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ORTHOGONAL PEXIDER HOM-DERIVATIONS IN BANACH ALGEBRAS

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Abstract. In the present paper, we introduce a new system of functional equations, known as orthogonal Pexider hom-derivation and Pexider hom-Pexider derivation (briefly, (Pexider) hom-derivation). Using the fixed point method, we investigate the stability of Pexider hom-derivations and (Pexider) hom-derivations on Banach algebras.

1. Introduction and preliminaries

A classical question in the sense of functional equation says that "when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?" Ulam [28] raised the stability of functional equations and Hyers [9] was the first one which gave an affirmative answer to the question of Ulam for additive mapping between Banach spaces. Hyers' Theorem was generalized by Rassias [27] for linear

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mappings. Therefore, Rassias [26] by using Rassias theorem changed the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^p$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. A generalization of the theorem of Rassias was obtained by Găvruta [7] by replacing the factor of Rassias theorem by a general control function $\varphi : X \times X \longrightarrow [0, \infty)$. The study of stability problem functional equations has been done by several authors on different functional equations (see [1, 4, 6, 10, 12, 13, 15, 16, 17, 19, 20, 21, 22, 23]).

There are several orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, (semi-)inner product, Singer, Carlsson, unitary–Boussouis, Roberts, Pythagorean and Diminnie (see [2, 3]). But we present the orthogonality concept introduced by Rätz [25]. This is given in the following definition.

Definition 1.1. ([25]) Suppose that X is a real vector space (or an algebra) with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

- (O_1) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;
- (O_2) independence: if $x, y \in X - \{0\}$, $x \perp y$, then x, y are linearly independent;
- (O_3) homogeneity: if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
- (O_4) the Thalesian property: if P is a $2-$dimensional subspace (subalgebra) of X, $x \in P$ and $\lambda \in R_+$, then there exists $u_x \in P$ such that $x \perp u_x$ and $x + u_x \perp \lambda x - u_x$.

The pair $(X, \perp)$ is called an orthogonality space (orthogonality algebra). By an orthogonality normed space (orthogonality normed algebra) we mean an orthogonality space (orthogonality algebra) having a normed structure. The orthogonal Cauchy functional equation $f(x + y) = f(x) + f(y)$, $x \perp y$ in which $\perp$ is an abstract orthogonality relation was first investigated in [8]. A generalized version of Cauchy equation is the equation of Pexider type $f_1(x+y) = f_2(x)+f_3(y)$. Jun *et al.* [11, 14] obtained the Hyers-Ulam stability of this Pexider equation. Park *et al.* [24] defined hom-derivation and proved the Hyers-Ulam stability of the hom-derivation in Banach algebras.

In this paper, we may define orthogonally Pexider hom-derivation associated to the Pexiderized Cauchy functional equation.

Definition 1.2. Let $(\mathfrak{A}, \perp)$ be an orthogonality normed algebra and let D, D_1, $D_2 : \mathfrak{A} \longrightarrow \mathfrak{A}$ be mappings satisfying

$$D(x + y) = D_1(x) + D_2(y),$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$. Then we call the triple (D, D_1, D_2) an orthogonal Pexider hom-derivation if there is a homomorphism $H : \mathfrak{A} \to \mathfrak{A}$ such that

$$D(xy) = D_1(x)H(y) + H(x)D_2(y)$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$.

Definition 1.3. Let $(\mathfrak{A}, \perp)$ be an orthogonality normed algebra and let D, D_1, $D_2 : \mathfrak{A} \longrightarrow \mathfrak{A}$ be mappings satisfying

$$D(x+y) = D_1(x) + D_2(y)$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$. Then we call the triple (D, D_1, D_2) an orthogonal Pexider hom-Pexider derivation (briefly, (Pexider) hom-derivation) if there are two homomorphisms $H_1, H_2 : \mathfrak{A} \to \mathfrak{A}$ such that

$$D(xy) = D_1(x)H_1(y) + H_2(x)D_2(y)$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$.

Theorem 1.4. ([18]) *Suppose that (X, d) is a complete generalized metric space and $T : X \to X$ is a strictly contractive mapping with the Lipschitz constant L. Then for any $x \in X$, either*

$$d(T^m x, T^{m+1} x) = \infty, \quad \forall m \geq 0,$$

or there exists a natural number m_0 such that

(1) *$d(T^m x, T^{m+1} x) < \infty$ for all $m \geq m_0$;*
(3) *the sequence $\{T^m x\}$ is convergent to a fixed point y^* of T;*
(3) *y^* is the unique fixed point of T in $\Lambda = \{y \in X : d(T^{m_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.*

In this paper, we prove the Hyers-Ulam stability of Pexider hom-derivations and (Pexider) hom-derivations on Banach algebras.

2. Main results

Throughout this paper, assume that $\mathfrak{A}$ is a Banach algebra. Suppose that φ and ϕ are two functions from $\mathfrak{A}^2$ into $[0, \infty)$ satisfying, for all $x, y \in \mathfrak{A}$ with $x \perp y$, $j \in \{-1, 1\}$,

$$\varphi(x, y) \leq 2^j L \varphi(\frac{1}{2^j} x, \frac{1}{2^j} y) \tag{2.1}$$

and

$$\phi(x, y) \leq 2^{2j} L \phi(\frac{1}{2^j} x, \frac{1}{2^j} y) \tag{2.2}$$

for some constant $0 < L = L(j) < 1$.

Now we are ready to prove the Hyers-Ulam stability of orthogonal Pexider hom-derivations on Banach algebras.

Theorem 2.1. *Suppose that $f, f_1, f_2, h : \mathfrak{A} \to \mathfrak{A}$ are mappings fulfilling the system of functional inequalities*

$$\|f(x+y) - f_1(x) - f_2(y)\| \leq \varphi(x,y), \tag{2.3}$$

$$\|h(x+y) - h(x) - h(y)\| \leq \varphi(x,y), \tag{2.4}$$

$$\|h(xy) - h(x)h(y)\| \leq \phi(x,y), \tag{2.5}$$

$$\|f(xy) - f_1(x)h(y) - h(x)f_2(y)\| \leq \phi(x,y), \tag{2.6}$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$, where φ and ϕ are defined as (2.1) and (2.2). If f is an odd mapping, $\varphi(0,0) = \phi(0,0) = 0$, such that for any fixed $x \in \mathfrak{A}$ and some $u_x \in \mathfrak{A}$ with $x \perp u_x$, the mapping

$$\begin{aligned} x \mapsto \psi(x,u_x) =& \varphi(\frac{x+u_x}{2}, \frac{x-u_x}{2}) + \varphi(0, \frac{x-u_x}{2}) \\ &+ \varphi(\frac{x+u_x}{2}, 0) + \varphi(\frac{x}{2}, \frac{u_x}{2}) + \varphi(\frac{x}{2}, \frac{-u_x}{2}) \\ &+ 2\varphi(\frac{x}{2}, 0) + \varphi(0, \frac{u_x}{2}) + \varphi(0, \frac{-u_x}{2}) \end{aligned} \tag{2.7}$$

has the property

$$\psi(x,u_x) \leq 2^j L \psi(\frac{x}{2^j}, \frac{u_x}{2^j}), \tag{2.8}$$

then there exist a unique orthogonal homomorphism $H : \mathfrak{A} \longrightarrow \mathfrak{A}$ and unique orthogonal hom-derivation $D : \mathfrak{A} \to \mathfrak{A}$ such that

$$\begin{aligned} &\|f(x) - D(x)\| \leq \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x,u_x), \\ &\|f_1(x) - f_1(0) - D(x)\| \leq \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x,u_x) + \varphi(x,0), \\ &\|f_2(x) - f_2(0) - D(x)\| \leq \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x,u_x) + \varphi(0,x) \end{aligned} \tag{2.9}$$

and

$$\|h(x) - H(x)\| \leq \frac{L}{1-L}\varphi(x,x), \tag{2.10}$$

for all $x \in \mathfrak{A}$.

Proof. By the same procedure as in the proof of [5, Theorem 2.1], there exists a unique Pexider additive mapping $D : \mathfrak{A} \to \mathfrak{A}$ satisfying (2.9) which is given by

$$D(x) = \lim_{n\to\infty} \frac{f(2^{nj}x)}{2^{nj}} = \lim_{n\to\infty} \frac{f_1(2^{nj}x)}{2^{nj}} = \lim_{n\to\infty} \frac{f_2(2^{nj}x)}{2^{nj}}.$$

Similarly, there exists a unique additive mapping $H : \mathfrak{A} \to \mathfrak{A}$ satisfying (2.10) which is given by

$$H(x) = \lim_{n\to\infty} \frac{h(2^{nj}x)}{2^{nj}}.$$

It follows from (2.5) that

$$\begin{aligned} \|H(xy) - H(x)H(y)\| &= \lim_{n\to\infty} \left\| \frac{h\big(2^{nj}(xy)\big)}{2^{nj}} - h\Big(\frac{2^{nj}x}{2^{nj}}\Big)h\Big(\frac{2^{nj}y}{2^{nj}}\Big) \right\| \\ &\le \lim_{n\to\infty} \frac{1}{2^{2nj}}\phi(2^{nj}x, 2^{nj}y) \\ &\le \lim_{n\to\infty} \frac{L}{2^{2nj}}\phi(2^{nj}x, 2^{nj}y) = 0 \end{aligned} \tag{2.11}$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$. Therefore

$$H(xy) = H(x)H(y)$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$. It follows from (2.6) that

$$\begin{aligned} &\|D(xy) - D_1(x)H(y) - H(x)D_2(y)\| \\ &\quad = \lim_{n\to\infty} \left\| \frac{f\big(2^{nj}(xy)\big)}{2^{nj}} - f_1\Big(\frac{2^{nj}x}{2^{nj}}\Big)h\Big(\frac{2^{nj}y}{2^{nj}}\Big) - h\Big(\frac{2^{nj}x}{2^{nj}}\Big)f_2\Big(\frac{2^{nj}y}{2^{nj}}\Big) \right\| \\ &\quad \le \lim_{n\to\infty} \frac{1}{2^{2nj}}\phi(2^{nj}x, 2^{nj}y) \\ &\quad \le \lim_{n\to\infty} \frac{L}{2^{2nj}}\phi(2^{nj}x, 2^{nj}y) = 0 \end{aligned} \tag{2.12}$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$. Therefore,

$$D(xy) = D_1(x)H(y) + H(x)D_2(y)$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$. The proof is completed. □

In the next theorem, we prove the Hyers-Ulam stability of orthogonal (Pexider) hom-derivations on Banach algebras.

Theorem 2.2. *Suppose that $f, f_1, f_2, h_1, h_2 : \mathfrak{A} \to \mathfrak{A}$ are odd mappings fulfilling the system of functional inequalities*

$$\|f(x+y) - f_1(x) - f_2(y)\| \le \varphi(x, y), \tag{2.13}$$

$$\|h(x+y) - h_1(x) - h_2(y)\| \le \varphi(x, y), \tag{2.14}$$

$$\|h(xy) - h_1(x)h_2(y)\| \le \phi(x, y), \tag{2.15}$$

$$\|f(xy) - f_1(x)h_1(y) - h_2(x)f_2(y)\| \le \phi(x, y), \tag{2.16}$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$, where φ and ϕ are defined as (2.1) *and* (2.2)*, such that for any fixed $x \in \mathfrak{A}$ and some $u_x \in \mathfrak{A}$ with $x \perp u_x$, the mapping*

$$\begin{aligned}\psi(x,u_x) &= \varphi(\frac{x+u_x}{2},\frac{x-u_x}{2}) + \varphi(0,\frac{x-u_x}{2}) \\ &+ \varphi(\frac{x+u_x}{2},0) + \varphi(\frac{x}{2},\frac{u_x}{2}) + \varphi(\frac{x}{2},\frac{-u_x}{2}) \\ &+ 2\varphi(\frac{x}{2},0) + \varphi(0,\frac{u_x}{2}) + \varphi(0,\frac{-u_x}{2})\end{aligned} \tag{2.17}$$

has the property

$$\psi(x,u_x) \le L2^j\psi(\frac{x}{2^j},\frac{u_x}{2^j}). \tag{2.18}$$

Then there exist a unique orthogonal homomorphism $H : \mathfrak{A} \longrightarrow \mathfrak{A}$ and a unique orthogonal hom-derivation $D : \mathfrak{A} \to \mathfrak{A}$ such that

$$\begin{aligned}&\|f(x) - D(x)\| \le \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x,u_x), \\ &\|f_1(x) - f_1(0) - D(x)\| \le \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x,u_x) + \varphi(x,0), \\ &\|f_2(x) - f_2(0) - D(x)\| \le \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x,u_x) + \varphi(0,x)\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}&\|h(x) - H(x)\| \le \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x,u_x), \\ &\|h_1(x) - h_1(0) - H(x)\| \le \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x,u_x) + \varphi(x,0), \\ &\|h_2(x) - h_2(0) - H(x)\| \le \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x,u_x) + \varphi(0,x)\end{aligned} \tag{2.20}$$

for all $x \in \mathfrak{A}$.

Proof. By the same reasoning as in the proof of Theorem 2.1, there are unique additive mappings $D, H_1, H_2 : \mathfrak{A} \to \mathfrak{A}$ satisfying (2.19) and (2.20), respectively, which are given by

$$\begin{aligned}D(x) &= \lim_{n\to\infty}\frac{f(2^{nj}x)}{2^{nj}} = \lim_{n\to\infty}\frac{f_1(2^{nj}x)}{2^{nj}} = \lim_{n\to\infty}\frac{f_2(2^{nj}x)}{2^{nj}}, \\ H(x) &= \lim_{n\to\infty}\frac{h(2^{nj}x)}{2^{nj}} = \lim_{n\to\infty}\frac{h_1(2^{nj}x)}{2^{nj}} = \lim_{n\to\infty}\frac{h_2(2^{nj}x)}{2^{nj}}\end{aligned} \tag{2.21}$$

for all $x \in \mathfrak{A}$. It follows from (2.15) and (2.21) that

$$\begin{aligned}\|H(xy) - H_1(x)H_2(y)\| &= \lim_{n\to\infty} \left\| \frac{h\big(2^{nj}(xy)\big)}{2^{nj}} - h_1\Big(\frac{2^{nj}x}{2^{nj}}\Big)h_2\Big(\frac{2^{nj}y}{2^{nj}}\Big) \right\| \\ &\le \lim_{n\to\infty} \frac{1}{2^{2nj}}\phi(2^{nj}x, 2^{nj}y) \\ &\le \lim_{n\to\infty} \frac{L}{2^{2nj}}\phi(2^{nj}x, 2^{nj}y) \\ &= 0\end{aligned}$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$. Therefore

$$H(xy) = H_1(x)H_2(y)$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$. It follows from (2.16) and (2.21) that

$$\begin{aligned}&\|D(xy) - D_1(x)H_1(y) - H_1(x)D_2(y)\| \\ &\quad= \lim_{n\to\infty} \left\| \frac{f\big(2^{nj}(xy)\big)}{2^{nj}} - f_1\Big(\frac{2^{nj}x}{2^{nj}}\Big)h_1\Big(\frac{2^{nj}y}{2^{nj}}\Big) - h_2\Big(\frac{2^{nj}x}{2^{nj}}\Big)f_2\Big(\frac{2^{nj}y}{2^{nj}}\Big) \right\| \\ &\quad\le \lim_{n\to\infty} \frac{1}{2^{2nj}}\phi(2^{nj}x, 2^{nj}y) \\ &\quad\le \lim_{n\to\infty} \frac{L}{2^{2nj}}\phi(2^{nj}x, 2^{nj}y) \\ &\quad= 0\end{aligned}$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$. Therefore

$$D(xy) = D_1(x)H_1(y) + H_2(x)D_2(y)$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$. The proof of Theorem 2.2 is now complete. □

Theorems 2.1 and 2.2 generalize the result of Rassias [27], that is, if we define in Theorems 2.1 and 2.2

$$\varphi(x, y) := \theta\Big(\|x\|^p + \|y\|^p\Big), \quad \phi(x, y) := \theta\Big(\|x\|^s + \|y\|^s\Big)$$

for all $\varepsilon, \theta \in \mathbb{R}^+$ and $p, s \neq 1$, then one gets the following corollaries.

Corollary 2.3. *Let $j \in \{-1, 1\}$ and $f, f_1, f_2, h : \mathfrak{A} \longrightarrow \mathfrak{A}$ be mappings satisfying*

$$\|f(x + y) - f_1(x) - f_2(y)\| \le \varepsilon(\|x\|^p + \|y\|^p),$$

$$\|h(x + y) - h(x) - h(y)\| \le \varepsilon(\|x\|^p + \|y\|^p),$$

$$\|h(xy) - h(x)h(y)\| \le \theta\|x\|^s\|y\|^s$$

and

$$\|f(xy) - f_1(x)h(y) - h(x)f_2(y)\| \le \theta\|x\|^s\|y\|^s$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$, $\varepsilon, \theta \geq 0$ and real numbers p, s such that $p, s < 1$ for $j = 1$. If f is an odd mapping, then there exist a unique orthogonal homomorphism $H : \mathfrak{A} \longrightarrow \mathfrak{A}$ and a unique orthogonal hom-derivation $D : \mathfrak{A} \to \mathfrak{A}$ such that

$$\|f(x) - D(x)\| \leq \frac{2^{\frac{j(1+j)(p-1)}{2}}}{1 - 2^{j(p-1)}} \varepsilon \left(2\|x + u_x\|^p + 2\|x - u_x\|^p + 4\|x\|^p + 4\|u_x\|^p\right),$$

$$\begin{aligned}&\|f_1(x) - f_1(0) - D(x)\| \\ &\quad \leq \varepsilon \left\{ \frac{2^{\frac{j(1+j)(p-1)}{2}}}{1 - 2^{j(p-1)}} \left(2\|x + u_x\|^p + 2\|x - u_x\|^p + 4\|x\|^p + 4\|u_x\|^p\right) + (\|x\|^p) \right\},\end{aligned}$$

$$\begin{aligned}&\|f_2(x) - f_2(0) - D(x)\| \\ &\quad \leq \varepsilon \left\{ \frac{2^{\frac{j(1+j)(p-1)}{2}}}{1 - 2^{j(p-1)}} \left(2\|x + u_x\|^p + 2\|x - u_x\|^p + 4\|x\|^p + 4\|u_x\|^p\right) + (\|x\|^p) \right\}\end{aligned}$$

and

$$\|f(x) - H(x)\| \leq \frac{2^{\frac{j(1+j)(p-1)}{2}}}{1 - 2^{j(p-1)}} \varepsilon \left(2\|x + u_x\|^p + 2\|x - u_x\|^p + 4\|x\|^p + 4\|u_x\|^p\right)$$

for any fixed $x \in \mathfrak{A}$ and some $u_x \in \mathfrak{A}$ with $x \perp u_x$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x, y) = \varepsilon(\|x\|^p + \|y\|^p) \quad \text{and} \quad \phi(x, y) = \theta\|x\|^q\|y\|^s$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$. Then we can choose $L = 2^{j(p-1)}$ and we get desired results. □

Corollary 2.4. *Let $j \in \{-1, 1\}$ and $f, f_1, f_2, h : \mathfrak{A} \longrightarrow \mathfrak{A}$ be mappings satisfying*

$$\|f(x + y) - f_1(x) - f_2(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p),$$

$$\|h(x + y) - h_1(x) - h_2(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p),$$

$$\|f(xy) - f_1(x)h_1(y) - h_2(x)f_2(y)\| \leq \theta\|x\|^s\|y\|^s$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$, $\varepsilon, \theta \geq 0$ and real numbers p, s such that $p, s < 1$ for $j = 1$. If f is an odd mapping, then there exist a unique orthogonal *homomorphism* $H : \mathfrak{A} \longrightarrow \mathfrak{A}$ *and a unique orthogonal hom-derivation* $D : \mathfrak{A} \to \mathfrak{A}$ *such that*

$$\|f(x) - D(x)\| \leq \frac{2^{\frac{j(1+j)(p-1)}{2}}}{1 - 2^{j(p-1)}} \varepsilon \left(2\|x + u_x\|^p + 2\|x - u_x\|^p + 4\|x\|^p + 4\|u_x\|^p\right),$$

$$\|f_1(x) - f_1(0) - D(x)\| \leq \varepsilon \left\{ \frac{2^{\frac{j(1+j)(p-1)}{2}}}{1 - 2^{j(p-1)}} \left(2\|x + u_x\|^p + 2\|x - u_x\|^p + 4\|x\|^p + 4\|u_x\|^p\right) + (\|x\|^p) \right\},$$

$$\|f_2(x) - f_2(0) - D(x)\| \leq \varepsilon \left\{ \frac{2^{\frac{j(1+j)(p-1)}{2}}}{1 - 2^{j(p-1)}} \left(2\|x + u_x\|^p + 2\|x - u_x\|^p + 4\|x\|^p + 4\|u_x\|^p\right) + (\|x\|^p) \right\},$$

$$\|h(x) - H(x)\| \leq \frac{2^{\frac{j(1+j)(p-1)}{2}}}{1 - 2^{j(p-1)}} \varepsilon \left(2\|x + u_x\|^p + 2\|x - u_x\|^p + 4\|x\|^p + 4\|u_x\|^p\right),$$

$$\|h_1(x) - h_1(0) - H(x)\| \leq \varepsilon \left\{ \frac{2^{\frac{j(1+j)(p-1)}{2}}}{1 - 2^{j(p-1)}} \left(2\|x + u_x\|^p + 2\|x - u_x\|^p + 4\|x\|^p + 4\|u_x\|^p\right) + (\|x\|^p) \right\}$$

and

$$\|h_2(x) - h_2(0) - H(x)\| \leq \varepsilon \left\{ \frac{2^{\frac{j(1+j)(p-1)}{2}}}{1 - 2^{j(p-1)}} \left(2\|x + u_x\|^p + 2\|x - u_x\|^p + 4\|x\|^p + 4\|u_x\|^p\right) + (\|x\|^p) \right\}$$

for any fixed $x \in \mathfrak{A}$ *and some* $u_x \in \mathfrak{A}$ *with* $x \perp u_x$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x, y) = \varepsilon(\|x\|^p + \|y\|^p) \quad \text{and} \quad \phi(x, y) = \theta\|x\|^q\|y\|^s$$

for all $x, y \in \mathfrak{A}$ with $x \perp y$. Then we can choose $L = 2^{j(p-1)}$ and we get desired results. □

3. Conclusion

In this paper, we have introduced a new system of orthogonal Pexider hom-derivation and Pexider hom-Pexider derivation (briefly, (Pexider) hom-derivation). Using the fixed point method, we have investigated the stability of Pexider hom-derivations and (Pexider) hom-derivations on Banach algebras.

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