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NEW PROOFS OF SOME FIXED POINT THEOREMS FOR MAPPINGS SATISFYING REICH TYPE CONTRACTIONS IN MODULAR METRIC SPACES

Godwin Amechi Okeke[1], Daniel Francis[2] and Jong Kyu Kim[3]

[1]Functional Analysis and Optimization Research Group Laboratory (FANORG),
Department of Mathematics, School of Physical Sciences,
Federal University of Technology Owerri, P.M.B. 1526 Owerri, Imo State, Nigeria
e-mail: `gaokeke1@yahoo.co.uk, godwin.okeke@futo.edu.ng`

[2]Functional Analysis and Optimization Research Group Laboratory (FANORG),
Department of Mathematics, School of Physical Sciences,
Federal University of Technology Owerri, P.M.B. 1526 Owerri, Imo State, Nigeria
e-mail: `francis.daniel@mouau.edu.ng`

[3]Department of Mathematics Education, Kyungnam University,
Changwon, Gyeongnam, 51767, Korea
e-mail: `jongkyuk@kyungnam.ac.kr`

Abstract. Our aim in this paper is to give some new proofs to fixed point theorems due to Abdou [1] for mappings satisfying Reich type contractions in modular metric spaces. We removed the restriction that ω satisfies the Δ_2-type condition imposed on the results of [1]. Furthermore, Lemma 2.6 of [1] which was crucial in the proofs of the results of [1] is not needed in the proofs of our results. Our method of proof is simpler and interesting.

1. Introduction

The famous Banach contraction mapping principle was proved in 1922 by Banach [3]. Interests in the results of Banach [3] is due to its applications in solving problems in physics, economics, computer science, engineering,

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[0]Corresponding author: G. A. Okeke(`godwin.okeke@futo.edu.ng`).

telecommunication and management sciences. Several well-known mathematicians have extended the Banach contraction mapping principle to spaces like multiplicative metric spaces, complex valued metric spaces, b-metric spaces, modular metric spaces, G-metric spaces, cone metric spaces and so on. The first mathematician who successfully extended the results of [3] to multivalued mappings was Nadler [7]. This extension was interesting because it found several applications in economics, differential inclusions, convex optimization and control theory. Consequently, several well-known authors have extensively studied the Nadler fixed point theorem [7].

In 1972, Reich [15] generalized Nadler's fixed point theorem. The following is the statement of Reich's fixed point theorem: a mapping $T : X \to \mathcal{K}(\mathcal{X})$, where $\mathcal{K}(\mathcal{X})$ is the family of every nonempty compact subsets of the set X, has a fixed point if it satisfies

$$H(Tx, Ty) \leq k(d(x, y))d(x, y)$$

for each $x, y \in X$ with $x \neq y$, where $k : (0, \infty) \to [0, 1)$ is such that $\limsup_{r \to t_+} k(r) < 1$ for each $t \in (0, \infty)$.

In 1974, Reich [16] posed an open question whether the results of [15] will hold if T takes values in $\mathcal{CB}(\mathcal{X})$ instead of $\mathcal{K}(\mathcal{X})$, where $\mathcal{CB}(\mathcal{X})$ is the family of every nonempty closed and bounded subsets of X. A partial answer to this Reich's open question was given in 1989 by Mizoguchi and Takahashi [6].

The concept of modular metric spaces was introduced in 2010 by Chistyakov [4, 5] as a generalization of the classical modulars over linear spaces such as Orlicz spaces. In 2016, Abdou [1] proved some interesting theorems for mappings satisfying Reich contraction in modular metric spaces. Readers interested in studies in this direction may consult ([8, 9, 10, 11, 12, 13, 14]) and the references therein.

Motivated by the results above, we give some new proofs to fixed point theorems due to Abdou [1] for mappings satisfying Reich type contractions in modular metric spaces. We removed the restriction that ω satisfies the Δ_2-type condition imposed on the results of [1]. Furthermore, Lemma 2.6 of [1] which was crucial in the proofs of the results of [1] is not needed in the proofs of our results. Our method of proof is simpler and interesting.

2. Preliminaries

We begin this section by recalling some definitions and results which will be useful in this paper.

Definition 2.1. ([4]) Let X be a nonempty set. A modular metric on a set X is a function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ satisfying, for all $x, y, z \in X$, the following three properties:

(1) $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
(2) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$;
(3) $\omega_{\lambda+\nu}(x,y) \leq \omega_\lambda(x,z) + \omega_\nu(z,y)$ for $\lambda,\nu > 0$.

Then the pair (X,ω) is called a modular metric space. Throughout this paper, we take X_ω or (X,ω) to be modular metric space. If, instead of (1), we have the following condition.

(1)' $\omega_\lambda(x,x) = 0$ for all $\lambda > 0$ and $x \in X_\omega$.

Then ω is said to be a pseudomodular metric on X_ω. A modular metric ω is said to be regular if the following weaker condition of (1) is satisfied:

(1)" $x = y$ if and only if $\omega_\lambda(x,x) = 0$, for some $\lambda > 0$.

Definition 2.2. ([4]) A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a convex modular metric on a set X if it satisfies the axioms (1) and (2) of Definition 2.1 as well as the following axiom:

(4) $\omega_{\lambda+\nu}(x,y) \leq \omega_{\frac{\lambda}{\lambda+\nu}}(x,z) + \omega_{\frac{\nu}{\lambda+\nu}}(z,y)$ for λ, $\nu > 0$ and $x, y, z \in X$.

If, instead of (1), we have only condition (1)' of Definition 2.1, then ω is called a convex pseudomodular metric on X.

Definition 2.3. ([4]) Given a pseudomodular ω on X, along with the modular set X_ω. For given $x_0 \in X$, set

$$X_\omega \equiv X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) = 0 \ as\ \lambda \to \infty\}$$

and

$$X^*_\omega \equiv X^*_\omega(x_0) = \{x \in X_\omega : \exists\ \lambda(x) = \lambda > 0 \text{ such that } \omega_\lambda(x,x_0) < \infty\}.$$

Then X_ω and X^*_ω are said to be modular spaces centered at x_0.

Observe that if ω is a modular on X, then $d_\omega(x,y) = \inf\{t > 0 : \omega_t(x,y) \leq t\}$ for any $x, y \in X_\omega$, defines a distance on X_ω. If ω is convex, then we have $X^*_\omega = X_\omega$, (see, e.g. [4, 5]). Observe that d^*_ω, given by $d^*_\omega(x,y) = \inf\{t > 0 : \omega_t(x,y) \leq 1\}$, for any $x, y \in X_\omega$ is a metric.

Remark 2.4. For any $x_i \in X$, the set

$$X_\omega(x_i) = \{x \in X \text{ such that } \lim_{\lambda\to\infty} \omega_\lambda(x,x_i) = 0\}$$

is called a modular metric space generated by x_i and induced by ω. If its generator x_i does not play any role in this case (that is, X_ω is independent of generators), we shall write X_ω instead of $X_\omega(x_i)$.

Remark 2.5. For any $x, y \in X_\omega$, if a modular metric ω on X_ω has a finite value and $\omega_\lambda(x, y) = \omega_\mu(x, y)$ for all $\lambda, \mu > 0$, then $\rho(x, y) = \lambda\omega_\lambda(x, y)$ is a metric on X_ω.

Definition 2.6. ([1]) Let ω be a modular metric defined on X.

(1) We say that $\{x_n\}_{n\in\mathbb{N}} \subset X_\omega$ is ω-convergent to $a \in X_\omega$ if $\omega_1(x_n, a) \to 0$, as $n \to \infty$. In this case, a is said to be an ω-limit of $\{x_n\}_{n\in\mathbb{N}}$.
(2) We say that $\{x_n\}_{n\in\mathbb{N}} \subset X_\omega$ is ω-Cauchy if $\omega_1(x_n, x_m) \to 0$, as $n, m \to \infty$.
(3) We say that $M \subset X_\omega$ is closed if the ω-limit of an ω-convergent sequence of M is in M.
(4) We say that $M \subset X_\omega$ is ω-complete if every ω-Cauchy sequence in M is ω-convergent and its ω-limit belong to M.
(5) We say that $M \subset X_\omega$ is ω-bounded provided
$$\delta_\omega(M) = \sup\{\omega_1(a, b); a, b \in M\} < \infty.$$
(6) We say that $M \subset X_\omega$ is ω-compact if for any $\{x_n\}$ in M there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in M$ such that $\omega_1(x_{n_k}, x) \to 0$.
(7) ω is said to satisfy the Fatou property if we have
$$\omega_1(x, y) \le \liminf_{n\to\infty} \omega_1(x_n, y)$$
for any $\{x_n\}_{n\in\mathbb{N}}$ in X_ω which is ω-convergent to x, and for any $y \in X_\omega$.

If $\lim_{n\to\infty} \omega_\alpha(x_n, x) = 0$, for some $\alpha > 0$, then $\lim_{n\to\infty} \omega_\alpha(x_n, x) = 0$ may not necessary happen for all $\alpha > 0$. We say that ω satisfies the Δ_2-condition if $\lim_{n\to\infty} \omega_\alpha(x_n, x) = 0$, for some $\alpha > 0$ implies that $\lim_{n\to\infty} \omega_\alpha(x_n, x) = 0$ for all $\alpha > 0$.

Definition 2.7. ([1]) Let (X_ω, ω) be a modular metric space, for all $\lambda > 0$, we say that ω satisfies Δ_2-type condition if for $\alpha > 0$, there exists $C_\alpha > 0$ such that
$$\omega_{\frac{\lambda}{\alpha}}(x, y) \le C_\alpha \omega_\lambda(x, y),$$
for all $x, y \in X_\omega$, with $y \ne x$, and any $\lambda > 0$.

Lemma 2.8. ([2]) *Let (X, ω) be a modular metric space where ω is convex and regular. Assume that ω satisfies the Δ_2-type condition. Let $\{x_n\}$ be in X_ω such that $\omega_1(x_{n+1}, x_n) \le K\alpha^n$, $n = 1, 2, \cdots$, where K is an arbitrary nonzero constant and $\alpha \in (0, 1)$. Then $\{x_n\}$ is Cauchy for both ω and d^*_ω.*

Following the notations in [1], let M be a nonempty subset of a modular metric space X_ω. Set

(1) $\mathcal{C}(M) = \{B :\ B \text{ is nonempty and } \ \omega-\text{closed subset of } M\}$;

(2) $\mathcal{CB}(M) = \{B : B \text{ is nonempty}, \omega-\text{closed and } \omega-\text{bounded subset of } M\}$;
(3) Define the Hausdorff modular metric on $\mathcal{CB}(M)$ by

$$H_\omega(C_1, C_2) = \max\{\sup_{a\in C_1} \omega_1(a, C_2), \sup_{b\in C_2} \omega_1(b, C_1)\},$$

where $\omega_1(a, C) = \inf_{b\in C} \omega_1(a, b)$.

Definition 2.9. ([1]) Let (X_ω, ω) be a modular metric space and M be a nonempty subset of X_ω. A mapping $T : M \to \mathcal{CB}(M)$ is called a Reich contraction mapping if there exists $k : (0, \infty) \to [0, 1)$ which satisfies $\limsup_{s\to t^+} k(s) < 1$ for any $t \in [0, \infty)$, such that for any different $a, b \in M$, we have

$$H_\omega(T(a), T(b)) \le k(\omega_1(a, b))\omega_1(a, b). \tag{2.1}$$

Lemma 2.10. ([2]) *Let (X_ω, ω) be a modular metric space and M be a nonempty subset of X_ω. Let $A, B \in \mathcal{CB}(M)$. Then, for each $\epsilon > 0$ and $x \in A$ there exists $y \in B$ such that $\omega_1(x, y) \le H_\omega(A, B) + \epsilon$. Moreover, if B is ω-compact and ω satisfies the Fatou property, then for any $x \in A$ there exists $y \in B$ such that $\omega_1(x, y) \le H_\omega(A, B)$.*

By the results of [1], Lemma 2.10 allows an alternative definition to Reich multivalued mappings as follows;

Definition 2.11. Let M be a nonempty subset of a modular metric space X_ω. The mapping $T : M \to \mathcal{CB}(M)$ is called a Reich-type multivalued mapping if there exists distinct $x, y \in M$, $a \in T(x)$ and $b \in T(y)$ such that

$$d(a, b) \le \beta(d(x, y))d(x, y), \tag{2.2}$$

where $\beta = \frac{1}{2}(1 + \alpha)$ satisfying $\limsup_{s\to t^+} \alpha(s) < 1$ for any $t \in [0, \infty)$.

A point $x \in M$ is said to be a fixed point of T if $x \in Tx$. The set of fixed points of T will be denoted by $Fix(T)$, that is, $Fix(T) = \{x \in M : x \in Tx\}$.

3. Main results

We shall give a proof of Reich multivalued mapping in modular metric space as discussed in [1] by relaxing the Δ_2-type condition and Lemma 2.8 will not be applied.

Theorem 3.1. *Let M be a nonempty ω-complete subset of a complete modular metric space (X_ω, ω). Let $F : M \to \mathcal{C}(M)$ be a multi-valued mapping such that there exists $\kappa : (0, \infty) \to [0, 1)$ with $\limsup_{s\to\tau^+} \kappa(s) < 1$, for any $\tau \in [0, \infty)$, for any distinct $x, y \in M, \lambda > 0$ and $a \in F(x)$, there exists $b(\neq a) \in F(y)$ such that*

$$\omega_\lambda(a, b) \le \kappa(\omega_\lambda(x, y))\omega_\lambda(x, y). \tag{3.1}$$

Then F has a fixed point $x^* \in M$.

Proof. We shall prove by contradiction. Suppose that F has no fixed point, that is, $x^* \notin F(x^*)$, then for any $\lambda > 0$, $\omega_\lambda(x^*, F(x^*)) > 0$ in $M \subset X_\omega$. For every $\tau > 0$, $\alpha, \beta > 0$, let $\alpha(\tau), \beta(\tau) > 0$ be such that $\tau < s \leq \tau + \beta(\tau)$ which implies that $\kappa(s) < \alpha(\tau) < 1$. Let $x_1 \in M \subset X_\omega$ so that $\omega_\lambda(x^*, x_1) < \infty$. Let $x_n \in F(x_{n-1})$ such that $0 \leq \epsilon_n < \frac{1}{n}$ for some $n \in \mathbb{N}$,

$$(1 + \epsilon_n)\kappa(\omega_\lambda(x_{n-1}, x_n)) < 1$$

and

$$\omega_\lambda(x_{n-1}, x_n) \leq (1 + \epsilon_n)\omega_\lambda(x_{n-1}, F(x_n)). \tag{3.2}$$

Set $\tau_n = \omega_\lambda(x_n, F(x_n))$, for $\delta_n > 0$, take $0 < \delta_n < \dfrac{1}{\kappa(\omega_\lambda(x_{n-1}, x_n))} - 1$ so that

$$\epsilon_{n+1} = \min\{\delta_n, \frac{1}{n+1}, \frac{1}{\alpha(\tau_n)} - 1, \frac{\beta(\tau_n)}{\tau_n}\},$$

for $n \in \mathbb{N}$, then choose $x_{n+1} \in F(x_n)$ such that

$$\omega_\lambda(x_n, x_{n+1}) \leq (1 + \epsilon_{n+1})\omega_\lambda(x_n, F(x_n)).$$

Hence, we have

$$\tau_n \leq \omega_\lambda(x_n, x_{n+1}) < \tau_n + \beta(\tau_n).$$

Suppose that there exists $y \in F(x_n)$ such that $\omega_\lambda(x_n, y) = \omega_\lambda(x_n, F(x_n))$, then we take $y = x_{n+1}$ and $\epsilon_{n+1} = 0$. Otherwise, we have $\tau_n < \omega_\lambda(x_n, x_{n-1})$, hence $\kappa(\omega_\lambda(x_n, x_{n+1})) < \alpha(\tau_n)$. Therefore, in any case, x_{n+1} satisfies, $(1 + \epsilon_{n+1})\kappa(\omega_\lambda(x_n, x_{n+1})) < 1$ and

$$\omega_\lambda(x_n, x_{n+1}) \leq (1 + \epsilon_{n+1})\omega_\lambda(x_n, F(x_n)). \tag{3.3}$$

From inequalities (3.2) and (3.3), we get

$$\begin{aligned}
\omega_\lambda(x_n, x_{n+1}) \leq& (1 + \epsilon_{n+1})\omega_\lambda(x_n, F(x_n)) \\
=& (1 + \epsilon_{n+1})\omega_\lambda(F(x_{n-1}), F(x_n)) \\
\leq& (1 + \epsilon_{n+1})\kappa(\omega_\lambda(x_{n-1}, x_n))\omega_\lambda(x_{n-1}, x_n) \\
\leq& (1 + \delta_n)\kappa(\omega_\lambda(x_{n-1}, x_n))\omega_\lambda(x_{n-1}, x_n) \\
<& \omega_\lambda(x_{n-1}, x_n).
\end{aligned} \tag{3.4}$$

Now, $-\omega_\lambda(x_{n+1}, F(x_{n+1})) \geq -\kappa(\omega_\lambda(x_n, x_{n+1}))\omega_\lambda(x_n, x_{n+1})$ for $\lambda > 0$, so that

$$\begin{aligned}
&\omega_\lambda(x_n, F(x_n)) - \omega_\lambda(x_{n+1}, F(x_{n+1})) \\
&\geq \omega_\lambda(x_n, F(x_n)) - \kappa(\omega_\lambda(x_n, x_{n+1}))\omega_\lambda(x_n, x_{n+1}) \\
&= \omega_\lambda(x_n, x_{n+1}) - \kappa(\omega_\lambda(x_n, x_{n+1}))\omega_\lambda(x_n, x_{n+1}) \\
&\geq \big(\frac{1}{1+\epsilon_{n+1}} - \kappa(\omega_\lambda(x_n, x_{n+1}))\big)\omega_\lambda(x_n, x_{n+1}) \\
&> 0.
\end{aligned} \tag{3.5}$$

Thus, by induction, we obtained a sequence $\{x_n\}_{n\geq 1} \subseteq M \subset X_\omega$ such that $x_{n+1} \in F(x_n)$, and $\{\omega_\lambda(x_n, x_{n+1})\}_{n\geq 1}$, $\{\omega_\lambda(x_n, F(x_n))\}_{n\geq 1}$ are both strictly decreasing sequences. Thus $\omega_\lambda(x_n, x_{n+1}) \downarrow \tau$ as $n \to \infty$ for some $\tau \geq 0,\ \lambda > 0$. Observe that by the hypothesis on κ, we have that

$$\limsup_{n\to\infty} \kappa(\omega_\lambda(x_n, x_{n+1})) < 1$$

for any $\lambda > 0$. Hence

$$\liminf_{n\to\infty} \big(\frac{1}{1+\epsilon_{n+1}} - \kappa(\omega_\lambda(x_n, x_{n+1}))\big) > 0.$$

Again, from inequality (3.5), suppose that there exists $\rho > 0$ such that for $n \geq 1$ large, we have

$$\rho\omega_\lambda(x_n, x_{n+1}) \leq \omega_\lambda(x_n, F(x_n)) - \omega_\lambda(x_{n+1}, F(x_{n+1})). \tag{3.6}$$

Since $\{\omega_\lambda(x_n, F(x_n))\}_{n\geq 1}$ is convergent, for $m > n$ large, we have

$$\begin{aligned}
\omega_\lambda(x_n, x_m) &\leq \omega_{\frac{\lambda}{m-n}}(x_n, x_{n-1}) + \omega_{\frac{\lambda}{m-n}}(x_{n-1}, x_{n-2}) + \omega_{\frac{\lambda}{m-n}}(x_{n-2}, x_{n-3}) \\
&\quad + \omega_{\frac{\lambda}{m-n}}(x_{n-3}, x_{n-4}) + \cdots + \omega_{\frac{\lambda}{m-n}}(x_{m-1}, x_m) \\
&\leq \omega_{\frac{\lambda}{m}}(x_n, x_{n-1}) + \omega_{\frac{\lambda}{m}}(x_{n-1}, x_{n-2}) + \omega_{\frac{\lambda}{m}}(x_{n-2}, x_{n-3}) \\
&\quad + \omega_{\frac{\lambda}{m}}(x_{n-3}, x_{n-4}) + \cdots + \omega_{\frac{\lambda}{m}}(x_{m-1}, x_m) \\
&\leq \sum_{k=n}^{m-1} \omega_\lambda(x_k, x_{k+1}) \\
&\leq \frac{1}{\rho}\sum_{k=n}^{m-1} \big(\omega_\lambda(x_k, F(x_k)) - \omega_\lambda(x_{k+1}, F(x_{k+1}))\big) \\
&\leq \frac{1}{\rho}\big(\omega_\lambda(x_n, F(x_n)) - \omega_\lambda(x_m, F(x_m))\big) \\
&\to 0,
\end{aligned}$$

as $n \to \infty$. Hence, $\{x_n\}_{n\geq 1}$ is a ω-Cauchy sequence, so $x_n \to x^* \in M \subset X_\omega$ as $n \to \infty$, since M is complete. Therefore, we have

$$\omega_\lambda(F(x_n), F(x^*)) \leq \kappa(\omega_\lambda(x_n, x^*))\omega_\lambda(x_n, x^*),$$

which implies that

$$\begin{aligned} \omega_\lambda(x^*, F(x^*)) &= \omega_\lambda(x_{n+1}, F(x^*)) \\ &\leq \kappa(\omega_\lambda(x_n, x^*))\omega_\lambda(x_n, x^*) \\ &\leq \omega_\lambda(x_n, x^*) \\ &\to 0, \end{aligned} \tag{3.7}$$

as $n \to \infty$. This means that for all $\lambda > 0$, $\omega_\lambda(x^*, F(x^*)) \leq 0$, a contradiction. Hence $x^* \in F(x^*)$. □

Next, we prove the following theorem.

Theorem 3.2. *Let M be a nonempty ω-complete subset of a complete modular metric space (X_ω, ω). Let $F : M \to \mathcal{C}(M)$ be a multi-valued mapping such that there exists $\kappa : (0, \infty) \to [0, 1)$ with $\limsup_{s\to t^+} \kappa(s) < 1$, for any $t \in [0, \infty)$, for any distinct $x, y \in M, \lambda > 0$ and for some $m \in \mathbb{N}$, $a \in F^m(x)$, there exists $b(\neq a) \in F^m(y)$ such that*

$$\omega_\lambda(a, b) \leq \kappa(\omega_\lambda(x, y))\omega_\lambda(x, y). \tag{3.8}$$

Then F has a fixed point $x^* \in M$ for some positive integer $m \geq 1$.

Proof. By Theorem 3.1, there exists exactly one $x^* \in M$ such that $x^* \in F^m(x^*)$ for some integer $m \geq 1$. This implies that $F^m(F(x^*)) = F^{m+1}(x^*) = F(x^*)$. Since $x^* \in F^m(x^*)$ for some integer $m \geq 1$, therefore $x^* \in F(x^*)$. □

Remark 3.3. The proofs of Theorem 3.1 is a new and alternative proof of Theorem 4.3 of Abdou [1]. Observe that the restriction that ω satisfies the Δ_2-type condition imposed on the results of [1] was removed. Furthermore, Lemma 2.6 of [1] which was crucial in the proofs of the results of [1] is not needed in the proofs of our results.

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References

[1] A.A.N. Abdou, *Some fixed point theorems in modular metric spaces,* J. Nonlinear. Sci. Appl., **9** (2016), 4381-4387.

[2] A.A. Abdou and M.A. Khamsi, *Fixed points of multivalued contraction mappings in modular metric spaces,* Fixed Point Theory Appl., 2014, **249** (2014), 10 pages.

[3] S. Banach, *Sur les operations dans les ensembles abstracts et leure application aux equations integrals,* Fund. Math., **3** (1922), 133-181.

[4] V.V. Chistyakov, *Metric Modular spaces, I Basic concepts*, Nonlinear Anal., **72** (2010), 1-14.

[5] V.V. Chistyakov, *Metric Modular spaces, II Applications to superposition operators,* Nonlinear Anal., **72** (2010), 15-30.

[6] N. Mizoguchi and W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces,* J. Math. Anal. Appl., **141**(1) (1989), 177-188.

[7] S.B. Nadler, *Multi-valued contraction mappings,* Pacific J. Math., **30**(2) (1969), 475-488.

[8] G.A. Okeke, S.A. Bishop and S.H. Khan, *Iterative approximation of fixed point of multivalued ρ-quasi-nonexpansive mappings in modular function spaces with applications,* J. Funct. Spaces, **2018** (2018), 9 pages.

[9] G.A. Okeke and D. Francis, *Fixed point theorems for Geraghty-type mappings applied to solving nonlinear Volterra-Fredholm integral equations in modular G-metric spaces,* Arab J. Math. Sci., **27**(2) (2021), 214-234.

[10] G.A. Okeke, D. Francis and A. Gibali, *On fixed point theorems for a class of $\alpha-\nu-$Meir Keeler type contraction mapping in modular extended b-metric spaces,* The J. Anal., **30**(3) (2022), 1257-1282.

[11] G.A. Okeke, D. Francis and M. de la Sen, *Some fixed point theorems for mappings satisfying rational inequality in modular metric spaces with applications,* Heliyon, **6** (2020), e04785.

[12] G.A. Okeke and S.H. Khan, *Approximation of fixed point of multivalued ρ-quasi-contractive mappings in modular function spaces*, Arab J. Math. Sci., **26**(1) (2020), 75-93.

[13] G.A. Okeke and J.K. Kim, *Approximation of common fixed point of three multi-valued ρ-quasi-nonexpansive mappings in modular function spaces*. Nonlinear Funct. Anal. Appl., **24**(4) (2019), 651-664.

[14] F. Ouzine, R. Azennar and D. Mentagui, *A fixed point theorem on some multi-valued maps in modular spaces,* Nonlinear Funct. Anal. Appl., **27**(3) (2022), 641-648.

[15] S. Reich, *Fixed points of contractive functions,* Bull. Un. Math. Ital., **5** (1972), 26-42.

[16] S. Reich, *Some fixed point problems*, Atti. Accad. Naz. Lineci. Rend. el. Sci. Fis. Mat. Natur., **57** (1974), 194-198.