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GENERALIZED FUSION FRAMES WITH C^*-VALUED BOUNDS

Mohamed Rossafi[1] **and Fakhr-dine Nhari**[2]

[1]LaSMA Laboratory Department of Mathematics, Faculty of Sciences Dhar El Mahraz,
University Sidi Mohamed Ben Abdellah, P. O. Box 1796 Fez Atlas, Morocco
e-mail: `rossafimohamed@gmail.com; mohamed.rossafi@usmba.ac.ma`

[2]Laboratory Analysis, Geometry and Applications Department of Mathematics,
Faculty of Sciences, University of Ibn Tofail, Kenitra, Morocco
e-mail: `nharidoc@gmail.com`

Abstract. In this paper, we introduced the notions of $* - g-$fusion frame and $* - K - g-$fusion frame in Hilbert C^*-modules, we gives some properties and study the tensor product of $* - g-$fusion frame. Non-trivial examples are further provided to support the hypotheses of our results.

1. Introduction

A frame is a set of vectors in a Hilbert space that can be used to reconstruct each vector in the space from its inner products with the frame vectors. These inner products are generally called the frame coefficients of the vector. But unlike an orthonormal basis each vector may have infinitely many different representations in terms of its frame coefficients. The motivation behind fusion frames comes from signal processing, more precisely, the desire to process and analyze large data sets efficiently. A natural idea is to split such data sets into suitable smaller blocks which can be treated independently.

Gabor [9] introduced a method using a family of elementary functions for reconstructing functions (signals) in 1946. The idea of frames originated in the

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[0]Corresponding author: Fakhr-dine Nhari(`nharidoc@gmail.com`).

1952 paper by Duffin and Schaeffer [7] to address some deep questions in non-harmonic Fourier series. In 2000, Frank-larson [8] introduced the concept of frames in Hilbet C^*-modules as a generalization of frames in Hilbert spaces. The basic idea was to consider modules over C^*-algebras of linear spaces and to allow the inner product to take values in the C^*-algebras [13]. Khosravi and Khosravi [12] introduced the fusion frames and $g-$frame theory in Hilbert C^*-modules. Afterwards, Alijani and Dehghan consider frames with C^*-valued bounds [2] in Hilbert C^*-modules. Bounader and Kabbaj [4] and Alijani [1] introduced the $*$-g-frames which are generalizations of g-frames in Hilbert C^*-modules. In 2016, Xiang and Li [21] give a generalization of $g-$frames for operators in Hilbert C^*-modules. Recently, Fakhr-dine Nhari et al. [14] introduced the concepts of $g-$fusion frame and $K-g-$fusion frame in Hilbert C^*-modules. For more about frames in Hilbert C^*-modules see [10, 15, 16, 17, 18, 19, 20].

Motivated by the above literature, we introduce and investigate some properties of $*-g-$fusion frame and $*-K-g-$fusion frame in Hilbert C^*-modules, we also generalize some known results for fusion frames to generalized fusion frames with C^*-valued bounds.

The paper is organized as follows, we continue this introductory section we briefly recall the definitions and basic properties of C^*-algebra and Hilbert C^*-modules. In section 2, we introduce the concept of $*-g-$fusion frame and gives some properties. In section 3, we introduced the notion of $K-*-g-$fusion frame in Hilbert C^*-modules.

Throughout this paper, H is considered to be a countably generated Hilbert $\mathcal{A}-$module. Let $\{H_i\}_{i\in I}$ be the collection of Hilbert $\mathcal{A}-$modules and $\{W_i\}_{i\in I}$ be a collection of closed orthogonally complemented submodules of H, where I is finite or countable index set. $End^*_{\mathcal{A}}(H,H_i)$ is a set of all adjointable operator from H to H_i. In particular $End^*_{\mathcal{A}}(H)$ denote the set of all adjointable operators on H. P_{W_i} denote the orthogonal projection onto the closed submodule orthogonally complemented W_i of H. Define the module

$$l^2(\{H_i\}_{i\in I}) = \Big\{\{x_i\}_{i\in I} : x_i \in H_i, \|\sum_{i\in I}\langle x_i, x_i\rangle\| < \infty\Big\},$$

with $\mathcal{A}-$valued inner product $\langle x,y\rangle = \sum_{i\in I}\langle x_i, y_i\rangle$, where $x=\{x_i\}_{i\in I}$ and $y=\{y_i\}_{i\in I}$, clearly $l^2(\{H_i\}_{i\in I})$ is a Hilbert $\mathcal{A}-$module.

In the following we briefly recall the definitions and basic properties of C^*-algebra, Hilbert $\mathcal{A}$-modules. Our reference for C^*-algebras is [5, 6]. For a C^*-algebra $\mathcal{A}$ if $a\in\mathcal{A}$ is positive we write $a\geq 0$ and $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 1.1. ([5]) If $\mathcal{A}$ is a Banach algebra, an involution is a map $a \mapsto a^*$ of $\mathcal{A}$ into itself such that for all a and b in $\mathcal{A}$ and all scalars α the following conditions hold:

(1) $(a^*)^* = a$.
(2) $(ab)^* = b^*a^*$.
(3) $(\alpha a + b)^* = \bar{\alpha}a^* + b^*$.

Definition 1.2. ([5]) A C^*-algebra $\mathcal{A}$ is a Banach algebra with involution such that :

$$\|a^*a\| = \|a\|^2,$$

for every a in $\mathcal{A}$.

Example 1.3. $\mathcal{B} = B(H)$ the algebra of bounded operators on a Hilbert space, is a C^*-algebra, where for each operator A, A^* is the adjoint of A.

Definition 1.4. ([11]) Let $\mathcal{A}$ be a unital C^*-algebra and H be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and U are compatible. H is a pre-Hilbert $\mathcal{A}$-module if H is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : H \times H \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(1) $\langle x, x\rangle \geq 0$, for all $x \in H$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.
(2) $\langle ax + y, z\rangle = a\langle x, z\rangle + \langle y, z\rangle$, for all $a \in \mathcal{A}$ and $x, y, z \in H$.
(3) $\langle x, y\rangle = \langle y, x\rangle^*$, for all $x, y \in H$.

For $x \in H$, we define $||x|| = ||\langle x, x\rangle||^{\frac{1}{2}}$. If H is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on H is defined by $|x| = \langle x, x\rangle^{\frac{1}{2}}$ for $x \in H$

Lemma 1.5. ([2]) *If $\phi : \mathcal{A} \to \mathcal{B}$ is an $*-$homomorphism between C^*-algebras then ϕ is increasing, that is, if $a \leq b$, then $\phi(a) \leq \phi(b)$.*

Lemma 1.6. ([3]) *Let H and K two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(H, K)$. Then the following statements are equivalent:*

(i) *T is surjective.*
(ii) *T^* is bounded below with respect to norm, that is, there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$, for all $x \in K$.*
(iii) *T^* is bounded below with respect to the inner product, that is, there is $m' > 0$ such that $\langle T^*x, T^*x\rangle \geq m'\langle x, x\rangle$, for all $x \in K$.*

Lemma 1.7. ([2]) *Let U and H be two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(U, H)$. Then, we have the following ststements.*

(i) *If T is injective and T has a closed range, then the adjointable map T^*T is invertible and*

$$\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2.$$

(ii) *If T is surjective, then the adjointable map TT^* is invertible and*

$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.$$

Lemma 1.8. ([14]) *Let $\{W_i\}_{i\in I}$ be a sequence of orthogonally complemented closed submodules of H and $T \in End^*_{\mathcal{A}}(H)$ be invertible. If $T^*TW_i \subset W_i$ for each $i \in I$, then $\{TW_i\}_{i\in I}$ is a sequence of orthogonally complemented closed submodules and $P_{W_i}T^* = P_{W_i}T^*P_{TW_i}$.*

2. $* - g-$fusion frame in Hilbert C^*-modules

Definition 2.1. Let $\{\Lambda_i\}_{i\in I} \subseteq \{End^*_{\mathcal{A}}(H, H_i), i \in I\}$, $\{W_i\}_{i\in I}$ be a family of closed orthogonally complemented submodules of H and $\{v_i\}_{i\in I}$ be a family of weights in $\mathcal{A}$, that is, each v_i is a positive invertible element from the center of C^*-algebra $\mathcal{A}$. We say that $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ is a $* - g-$fusion frame for H if there exist A, B strictly non-zero of $\mathcal{A}$ such that for each $x \in H$

$$A\langle x, x\rangle A^* \leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \leq B\langle x, x\rangle B^*.$$

The element A and B are called the lower and upper $* - g-$fusion frame bounds respectively. If $\lambda = A = B$, then the $* - g-$fusion frame is said to be a $\lambda-$tight $* - g-$fusion frame for H and if $A = B = 1_{\mathcal{A}}$ is called a Parseval $* - g-$fusion frame for H.

Example 2.2. Let H be an ordinary inner product space, $I = \mathbb{N}^*$ be the set of all nonnegative integers and $\{e_i\}_{i\in\mathbb{N}^*}$ be an orthonormal basis for Hilbert C^*-module H. Then, we construct $H_i = \overline{span}\{e_1, ..., e_i\}$ and $W_i = \overline{span}\{e_i\}$, $i \in I$.

Define $\Lambda_i : H \to H_i$ by

$$\Lambda_i x = \sum_{k=1}^{i} \langle x, \frac{e_k}{\sqrt{i}}\rangle e_k,$$

and the adjoint operator $\Lambda_i^* : H_i \to H$ define as

$$\Lambda_i^* x = \sum_{k=1}^{i} \langle x, \frac{e_k}{\sqrt{i}}\rangle e_i,$$

then for each $x \in H$, we have

$$\begin{aligned}\langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle &= \langle \sum_{k=1}^{i} \langle x, \frac{e_k}{\sqrt{i}}\rangle e_k, \sum_{k=1}^{i} \langle x, \frac{e_k}{\sqrt{i}}\rangle e_k\rangle \\ &= \langle x, \frac{e_i}{\sqrt{i}}\rangle \langle x, \frac{e_i}{\sqrt{i}}\rangle^* \sum_{k=1}^{i} \|e_k\|^2 \\ &= \langle x, e_i\rangle \langle x, e_i\rangle^*,\end{aligned}$$

so,

$$\sum_{i\in\mathbb{N}^*} \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle = \sum_{i\in\mathbb{N}^*} \langle x, e_i\rangle \langle x, e_i\rangle^* = \langle x, x\rangle.$$

This shows that $\{W_i, \Lambda_i, 1\}_{i\in\mathbb{N}^*}$ is a Parseval $*-g-$fusion frame for H.

Corollary 2.3. *Every $g-$fusion frame for H is a $*-g-$fusion frame for H.*

Proof. Let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $g-$fusion frame for H with bounds A and B. Then for each $x \in H$

$$A\langle x, x\rangle \leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \leq B\langle x, x\rangle,$$

so,

$$(\sqrt{A})1_{\mathcal{A}}\langle x, x\rangle(\sqrt{A})1_{\mathcal{A}} \leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \leq (\sqrt{B})1_{\mathcal{A}}\langle x, x\rangle(\sqrt{B})1_{\mathcal{A}}.$$

Hence Λ is a $*-g-$fusion frame for H with frame bounds $(\sqrt{A})1_{\mathcal{A}}$ and $(\sqrt{B})1_{\mathcal{A}}$. □

Lemma 2.4. *Let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $*-g-$fusion bessel sequence for H with bound B. Then for each sequence $\{x_i\}_{i\in I} \in l^2(\{H_i\}_{i\in I})$, the series $\sum_{i\in I} v_i P_{W_i} \Lambda_i^* x_i$ is convergent unconditionally.*

Proof. Let J be a finite subset of I. Then

$$\begin{aligned}||\sum_{i\in J} v_i P_{W_i} \Lambda_i^* x_i|| &= \sup_{||y||=1} ||\langle \sum_{i\in J} v_i P_{W_i} \Lambda_i^* x_i, y\rangle|| \\ &\leq ||\sum_{i\in J} \langle x_i, x_i\rangle||^{\frac{1}{2}} \sup_{||y||=1} ||\sum_{i\in J} v_i^2 \langle \Lambda_i P_{W_i} y, \Lambda_i P_{W_i} y\rangle||^{\frac{1}{2}} \\ &\leq \|B\|||\sum_{i\in J} \langle x_i, x_i\rangle||^{\frac{1}{2}}.\end{aligned}$$

It implies that $\sum_{j\in I} v_j P_{W_j} \Lambda_j^* f_j$ is unconditionally convergent in H. □

Now, we can define the synthesis operator by Lemma 2.4.

Definition 2.5. Let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $* - g-$fusion bessel sequence for H. Then the operator $T_\Lambda : l^2(\{H_i\}_{i\in I}) \to H$ defined by

$$T_\Lambda(\{x_i\}_{i\in I}) = \sum_{i\in I} v_i P_{W_i}\Lambda_i^* x_i, \qquad \forall \{x_i\}_{i\in I} \in l^2(\{H_i\}_{i\in I})$$

is called asynthesis operator. We say the adjoint T_Λ^* of the synthesis operator is the analysis operator and it is defined by $T_\Lambda^* : \mathcal{H} \to l^2(\{H_i\}_{i\in I})$ such that

$$T_\Lambda^*(x) = \{v_i \Lambda_i P_{W_i}(x)\}_{i\in I}, \qquad \forall x \in H.$$

The operator $S_\Lambda : H \to H$ defined by

$$S_\Lambda x = T_\Lambda T_\Lambda^* x = \sum_{j\in I} v_i^2 P_{W_i}\Lambda_i^*\Lambda_i P_{W_i}(x), \qquad \forall x \in H$$

is called a $* - g-$fusion frame operator. It can be easily verify that

$$\langle S_\Lambda x, x\rangle = \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i}(x), \Lambda_i P_{W_i}(x)\rangle, \qquad \forall x \in H. \tag{2.1}$$

Furthermore, if Λ is a $* - g-$fusion frame with bounds A and B, then

$$A\langle x, x\rangle A^* \leq \langle S_\Lambda x, x\rangle \leq B\langle x, x\rangle B^*, \qquad \forall x \in H.$$

Theorem 2.6. *Let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be $* - g-$fusion frame for H with bounds A and B. Then the synthesis operator T_Λ is surjective and the analysis operator T_Λ^* is injective closed rang with $\|T_\Lambda^*\| \leq \|B\|$.*

Proof. We have for each $x \in H$,

$$A\langle x, x\rangle A^* \leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle B\langle x, x\rangle B^*,$$

then,

$$A\langle x, x\rangle A^* \leq \langle T_\Lambda^* x, T_\Lambda^* x\rangle \leq B\langle x, x\rangle B^*. \tag{2.2}$$

Hence,

$$\|A^{-1}\|^{-1}\|x\| \leq \|T_\Lambda^* x\|, \tag{2.3}$$

so, by Lemma 1.6, T_Λ is surjective.

If $T_\Lambda^* x = 0$, then from (2.3), $x = 0$, hence T_Λ^* is injective. And from (2.3)

$$\|T_\Lambda^*\| \leq \|B\|\|x\|,$$

therefore, $\|T_\Lambda^*\| \leq \|B\|$.

Now show that the range of T_Λ^* is closed. Let $\{T_\Lambda^* x_n\}_{n\in\mathbb{N}}$ be a sequence in the range of T_Λ^* such that $\lim_{n\to\infty} T_\Lambda^* x_n = y$. Then, by (2.3) we have, for $n, m \in \mathbb{N}$,

$$\begin{aligned}\|A\langle x_n - x_m, x_n - x_m\rangle A^*\| &\leq \|\langle T_\Lambda^*(x_n - x_m), T_\Lambda^*(x_n - x_m)\rangle\| \\ &= \|T_\Lambda^*(x_n - x_m)\|^2.\end{aligned}$$

Since $\{T_\Lambda^* x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in H,

$$\|A\langle x_n - x_m, x_n - x_m\rangle A^*\| \to 0, \quad n, m \to \infty.$$

Note that for $n, m \in \mathbb{N}$,

$$\begin{aligned}\|\langle x_n - x_m, x_n - x_m\rangle\| &= \|A^{-1}A\langle x_n - x_m, x_n - x_m\rangle A^*(A^*)^{-1}\| \\ &\leq \|A^{-1}\|^2 \|A\langle x_n - x_m, x_n - x_m\rangle A^*\|.\end{aligned}$$

So the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy and hence there exists $x \in H$ such that $x_n \to x$ as $n \to \infty$. Again by (2.2) we have

$$\|T_\Lambda^*(x_n - x)\|^2 \leq \|B\|^2 \|\|\langle x_n - x, x_n - x\rangle\|.$$

Thus $\|T_\Lambda^* x_n - T_\Lambda^* x\| \to 0$ as $n \to \infty$ implies that $T_\Lambda^* x = y$. It concludes that the range of T_Λ^* is closed. □

Theorem 2.7. *Let $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $* - g-$fusion frame for H with bounds A and B. Then the $* - g-$fusion frame operator S_Λ is positive, self-adjoint, invertible and $\|A^{-1}\|^{-2} \leq \|S\| \leq \|B\|^2$.*

Proof. From the definition of S_Λ^*, the operator S_Λ^* is positive and self-adjoint. By Lemma 1.7 and Theorem 2.6, S_Λ^* is invertible. And we have, for each $x \in H$

$$\langle S_\Lambda^* x, x\rangle = \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \leq B\langle x, x\rangle B^*$$

and

$$\langle x, x\rangle \leq A^{-1} \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle (A^*)^{-1}.$$

Thus,

$$\|A^{-1}\|^2 \|\langle x, x\rangle\| \leq \|\langle S_\Lambda x, x\rangle\| \leq \|B\|^2 \|\langle x, x\rangle\|,$$

so, we have

$$\|A^{-1}\|^2 \leq \|S_\Lambda\| \leq \|B\|^2.$$

□

Theorem 2.8. *Let for every $i \in I$, $\Lambda_i \in End_{\mathcal{A}}^*(H, H_i)$ and $\{y_{i,j}, j \in I_i\}$ (I_i is finite or countable index set) be a $*-$frame for H_i with frame bounds A_i, B_i and there exist A, B strictly nonzero of $\mathcal{A}$ such that*

$$AaA^* \leq A_i a A_i^* \quad \textit{and} \quad B_i a B_i^* \leq BaB^*,$$

for all positive a of $\mathcal{A}$. Then the following statements are equivalent.

(1) *$\{v_i P_{W_i} \Lambda_i^*(y_{i,j}), j \in I_i\}$ is a $*-$frame for H.*

(2) *$\{W_i, \Lambda_i, v_i\}_{i\in I}$ is a $* - g-$fusion frame for H.*

Proof. For each $i \in I$, since $\{y_{i,j}, j \in I_i\}$ is a $*-$frame for H_i with bounds A_i and B_i, we have for each $x \in H$,

$$\begin{aligned} A_i\langle v_i\Lambda_i P_{W_i}x, v_i\Lambda_i P_{W_i}x\rangle A_i^* &\leq \sum_{j\in I_i}\langle v_i\Lambda_i P_{W_i}x, y_{i,j}\rangle\langle y_{i,j}, v_i\Lambda_i P_{W_i}x\rangle \\ &\leq B_i\langle v_i\Lambda_i P_{W_i}x, v_i\Lambda_i P_{W_i}x\rangle B_i^*. \end{aligned}$$

So,

$$\begin{aligned} A_i\langle v_i\Lambda_i P_{W_i}x, v_i\Lambda_i P_{W_i}x\rangle A_i^* &\leq \sum_{j\in I_i}\langle x, v_i P_{W_i}\Lambda_i^* y_{i,j}\rangle\langle v_i P_{W_i}\Lambda_i^* y_{i,j}, x\rangle \\ &\leq B_i\langle v_i\Lambda_i P_{W_i}x, v_i\Lambda_i P_{W_i}x\rangle B_i^*, \end{aligned}$$

then,

$$\begin{aligned} A\langle v_i\Lambda_i P_{W_i}x, v_i\Lambda_i P_{W_i}x\rangle A^* &\leq \sum_{j\in I_i}\langle x, v_i P_{W_i}\Lambda_i^* y_{i,j}\rangle\langle v_i P_{W_i}\Lambda_i^* y_{i,j}, x\rangle \\ &\leq B\langle v_i\Lambda_i P_{W_i}x, v_i\Lambda_i P_{W_i}x\rangle B^*. \end{aligned}$$

Hence,

$$\begin{aligned} A\sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle A^* &\leq \sum_{i\in I}\sum_{j\in I_i}\langle x, v_i P_{W_i}\Lambda_i^* y_{i,j}\rangle\langle v_i P_{W_i}\Lambda_i^* y_{i,j}, x\rangle \\ &\leq B\sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle B^*. \end{aligned} \tag{2.4}$$

And, we suppose that $\{v_i P_{W_i}\Lambda_i^* y_{i,j}, j \in I_i\}$ is a $*-$frame for H with frame bounds $A^{'}$ and $B^{'}$. Then we have, for each $x \in H$

$$\begin{aligned} A^{'}\langle x, x\rangle(A^{'})^* &\leq \sum_{i\in I}\sum_{j\in I_i}\langle x, v_i P_{W_i}\Lambda_i^* y_{i,j}\rangle\langle v_i P_{W_i}\Lambda_i^* y_{i,j}, x\rangle \\ &\leq B^{'}\langle x, x\rangle(B^{'})^*. \end{aligned} \tag{2.5}$$

Hence, by combining (2.4) and (2.5), we have

$$A\sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle A^* \leq B^{'}\langle x, x\rangle(B^{'})^*$$

and

$$A^{'}\langle x, x\rangle(A^{'})^* \leq B\sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle B^*.$$

Therefore, we have

$$\begin{aligned} B^{-1}A^{'}\langle x, x\rangle(A^{'})^*(B^*)^{-1} &\leq \sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle \\ &\leq A^{-1}B^{'}\langle x, x\rangle(B^{'})^*(A^*)^{-1}. \end{aligned}$$

Conversely, suppose that $\{W_i, \Lambda_i, v_i\}_{i\in I}$ is a $* - g-$fusion frame for H with bounds $A^{'}$ and $B^{'}$. Then for each $x \in H$, we have

$$\begin{aligned} A^{'}\langle x, x\rangle (A^{'})^* &\leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \\ &\leq B^{'}\langle x, x\rangle (B^{'})^*. \end{aligned}$$

It follows from (2.4) that

$$AA^{'}\langle x, x\rangle (A^{'})^* A^* \leq A \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle A^*$$

and

$$B \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle B^* \leq BB^{'}\langle x, x\rangle (B^{'})^* B^*.$$

Hence we have

$$\begin{aligned} AA^{'}\langle x, x\rangle (A^{'})^* A^* &\leq \sum_{i\in I} \sum_{j\in I_i} \langle x, v_i P_{W_i} \Lambda_i^* y_{i,j}\rangle \langle v_i P_{W_i} \Lambda_i^* y_{i,j}, x\rangle \\ &\leq BB^{'}\langle x, x\rangle (B^{'})^* B^*. \end{aligned}$$

□

Corollary 2.9. *Let for each $i \in I$, $\Lambda_i \in End^*_{\mathcal{A}}(H, H_i)$ and $\{x_{i,j}, j \in I_i\}$ be a parseval $*-$frame for H_i. Then, we have that*

1. *$\{W_i, \Lambda_i, v_i\}_{i\in I}$ is a $* - g-$fusion frame for H if and only if $\{v_i P_{W_i} \Lambda_i^* x_{i,j}, i \in I, j \in I_i\}$ is a $*-$frame for H.*
2. *the $* - g-$fusion frame operator of $\{W_i, \Lambda_i, v_i\}_{i\in I}$ is the $*-$frame operator of $\{v_i P_{W_i} \Lambda_i^* x_{i,j}, i \in I, j \in I_i\}$.*

Proof. (1) It follows from Theorem 2.8 that (1) is true.

(2) Letting $x, y \in H$, then we get

$$\begin{aligned} \langle v_i P_{W_i} \Lambda_i^* y, x\rangle &= \langle y, v_i \Lambda_i P_{W_i} x\rangle \\ &= \sum_{j\in I_i} \langle y, x_{i,j}\rangle \langle x_{i,j}, v_i \Lambda_i P_{W_i} x\rangle \\ &= \sum_{j\in I_i} \langle y, x_{i,j}\rangle \langle v_i P_{W_i} \Lambda_i^* x_{i,j}, x\rangle, \end{aligned}$$

so, we have

$$v_i P_{W_i} \Lambda_i^* y = \sum_{j\in I_i} \langle y, x_{i,j}\rangle v_i P_{W_i} \Lambda_i^* x_{i,j}.$$

Hence,

$$\begin{aligned}\sum_{i\in I} v_i^2 P_{W_i}\Lambda_i^*\Lambda_i P_{W_i}x &= \sum_{i\in I} v_i P_{W_i}\Lambda_i^*(v_i\Lambda_i P_{W_i}x)\\ &= \sum_{i\in I}\sum_{j\in I_i}\langle v_i\Lambda_i P_{W_i}x, x_{i,j}\rangle v_i P_{W_i}\Lambda_i^* x_{i,j}\\ &= \sum_{i\in I}\sum_{j\in I_i}\langle x, v_i P_{W_i}\Lambda_i^* x_{i,j}\rangle v_i P_{W_i}\Lambda_i^* x_{i,j}.\end{aligned}$$

Then the $*-g-$fusion frame operator of $\{W_i, \Lambda_i, v_i\}_{i\in I}$ is the $*-$frame operator of $\{v_i P_{W_i}\Lambda_i^* x_{i,j}, i \in I, j \in I_i\}$. □

Theorem 2.10. *Let $\{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $*-g-$fusion frame for H with bounds A, B and the $*-g-$fusion frame operator S. If $\theta \in End_{\mathcal{A}}^*(H)$ is injective which has closed range commute with P_{W_i} for each $i \in I$, then $\{W_i, \Lambda_i\theta, v_i\}_{i\in I}$ is a $*-g-$fusion frame for H with $*-g-$fusion frame operator $\theta^* S\theta$.*

Proof. For all $x \in H$, we have

$$A\langle x, x\rangle A^* \leq \sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle \leq B\langle x, x\rangle B^*$$

and

$$\sum_{i\in I} v_i^2\langle \Lambda_i\theta P_{W_i}x, \Lambda_i\theta P_{W_i}x\rangle = \sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}\theta x, \Lambda_i P_{W_i}\theta x\rangle.$$

Then,

$$\begin{aligned}\sum_{i\in I} v_i^2\langle \Lambda_i\theta P_{W_i}x, \Lambda_i\theta P_{W_i}x\rangle &\leq B\langle \theta x, \theta x\rangle B^*\\ &\leq (\|\theta\|B)\langle x, x\rangle(\|\theta\|B)^*. \qquad (2.6)\end{aligned}$$

And also, for each $x \in H$, we have

$$A\langle \theta x, \theta x\rangle A^* \leq \sum_{i\in I} v_i^2\langle \Lambda_i\theta P_{W_i}x, \Lambda_i\theta P_{W_i}x\rangle.$$

Since θ is injective which has closed range,

$$A\|(\theta^*\theta)^{-1}\|^{-1}\langle x, x\rangle A^* \leq A\langle \theta x, \theta x\rangle A^*$$

or,

$$\|\theta^{-1}\|^{-2} \leq \|(\theta^*\theta)^{-1}\|^{-1}.$$

So, we have,

$$(\|\theta^{-1}\|^{-1}A)\langle x, x\rangle(\|\theta^{-1}\|^{-1}A)^* \leq A\langle \theta x, \theta x\rangle,$$

Hence,

$$(\|\theta^{-1}\|^{-1}A)\langle x, x\rangle(\|\theta^{-1}\|^{-1}A)^* \leq \sum_{i\in I} v_i^2\langle \Lambda_i\theta P_{W_i}x, \Lambda_i\theta P_{W_i}x\rangle. \qquad (2.7)$$

From (2.6) and (2.7), we conclude that $\{W_i, \Lambda_i\theta, v_i\}_{i\in I}$ is a $*-g-$fusion frame for H. Let $x \in H$. Then,

$$\begin{aligned}\theta^* S\theta &= \theta^* \sum_{i\in I} v_i^2 P_{W_i}\Lambda_i^*\Lambda_i P_{W_i}\theta x \\ &= \sum_{i\in I} v_i^2 \theta^* P_{W_i}\Lambda_i^*\Lambda_i P_{W_i}\theta x \\ &= \sum_{i\in I} v_i^2 P_{W_i}\theta^*\Lambda_i^*\Lambda_i\theta P_{W_i} x \\ &= \sum_{i\in I} v_i^2 P_{W_i}(\Lambda_i\theta)^*(\Lambda_i\theta) P_{W_i} x.\end{aligned}$$

This completes the proof. □

Corollary 2.11. *Let $\{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $*-g-$fusion frame for H. Then $\{W_i, \Lambda_i S^{-1}, v_i\}_{i\in I}$ is a $*-g-$fusion frame for H with the $*-g-$fusion frame operator S^{-1}.*

Proof. Taking $\theta = S^{-1}$. □

In the next theorem we take $H_i \subseteq H$ for each $i \in I$.

Theorem 2.12. *Let $(H, \mathcal{A}, \langle ., .\rangle_{\mathcal{A}})$ and $(H, \mathcal{B}, \langle ., .\rangle_{\mathcal{B}})$ be two Hilbert C^*-modules and let $\phi : \mathcal{A} \to \mathcal{B}$ be a $*-$homomorphism and θ be a map on H such that $\langle \theta x, \theta y\rangle_{\mathcal{B}} = \phi(\langle x, y\rangle_{\mathcal{A}})$ for all $x, y \in H$. Also suppose that $\{W_i, \Lambda_i, v_i\}_{i\in I}$ is a $*-g-$fusion frame for $(H, \mathcal{A}, \langle ., .\rangle_{\mathcal{A}})$ with $*-g-$fusion frame operator $S_{\mathcal{A}}$ and lower and upper $*-g-$fusion frame bounds A and B, respectively. If θ is surjective and $\theta\Lambda_i P_{W_i} = \Lambda_i P_{W_i}\theta$, for each $i \in I$, then $\{W_i, \Lambda_i, \phi(v_i)\}_{i\in I}$ is a $*-g-$fusion frame for $(H, \mathcal{B}, \langle ., .\rangle_{\mathcal{B}})$ with $*-g-$fusion frame operator $S_{\mathcal{B}}$ and lower and upper $*-g-$fusion frame bounds $\phi(A)$ and $\phi(B)$, respectively, and $\langle S_{\mathcal{B}}\theta x, \theta y\rangle_{\mathcal{B}} = \phi(\langle S_{\mathcal{A}} x, y\rangle_{\mathcal{A}})$.*

Proof. For $y \in H$, there exists $x \in H$ such that $\theta x = y$, and by definition of $*-g-$fusion frame we have

$$A\langle x, x\rangle_{\mathcal{A}} A^* \leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}} B^*.$$

By Lemma 1.5, we have,

$$\begin{aligned}\phi(A)\phi(\langle x, x\rangle_{\mathcal{A}})\phi(A^*) &\leq \phi\left(\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle_{\mathcal{A}}\right) \\ &\leq \phi(B)\phi(\langle x, x\rangle_{\mathcal{A}})\phi(B^*).\end{aligned}$$

From the definition of $*-$homomorphism, we have

$$\begin{aligned}\phi(A)\langle\theta x,\theta x\rangle_{\mathcal{B}}\phi(A)^* &\leq \sum_{i\in I}\phi(v_i^2)\langle\theta\Lambda_iP_{W_i}x,\theta\Lambda_iP_{W_i}x\rangle_{\mathcal{B}}\\ &\leq \phi(B)\langle\theta x,\theta x\rangle_{\mathcal{B}}\phi(B)^*.\end{aligned}$$

Hence,

$$\begin{aligned}\phi(A)\langle\theta x,\theta x\rangle_{\mathcal{B}}\phi(A)^* &\leq \sum_{i\in I}\phi(v_i)^2\langle\Lambda_iP_{W_i}\theta x,\Lambda_iP_{W_i}\theta x\rangle_{\mathcal{B}}\\ &\leq \phi(B)\langle\theta x,\theta x\rangle_{\mathcal{B}}\phi(B)^*,\end{aligned}$$

so, we have

$$\begin{aligned}\phi(A)\langle y,y\rangle_{\mathcal{B}}\phi(A)^* &\leq \sum_{i\in I}\phi(v_i)^2\langle\Lambda_iP_{W_i}y,\Lambda_iP_{W_i}y\rangle_{\mathcal{B}}\\ &\leq \phi(B)\langle y,y\rangle_{\mathcal{B}}\phi(B)^*.\end{aligned}$$

And also, we have

$$\begin{aligned}\phi(\langle S_{\mathcal{A}}x,y\rangle_{\mathcal{A}}) &= \phi\bigg(\langle\sum_{i\in I}v_i^2P_{W_i}\Lambda_i^*\Lambda_iP_{W_i}x,y\rangle_{\mathcal{A}}\bigg)\\ &= \sum_{i\in I}\phi\bigg(v_i^2\langle\Lambda_iP_{W_i}x,\Lambda_iP_{W_i}y\rangle_{\mathcal{A}}\bigg)\\ &= \sum_{i\in I}\phi(v_i)^2\langle\theta\Lambda_iP_{W_i}x,\theta\Lambda_iP_{W_i}y\rangle_{\mathcal{B}}\\ &= \sum_{i\in I}\phi(v_i)^2\langle\Lambda_iP_{W_i}\theta x,\Lambda_iP_{W_i}\theta y\rangle_{\mathcal{B}}\\ &= \langle\sum_{i\in I}\phi(v_i)^2P_{W_i}\Lambda_i^*\Lambda_iP_{W_i}\theta x,\theta y\rangle_{\mathcal{B}}\\ &= \langle S_{\mathcal{B}}\theta x,\theta y\rangle_{\mathcal{B}}.\end{aligned}$$

This completes the proof. □

Theorem 2.13. *If the families $\Lambda=\{W_i,\Lambda_i,v_i\}_{i\in I}$ and $\Gamma=\{V_j,\Gamma_j,w_j\}_{j\in J}$ are $*-g-$fusion frames for H and K with bounds (A,B) and (C,D), respectively, then $\Lambda\otimes\Gamma=\{W_i\otimes V_j,\Lambda_i\otimes\Gamma_j,v_iw_j\}_{i,j}$ is a $*-g-$fusion frame for $H\otimes K$.*

Proof. For each $x\in H$ and $y\in K$, we have

$$A\langle x,x\rangle_{\mathcal{A}}A^*\leq\sum_{i\in I}v_i^2\langle\Lambda_iP_{W_i}x,\Lambda_iP_{W_i}x\rangle_{\mathcal{A}}\leq B\langle x,x\rangle_{\mathcal{A}}B^*,$$

and

$$C\langle y,y\rangle_{\mathcal{B}}C^* \leq \sum_{j\in J} w_j^2\langle \Gamma_j P_{V_j}y, \Gamma_j P_{V_j}y\rangle_{\mathcal{B}} \leq D\langle y,y\rangle_{\mathcal{B}}D^*.$$

Therefore,

$$\begin{aligned}&(A\langle x,x\rangle_{\mathcal{A}}A^*)\otimes(C\langle y,y\rangle_{\mathcal{B}}C^*)\\ &\quad\leq \sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle_{\mathcal{A}} \otimes \sum_{j\in J} w_j^2\langle \Gamma_j P_{V_j}y, \Gamma_j P_{V_j}y\rangle_{\mathcal{B}}\\ &\quad\leq (B\langle x,x\rangle_{\mathcal{A}}B^*)\otimes(D\langle y,y\rangle_{\mathcal{B}}D^*).\end{aligned}$$

Hence, we have

$$\begin{aligned}&(A\otimes C)(\langle x,x\rangle_{\mathcal{A}}\otimes\langle y,y\rangle_{\mathcal{B}})(A^*\otimes C^*)\\ &\quad\leq \sum_{i,j} v_i^2 w_j^2\langle \Lambda_i P_{W_i}x\otimes\Gamma_j P_{V_j}y, \Lambda_i P_{W_i}x\otimes\Gamma_j P_{V_j}y\rangle_{\mathcal{A}\otimes\mathcal{B}}\\ &\quad\leq (B\otimes D)(\langle x,x\rangle_{\mathcal{A}}\otimes\langle y,y\rangle_{\mathcal{B}})(B^*\otimes D^*),\end{aligned}$$

it implies that

$$\begin{aligned}&(A\otimes C)\langle x\otimes y, x\otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}}(A\otimes C)^*\\ &\quad\leq \sum_{i,j} v_i^2 w_j^2\langle (\Lambda_i\otimes\Gamma_j)P_{W_i\otimes V_j}(x\otimes y), (\Lambda_i\otimes\Gamma_j)P_{W_i\otimes V_j}(x\otimes y)\rangle_{\mathcal{A}\otimes\mathcal{B}}\\ &\quad\leq (B\otimes D)\langle x\otimes y, x\otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}}(B\otimes D)^*.\end{aligned}$$

Therefore $\Lambda\otimes\Gamma$ is a $*-g-$fusion frame for $H\otimes K$ with bounds $A\otimes C$ and $B\otimes D$. □

3. $*-K-g-$fusion frame in Hilbert C^*-module

Definition 3.1. Let $K\in End_{\mathcal{A}}^*(H)$, $\{\Lambda_i\}_{i\in I}\subseteq\{End_{\mathcal{A}}^*(H,H_i), i\in I\}$, $\{W_i\}_{i\in I}$ be a family of closed orthogonally complemented submodules of H and $\{v_i\}_{i\in I}$ be a family of weights in $\mathcal{A}$, that is, each v_i is a positive invertible element from the center of C^*-algebra $\mathcal{A}$. We say $\Lambda=\{W_i,\Lambda_i,v_i\}_{i\in I}$ is a $*-K-g-$fusion frame for H if there exist A,B strictly non-zeros of $\mathcal{A}$ such that for each $x\in H$

$$A\langle K^*x,K^*x\rangle A^* \leq \sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle \leq B\langle x,x\rangle B^*.$$

Example 3.2. Let H be an ordinary inner product space, $I=\mathbb{N}^*$ be the set of all nonnegative integers and $\{e_i\}_{i\in\mathbb{N}^*}$ be an orthonormal basis for Hilbert C^*-module H.

We construct $H_i = \overline{span}\{e_1, ..., e_i\}$ and $W_i = \overline{span}\{e_i\}$, $i \in I$. Define $\Lambda_i : H \to H_i$ by

$$\Lambda_i x = \sum_{k=1}^{i} \langle x, \frac{e_k}{\sqrt{i}} \rangle e_k,$$

the adjoint operator $\Lambda_i^* : H_i \to H$ define as

$$\Lambda_i^* x = \sum_{k=1}^{i} \langle x, \frac{e_k}{\sqrt{i}} \rangle e_i.$$

Then for each $x \in H$, we have

$$\begin{aligned}\langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x \rangle &= \langle \sum_{k=1}^{i} \langle x, \frac{e_k}{\sqrt{i}} \rangle e_k, \sum_{k=1}^{i} \langle x, \frac{e_k}{\sqrt{i}} \rangle e_k \rangle \\ &= \langle x, \frac{e_i}{\sqrt{i}} \rangle \langle x, \frac{e_i}{\sqrt{i}} \rangle^* \sum_{k=1}^{i} \|e_k\|^2 \\ &= \langle x, e_i \rangle \langle x, e_i \rangle^*,\end{aligned}$$

it implies that

$$\begin{aligned}\sum_{i \in \mathbb{N}^*} \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x \rangle &= \sum_{i \in \mathbb{N}^*} \langle x, e_i \rangle \langle x, e_i \rangle^* \\ &= \langle x, x \rangle.\end{aligned}$$

Fix $N \in \mathbb{N}^*$ and define $K : H \to H$ by

$$Ke_i = \begin{cases} ie_i & if \quad i \leq N, \\ 0 & if \quad i > N. \end{cases}$$

Then it is easy to check that K is adjointable and satisfies

$$K^* e_i = \begin{cases} ie_i & if \quad i \leq N, \\ 0 & if \quad i > N. \end{cases}$$

For any $x \in H$ we have

$$\begin{aligned}\frac{1}{N^2} \langle K^* x, K^* x \rangle &\leq \sum_{i \in \mathbb{N}^*} \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x \rangle \\ &= \langle x, x \rangle.\end{aligned}$$

This shows that $\{W_i, \Lambda_i, 1\}_{i \in \mathbb{N}^*}$ is a $* - K - g-$fusion frame with bounds $\frac{1}{N^2}$ and 1.

Theorem 3.3. *Let $U \in End^*_{\mathcal{A}}(H)$ be invertible, $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $* - g-$fusion frame for H with bounds A and B. Suppose that $U^*UW_i \subseteq W_i$, for all $i \in I$. Then $\Gamma = \{UW_i, \Lambda_i P_{W_i} U^*, v_i\}_{i\in I}$ is a $* - UKU^* - g-$fusion frame for H.*

Proof. For each $x \in H$, we have

$$\begin{aligned} A\langle K^*x, K^*x\rangle A^* &\leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \\ &\leq B\langle x, x\rangle B^*. \end{aligned}$$

And for each $x \in h$

$$\begin{aligned} &\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} U^* P_{UW_i} x, \Lambda_i P_{W_i} U^* P_{UW_i} x\rangle \\ &\quad = \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} U^* x, \Lambda_i P_{W_i} U^* x\rangle \\ &\quad \leq B\langle U^*x, U^*x\rangle B^* \\ &\quad \leq (B\|U\|)\langle x, x\rangle (B\|U\|)^*. \end{aligned} \tag{3.1}$$

On the other hand, for each $x \in H$

$$\begin{aligned} &A\langle (UKU^*)^*x, (UKU^*)^*x\rangle A^* \\ &\quad = A\langle UK^*U^*x, UK^*U^*x\rangle A^* \\ &\quad \leq A\|U\|^2 \langle K^*U^*x, K^*U^*x\rangle A^* \\ &\quad \leq \|U\|^2 \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} U^* x, \Lambda_i P_{W_i} U^* x\rangle \\ &\quad = \|U\|^2 \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} U^* P_{UW_i} x, \Lambda_i P_{W_i} U^* P_{UW_i} x\rangle \end{aligned}$$

and

$$\begin{aligned} &(\|U\|^{-1}A)\langle (UKU^*)^*x, (UKU^*)^*x\rangle (\|U\|^{-1}A)^* \\ &\quad \leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} U^* P_{UW_i} x, \Lambda_i P_{W_i} U^* P_{UW_i} x\rangle. \end{aligned} \tag{3.2}$$

From (3.1) and (3.3), we conclude that Γ is a $* - UKU^* - g-$fusion frame for H. □

Theorem 3.4. *Let $U \in End^*_{\mathcal{A}}(H)$ be invertible and $\Gamma = \{UW_i, \Lambda_i P_{W_i} U^*, v_i\}_{i\in I}$ be a $K - g-$fusion frame for H with bounds A and B for some $K \in End^*_{\mathcal{A}}(H)$. Suppose that $U^*UW_i \subseteq W_i$. Then $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ is a $U^{-1}KU - g-$fusion frame for H.*

Proof. For each $x \in H$, we have

$$\begin{aligned}
&\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \\
&\qquad = \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} U^*(U^*)^{-1}x, \Lambda_i P_{W_i} U^*(U^*)^{-1}x\rangle \\
&\qquad = \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} U^* P_{UW_i}(U^*)^{-1}x, \Lambda_i P_{W_i} U^* P_{UW_i}(U^*)^{-1}x\rangle \\
&\qquad \leq B\langle (U^*)^{-1}x, (U^*)^{-1}x\rangle B^* \\
&\qquad \leq (\|U^{-1}\|B)\langle x, x\rangle (\|U^{-1}\|B)^*. \qquad (3.3)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&A\langle (U^{-1}KU)^*x, (U^{-1}KU)^*x\rangle A^* = A\langle U^*K^*(U^{-1})^*x, U^*K^*(U^{-1})^*x\rangle A^* \\
&\quad \leq A\|U\|^2\langle K^*(U^{-1})^*x, K^*(U^{-1})^*x\rangle A^* \\
&\quad \leq \|U\|^2 \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} U^* P_{UW_i}(U^{-1})^*x, \Lambda_i P_{W_i} U^* P_{UW_i}(U^{-1})^*x\rangle \\
&\quad = \|U\|^2 \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} U^*(U^{-1})^*x, \Lambda_i P_{W_i} U^*(U^{-1})^*x \\
&\quad = \|U\|^2 \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle,
\end{aligned}$$

it implies that

$$\begin{aligned}
&(\|U\|^{-1}A)\langle (U^{-1}KU)^*x, (U^{-1}KU)^*x\rangle (\|U\|^{-1}A)^* \\
&\qquad \leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle. \qquad (3.4)
\end{aligned}$$

From (3.3) and (3.4), Λ is a $* - U^{-1}KU - g-$fusion frame for H. □

Theorem 3.5. *Let $K \in End_{\mathcal{A}}^*(H)$ be invertible and $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $* - g-$fusion frame for H with bounds A and B and S_Λ be the associated $* - g-$fusion frame operator. Suppose that $T^*TW_i \subseteq W_i$ where $T = KS_\Lambda^{-1}$. Then $\Gamma = \{TW_i, \Lambda_i P_{W_i} T^*, v_i\}_{i\in I}$ is a $K - g-$fusion frame for H with corresponding $* - g-$fusion frame operator $KS_\Lambda^{-1}K^*$.*

Proof. For each $x \in H$, wer have

$$\begin{aligned}
\langle K^*x, K^*\rangle &= \langle S_\Lambda S_\Lambda^{-1}K^*x, S_\Lambda S_\Lambda^{-1}K^*x\rangle \\
&\leq \|S_\Lambda\|^2 \langle S_\Lambda^{-1}K^*x, S_\Lambda^{-1}K^*x\rangle \\
&\leq \|B\|^2 \langle S_\Lambda^{-1}K^*x, S_\Lambda^{-1}K^*x\rangle.
\end{aligned}$$

And also, for each $x \in H$,

$$\begin{aligned}\sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i} T^* P_{TW_i}x, \Lambda_i P_{W_i} T^* P_{TW_i}x\rangle &= \sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i} T^* x, \Lambda_i P_{W_i} T^* x\rangle \\ &\leq B\langle T^*x, T^*x\rangle B^* \\ &\leq (\|T\|B)\langle x, x\rangle(\|T\|B)^*. \qquad (3.5)\end{aligned}$$

On the other hand, for each $x \in H$

$$\begin{aligned}&\sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i} T^* P_{TW_i}x, \Lambda_i P_{W_i} T^* P_{TW_i}x\rangle \\ &\quad = \sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i} T^* x, \Lambda_i P_{W_i} T^* x\rangle \\ &\quad \geq A\langle T^*x, T^*x\rangle A^* \\ &\quad = A\langle S_\Lambda^{-1}K^*x, S_\Lambda^{-1}K^*x\rangle A^* \\ &\quad \geq (\|B\|^{-1}A)\langle K^*x, K^*x\rangle(\|B\|^{-1}A)^*. \qquad (3.6)\end{aligned}$$

From (3.5) and (3.6), $\Gamma = \{TW_i, \Lambda_i P_{W_i} T^*, v_i\}_{i\in I}$ is a $* - K - g-$fusion frame for H.

Next, for each $x \in H$,

$$\begin{aligned}&\sum_{i\in I} v_i^2 P_{TW_i}(\Lambda_i P_{W_i} T^*)^*(\Lambda_i P_{W_i} T^*)^* P_{TW_i}x \\ &\quad = \sum_{i\in I} v_i^2 P_{TW_i} T P_{W_i}\Lambda_i^*(\Lambda_i P_{W_i} T^*)P_{TW_i}x \\ &\quad = \sum_{i\in I} v_i^2 (P_{W_i} T^* P_{TW_i})^*\Lambda_i^*\Lambda_i(P_{W_i} T^* P_{TW_i})x \\ &\quad = T\big(\sum_{i\in I} v_i^2 P_{W_i}\Lambda_i^*\Lambda_i P_{W_i} T^* x\big) \\ &\quad = TS_\Lambda T^*x = KS_\Lambda^{-1}K^*x.\end{aligned}$$

Thus, $KS_\Lambda^{-1}K^*$ is the associated $* - g-$fusion frame operator. □

Theorem 3.6. *Let $K \in End_{\mathcal{A}}^*(H)$ and $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ be a $* - K - g-$fusion frame for H with frame operator S_Λ and frame bounds A and B. Let $U \in End_{\mathcal{A}}^*(H)$ be an invertible operator on H, and $U^*UW_i \subset W_i$, for all $i \in I$. Then the following statements are equivalent:*

(1) *$\Gamma = \{UW_i, \Lambda_i P_{W_i} U^*, v_i\}_{i\in I}$ is $* - UK - g-$fusion frame for H.*
(2) *The quotient operator $\left[(UK)^*/S_\Lambda^{\frac{1}{2}}U^*\right]$ is bounded.*
(3) *The quotient operator $\left[(UK)^*/(US_\Lambda U^*)^{\frac{1}{2}}\right]$ is bounded.*

Proof. (1) $\implies$ (2) Since Γ is $*-UK-g-$fusion frame for H, there exist $A, B \in \mathcal{A}-\{0\}$ such that for each $x \in H$,

$$\begin{aligned} A\langle (UK)^*x, (UK)^*x\rangle A^* &\leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} U^* P_{UW_i} x, \Lambda_i P_{W_i} U^* P_{UW_i} x\rangle \\ &\leq A\langle x, x\rangle A^* \end{aligned}$$

and

$$\begin{aligned} \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} U^* P_{UW_i} x, \Lambda_i P_{W_i} U^* P_{UW_i} x\rangle &= \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} U^* x, \Lambda_i P_{W_i} U^* x\rangle \\ &= \langle S_\Lambda U^* x, U^* x\rangle \\ &= \langle S_\Lambda^{\frac{1}{2}} U^* x, S_\Lambda^{\frac{1}{2}} U^* x\rangle. \end{aligned}$$

Hence we have

$$A\langle (UK)^*x, (UK)^*x\rangle A^* \leq \langle S_\Lambda^{\frac{1}{2}} U^* x, S_\Lambda^{\frac{1}{2}} U^* x\rangle.$$

We define the operator $T : \mathcal{R}(S^{\frac{1}{2}} U^*) \to \mathcal{R}((UK)^*)$ by

$$T(S^{\frac{1}{2}} U^* x) = (UK)^* x.$$

Then, from $Ker(S_\Lambda^{\frac{1}{2}} U^*) \subseteq Ker((UK)^*)$, T is a well-defined quotient operator. And for each $x \in H$,

$$\begin{aligned} \|T(S_\Lambda^{\frac{1}{2}} U^* x)\| &= \|(UK)^* x\| \\ &= \|\langle (UK)^* x, (UK)^* x\rangle\|^{\frac{1}{2}} \\ &\leq \|A^{-1}\langle S_\Lambda^{\frac{1}{2}} U^* x, S_\Lambda^{\frac{1}{2}} U^* x\rangle (A^*)^{-1}\|^{\frac{1}{2}} \\ &\leq \|A^{-1}\| \|\langle S_\Lambda^{\frac{1}{2}} U^* x, S_\Lambda^{\frac{1}{2}} U^* x\rangle\|^{\frac{1}{2}} \\ &\leq \|A^{-1}\| \|S_\Lambda^{\frac{1}{2}} U^* x\|, \end{aligned}$$

it implies that T is bounded.

(2) $\Longrightarrow$ (3) Suppose that the quotient operator $\left[(UK)^*/S_\Lambda^{\frac{1}{2}}U^*\right]$ is bounded. Then for all $x \in H$, there exist $C > 0$ such that

$$\begin{aligned}\|(UK)^*x\| &\leq C\|S_\Lambda^{\frac{1}{2}}U^*x\| \\ &= C\|\langle S_\Lambda^{\frac{1}{2}}U^*x, S_\Lambda^{\frac{1}{2}}U^*x\rangle\|^{\frac{1}{2}} \\ &= C\|\langle US_\Lambda U^*x, x\rangle\|^{\frac{1}{2}} \\ &= C\|\langle (US_\Lambda U^*)^{\frac{1}{2}}x, (US_\Lambda U^*)^{\frac{1}{2}}x\rangle\|^{\frac{1}{2}} \\ &= C\|(US_\Lambda U^*)^{\frac{1}{2}}x\|.\end{aligned}$$

(3) $\Longrightarrow$ (1) Since Λ is $* - K - g-$fusion frame for H with bounds A and B, for each $x \in H$,

$$A\langle K^*x, K^*x\rangle A^* \leq \sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}x, \Lambda_i P_{W_i}x\rangle \leq B\langle x, x\rangle B^*.$$

And also, for each $x \in H$,

$$\begin{aligned}\sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}U^*P_{UW_i}x, \Lambda_i P_{W_i}U^*P_{UW_i}x\rangle &= \sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}U^*x, \Lambda_i P_{W_i}U^*x\rangle \\ &\leq B\langle U^*x, U^*x\rangle B^* \\ &\leq (B\|U\|)\langle x, x\rangle (B\|U\|)^*. \qquad (3.7)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}U^*P_{UW_i}x, \Lambda_i P_{W_i}U^*P_{UW_i}x\rangle &= \sum_{i\in I} v_i^2\langle \Lambda_i P_{W_i}U^*x, \Lambda_i P_{W_i}U^*x\rangle \\ &\geq A\langle K^*U^*x, K^*U^*x\rangle A^* \\ &= A\langle (UK)^*x, (UK)^*x\rangle A^*. \qquad (3.8)\end{aligned}$$

Thus, from (3.7) and (3.8), we conclude that $\Gamma = \{UW_i, \Lambda_i P_{W_i}U^*, v_i\}_{i\in I}$ is $* - UK - g-$fusion frame for H. This completes the proof. □

4. Conclusions

In this work, we present the notions of $* - g-$fusion frame and $* - K - g-$fusion frame in Hilbert C^*-modules, we gives some properties and study the tensor product of $* - g-$fusion frame. We also give illustrative examples to exhibit the utility of our results. Our results generalize and extend various results in the existing literature.

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