansive mappings with* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$*. Let* $\{u_n\}$ *and* $\{v_n\}$ *be two bounded sequences in* C*, and let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha_n}\}$, $\{\hat{\beta_n}\}$ *and* $\{\hat{\gamma_n}\}$ *be six sequences in* $[0, 1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = \hat{\alpha_n} + \hat{\beta_n} + \hat{\gamma_n} = 1$;
(ii) $0 < \tau_1 = \inf\{\beta_n : n \geq 1\} \leq \sup\{\beta_n : n \geq 1\} = \tau_2 < 1$;
(iii) $\hat{\beta_n} \to 0$, $(n \to \infty)$;
(iv) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \hat{\gamma_n} < \infty$;
(v) $0 \leq \delta = \sup\{\hat{\beta_n} : n \geq 1\} < 1$.

Then the implicit iterative sequence $\{x_n\}$ *defined by* (1.7) *converges weakly to a common fixed point of the mappings* $\{T_1, T_2, \ldots, T_N\}$.

Proof. As we know that each nonexpansive mapping from C into C is an asymptotically nonexpansive mapping from $C \to C$, with $a_n = 1$, for all $n \geq 1$ and $L = 1$. Therefore all conditions in Theorem 2.3 are satisfied. The conclusion of Theorem 2.5 can be obtained from Theorem 2.3 immediately. This completes the proof of Theorem 2.5. □

Theorem 2.6. *Let* E *be a real uniformly convex Banach space and* C *be a nonempty closed convex subset of* E*. Let* $\{T_i\}_{i=1}^{N} : C \to C$ *be* N *asymptotically nonexpansive mappings with* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ *and there exists an* T_l, $1 \leq l \leq N$*, which is semi-compact (without loss of generality, we can assume that* T_1 *is semi-compact). Let* $\{u_n\}$ *and* $\{v_n\}$ *be two bounded sequences in* C*, and let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha_n}\}$, $\{\hat{\beta_n}\}$ *and* $\{\hat{\gamma_n}\}$ *be six sequences in* $[0, 1]$ *and* $\{a_n\}$ *be the sequence defined by* (1.3) *and* $L = \sup_{n \geq 1} a_n \geq 1$ *satisfying the conditions* (i)-(vi) *as in Theorem* 2.3*. Then the implicit iterative sequence* $\{x_n\}$ *defined by* (1.7) *converges strongly to a common fixed point of the mappings* $\{T_1, T_2, \ldots, T_N\}$ *in* C.

Proof. For any given $p \in F = \bigcap_{i=1}^{N} F(T_i)$, by the same method as given in proving Lemma 1.4 and (2.23), we can prove that

$$\lim_{n\to\infty} \|x_n - p\| = d, \tag{2.24}$$

where $d \geq 0$ is some nonnegative number, and

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \tag{2.25}$$

for all $l = 1, 2, \ldots, N$. Especially, we have

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0. \tag{2.26}$$

By the assumption of the theorem, T_1 is semi-compact, therefore it follows from (2.26) that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^* \in C$. Hence from (2.25) we have that

$$\|x^* - T_l x^*\| = \lim_{n_i\to\infty} \|x_{n_i} - T_l x_{n_i}\| = 0,$$

for all $l = 1, 2, \ldots, N$, which implies that

$$x^* \in F = \bigcap_{i=1}^{N} F(T_i).$$

Take $p = x^*$ in (2.24), similarly we can prove that

$$\lim_{n\to\infty} \|x_n - x^*\| = d_1,$$

where $d_1 \geq 0$ is some nonnegative number. From $x_{n_i} \to x^*$ we know that $d_1 = 0$, i.e., $x_n \to x^*$. This completes the proof of Theorem 2.6. □

Theorem 2.7. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{N} : C \to C$ be N asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and there exists an T_l, $1 \leq l \leq N$, which is semi-compact (without loss of generality, we can assume that T_1 is semi-compact). Let $\{u_n\}$ be a bounded sequence in C, and let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $[0,1]$ and $\{a_n\}$ be the sequence defined by* (1.3) *and $L = \sup_{n\geq 1} a_n \geq 1$ satisfying the conditions* (i)-(iv) *as in Theorem* 2.4. *Then the explicit iterative sequence $\{x_n\}$ defined by* (1.8) *converges weakly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$ in C.*

Proof. Taking $\hat{\beta_n} = \hat{\gamma_n} = 0$, for all $n \geq 1$ in Theorem 2.6, then the conclusion of the Theorem 2.7 can be obtained from Theorem 2.6 immediately. This completes the proof of Theorem 2.7. □

Theorem 2.8. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N : C \to C$ be N nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and there exists an T_l, $1 \leq l \leq N$, which is semi-compact (without loss of generality, we can assume that T_1 is semi-compact). Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in C, and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha_n}\}$, $\{\hat{\beta_n}\}$ and $\{\hat{\gamma_n}\}$ be six sequences in* $[0,1]$ *satisfying the conditions* (i)-(v) *as in Theorem* 2.5. *Then the implicit iterative sequence $\{x_n\}$ defined by* (1.7) *converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$ in C.*

Proof. As we know that each nonexpansive mapping from C into C is an asymptotically nonexpansive mapping from $C \to C$, with $a_n = 1$, for all $n \geq 1$ and $L = 1$. Therefore all conditions in Theorem 2.6 are satisfied. The conclusion of Theorem 2.8 can be obtained from Theorem 2.6 immediately. This completes the proof of Theorem 2.8. □

Remark 2.1. Since $0 \leq (a_n - 1)\beta_n \leq a_n - 1$, therefore it is easy to see that if condition (ii) is replaced by (ii)':

(ii)' $\sum_{n=1}^{\infty}(a_n - 1) < \infty$,

then the conclusion of Theorem 2.1 - 2.4, 2.6 and 2.7 all are holds.

Remark 2.2. Theorem 2.1, 2.3, 2.5, 2.6 and 2.8 also improve and extend the corresponding results of of Chang and Cho [2] and the key condition (v) in [[2], Theorem 3.1]: there exist constants $L > 0$ and $\alpha > 0$ such that, for any $i, j \in \{1, 2, \ldots, N\}$ with $i \neq j$

$$\left\|T_i^n x - T_j^n y\right\| \leq L \left\|x - y\right\|^{\alpha}, \quad \forall n \geq 1,$$

for all $x, y \in C$ is deleted.

Remark 2.3. Theorem 2.3 and 2.5 also improve and extend Theorem 1 of Zhou and Chang [18] and the key condition (v) in [[18], Theorem 1]: there exists a constant $L > 0$ such that for any $i, j \in \{1, 2, \ldots, N\}$ with $i \neq j$

$$\left\|T_i^n x - T_j^n y\right\| \leq L \left\|x - y\right\|, \quad \forall n \geq 1,$$

for all $x, y \in C$ is deleted.

Remark 2.4. Theorem 2.2 - 2.8 also improve and extend the corresponding results of [1, 4, 5, 7, 8, 10, 11, 13, 14, 16] to the case of more general class of spaces, mappings and iteration schemes considered in this paper.

Remark 2.5. Our results also extend the corresponding results of Gu [6] to the case of implicit iteration process with bounded errors considered in this paper.

Example 1. Let $X = \ell_2 = \{\bar{x} = \{x_i\}_{i=1}^{\infty} : x_i \in C, \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$, and let $\bar{B} = \{\bar{x} \in \ell_2 : \|x\| \leq 1\}$. Define $T\colon \bar{B} \to \ell_2$ by

$$T\bar{x} = (0, x_1^2, a_2x_2, a_3x_3, \dots),$$

where $\{a_j\}_{j=1}^{\infty}$ is a real sequence satisfying: $a_2 > 0$, $0 < a_j < 1$, $j \neq 2$, and $\prod_{j=2}^{\infty} a_j = 1/2$. Then

$$\begin{aligned} \|T^n\bar{x} - T^n\bar{y}\| &\leq 2\Big(\prod_{j=2}^{n} a_j\Big) \|\bar{x} - \bar{y}\| \\ &\leq k_n \|\bar{x} - \bar{y}\| \end{aligned}$$

where $k_n = 2\Big(\prod_{j=2}^{n} a_j\Big)$ and $\bar{x}, \bar{y} \in X$. Since $\lim_{n\to\infty} k_n = \lim_{n\to\infty} 2\Big(\prod_{j=2}^{n} a_j\Big) = 1$, it follows that T is an asymptotically nonexpansive mapping. But it is not nonexpansive mapping.

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