Nonlinear Functional Analysis and Applications
Vol. 18, No. 2 (2013), pp. 163-176

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

HERMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE OF CONVEXITIES

Bo-Yan Xi[1] and Feng Qi[2]

[1]College of Mathematics, Inner Mongolia University for Nationalities
Tongliao City, Inner Mongolia Autonomous Region, 028043, China
e-mail: baoyintu78@qq.com, baoyintu68@sohu.com, baoyintu78@imun.edu.cn

[2]School of Mathematics and Informatics, Henan Polytechnic University
Jiaozuo City, Henan Province, 454010, China
e-mail: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
URL: http://qifeng618.wordpress.com

Abstract. In the paper, the authors offer some new inequalities of Hermite-Hadamard type for functions whose derivatives are of some geometric convexities.

1. Introduction

We first recall several definitions.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.2. A function $f : I \subseteq \mathbb{R}_0 = [0, \infty) \to \mathbb{R}_+ = (0, \infty)$ is said to be geometrically convex if

$$f\left(x^t y^{1-t}\right) \le [f(x)]^t [f(y)]^{1-t} \tag{1.2}$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

---

[0]Received October 6, 2012. Revised January 25, 2013.
[0]2010 Mathematics Subject Classification: 26A51, 26D15, 41A55.
[0]Keywords: Hermite-Hadamard's integral inequality, geometrically convex function, (α, m)-geometrically convex function, s-geometrically convex function.

Definition 1.3. ([15]) Let $f(x)$ be a positive function on $[0,b]$ and $m\in(0,1]$. If

$$f\big(x^t y^{m(1-t)}\big)\le [f(x)]^t[f(y)]^{m(1-t)} \tag{1.3}$$

holds for all $x,y\in[0,b]$ and $t\in[0,1]$, then we say that the function $f(x)$ is m-geometrically convex on $[0,b]$.

Definition 1.4. ([15]) Let $f(x)$ be a positive function on $[0,b]$ and $(\alpha,m)\in(0,1]^2$. If

$$f\big(x^t y^{m(1-t)}\big)\le [f(x)]^{t^\alpha}[f(y)]^{m(1-t^\alpha)} \tag{1.4}$$

holds for all $x,y\in[0,b]$ and $t\in[0,1]$, then we say that the function $f(x)$ is (α,m)-geometrically convex on $[0,b]$.

Definition 1.5. ([21]) A function $f:I\subseteq\mathbb{R}_0\to\mathbb{R}_+$ is said to be s-geometrically convex for some $s\in(0,1]$ if

$$f\big(x^t y^{1-t}\big)\le [f(x)]^{t^s}[f(y)]^{(1-t)^s} \tag{1.5}$$

holds for all $x,y\in I$ and $t\in[0,1]$.

We now recall some inequalities of Hermite-Hadamard type.

Theorem 1.6. ([4, Theorem 2.2]) *Let $f:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$. If $|f'(x)|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{1.6}$$

Theorem 1.7. ([10, Theorems 1 and 2]) *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable on I° and $a,b\in I$ with $a<b$. If $|f'(x)|^q$ is convex on $[a,b]$ and $q\ge1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q} \tag{1.7}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}. \tag{1.8}$$

Theorem 1.8. ([8, Theorem 2.3]) *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I$ with $a<b$. If the mapping $|f'(x)|^{p/(p-1)}$ is convex on $[a,b]$ for $p>1$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-f\left(\frac{a+b}{2}\right)\right|$$

$$\le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\Big\{\big[|f'(a)|^{p/(p-1)}+3|f'(b)|^{p/(p-1)}\big]^{1-1/p} + \big[3|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}\big]^{1-1/p}\Big\}. \tag{1.9}$$

In recent years, some other kinds of Hermite-Hadamard type inequalities were generated in, for example, [1]-[3], [7], [11]–[22]. For more systematic information, please refer to monographs [5, 6, 9] and related references therein.

In this paper, we will establish some new inequalities of Hermite-Hadamard type for functions whose derivatives are of some geometric convexities.

2. Lemmas

For any mapping $\varphi : \mathbb{R}_0 \to \mathbb{R}_+$ and for $u, v, \xi \in \mathbb{R}_+$ and $\alpha, m \in (0, 1]$, define

$$\varphi_{u,v} = \frac{\varphi(u)}{[\varphi(v^{1/m})]^m}, \quad \beta(\alpha;\xi) = \begin{cases} 0, & 0<\xi\le 1, \\ 1-\alpha, & \xi\ge 1, \end{cases} \tag{2.1}$$

$$L(u,v) = \begin{cases} \dfrac{v-u}{\ln v - \ln u}, & u\ne v, \\ u, & u=v, \end{cases} \tag{2.2}$$

and

$$G(u,v) = \begin{cases} \dfrac{2\sqrt{v}}{\ln v-\ln u}\big[L\big(\sqrt{u},\sqrt{v}\big)-\sqrt{u}\big], & u\ne v, \\ \dfrac{u}{2}, & u=v. \end{cases} \tag{2.3}$$

Lemma 2.1. *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $0 < a < b$. If $f' \in L([a, b])$, then*

$$\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x - f\big(\sqrt{ab}\big) = \frac{\ln b-\ln a}{4} \times \int_0^1 t\big[a^{t/2}b^{1-t/2}f'\big(a^{t/2}b^{1-t/2}\big) - a^{1-t/2}b^{t/2}f'\big(a^{1-t/2}b^{t/2}\big)\big]\,\mathrm{d}t. \tag{2.4}$$

Proof. Letting $x = a^{1-t/2}b^{t/2}$ for $0 \le t \le 1$ and integrating by part give

$$\begin{aligned} &\frac{\ln b-\ln a}{2}\int_0^1 ta^{1-t/2}b^{t/2}f'\big(a^{1-t/2}b^{t/2}\big)\,\mathrm{d}t \\ &= \int_0^1 t\,\mathrm{d}[f\big(a^{1-t/2}b^{t/2}\big)] \\ &= tf\big(a^{1-t/2}b^{t/2}\big)\big|_0^1 - \int_0^1 f\big(a^{1-t/2}b^{t/2}\big)\,\mathrm{d}t \end{aligned}$$

$$= f\big(\sqrt{ab}\,\big) - \frac{2}{\ln b - \ln a}\int_a^{\sqrt{ab}} \frac{f(x)}{x}\,\mathrm{d}x.$$

Further putting $x = a^{t/2}b^{1-t/2}$ for $0 \le t \le 1$ yields

$$\begin{aligned}
&\frac{\ln b - \ln a}{2}\int_0^1 t a^{t/2}b^{1-t/2} f'\big(a^{t/2}b^{1-t/2}\big)\,\mathrm{d}t\\
&= -\int_0^1 t\,\mathrm{d}\big[f\big(a^{t/2}b^{1-t/2}\big)\big]\\
&= -tf\big(a^{t/2}b^{1-t/2}\big)\big|_0^1 + \int_0^1 f\big(a^{t/2}b^{1-t/2}\big)\,\mathrm{d}t\\
&= -f\big(\sqrt{ab}\,\big) + \frac{2}{\ln b - \ln a}\int_{\sqrt{ab}}^b \frac{f(x)}{x}\,\mathrm{d}x.
\end{aligned}$$

Lemma 2.1 is thus proved. □

Lemma 2.2. *For $u, v > 0$ and $u \ne v$, we have*

$$\int_0^1 t u^{t/2} v^{1-t/2}\,\mathrm{d}t = \frac{2\sqrt{v}}{\ln v - \ln u}\big[L\big(\sqrt{u}\,, \sqrt{v}\,\big) - \sqrt{u}\,\big] \tag{2.5}$$

and

$$\int_0^1 t\big(u^{t/2}v^{1-t/2} + u^{1-t/2}v^{t/2}\big)\,\mathrm{d}t = \big[L\big(\sqrt{u}\,, \sqrt{v}\,\big)\big]^2, \tag{2.6}$$

where $L(u, v)$ is defined by (2.2)*.*

Proof. The proof is straightforward. □

3. Some inequalities of Hermite-Hadamard type

Now we are in a position to establish some inequalities of Hermite-Hadamard type for functions whose derivatives are of the geometric convexity, the m- and (α, m)-geometric convexities, and the s-geometric convexity.

Theorem 3.1. *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $0 < a < b$. If $|f'(x)|$ is geometrically convex on $[a, b]$, then*

$$\begin{aligned}
&\left|\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x - f\big(\sqrt{ab}\,\big)\right|\\
&\le \frac{\ln b - \ln a}{4}\big\{L\big([a|f'(a)|]^{1/2}, [b|f'(b)|]^{1/2}\big)\big\}^2,
\end{aligned} \tag{3.1}$$

where $L(u, v)$ is defined by (2.2)*.*

Proof. From Lemma 2.1, the geometric convexity of $|f'(x)|$ on $[a,b]$, and Lemma 2.2, we have

$$\begin{aligned}
&\left|\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x - f\big(\sqrt{ab}\big)\right|\\
&\le \frac{\ln b-\ln a}{4}\int_0^1 t\big[a^{t/2}b^{1-t/2}\big|f'\big(a^{t/2}b^{1-t/2}\big)\big| + a^{1-t/2}b^{t/2}\big|f'\big(a^{1-t/2}b^{t/2}\big)\big|\big]\,\mathrm{d}t\\
&\le \frac{\ln b-\ln a}{4}\int_0^1 t\big\{[a|f'(a)|]^{t/2}[b|f'(b)|]^{1-t/2} + [a|f'(a)|]^{1-t/2}[b|f'(b)|]^{t/2}\big\}\,\mathrm{d}t\\
&= \frac{\ln b-\ln a}{4}\big\{L\big([a|f'(a)|]^{1/2},[b|f'(b)|]^{1/2}\big)\big\}^2.
\end{aligned}$$

The proof of Theorem 3.1 is complete. □

Corollary 3.2. *For $b>a>0$ and $r>0$, we have*

$$0 < L\big(a^{2r},b^{2r}\big) - (ab)^r \le \frac{b^r-a^r}{2}L(a^r,b^r). \tag{3.2}$$

Proof. This follows from letting $f(x)=x^{2r}$ for $x\in\mathbb{R}_+$ and $r>0$ in Theorem 3.1. □

Theorem 3.3. *Let $f:\mathbb{R}_0\to\mathbb{R}$ be a differentiable function on $\mathbb{R}_0$, $(\alpha,m)\in(0,1]^2$, and $f'\in L([a,b])$ for $a,b\in\mathbb{R}_0$ with $0<a<b$. If $|f'(x)|$ is (α,m)-geometrically convex on the closed interval $\big[0,\max\{b,b^{1/m}\}\big]$, then*

$$\begin{aligned}
&\left|\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x - f\big(\sqrt{ab}\big)\right|\\
&\le \frac{\ln b-\ln a}{4}\bigg\{b\big|f'\big(b^{1/m}\big)\big|^m|f'|_{a,b}^{\beta(\alpha;|f'|_{a,b})}G\bigg(1,\frac{a}{b}|f'|_{a,b}^{\alpha}\bigg)\\
&\quad + a\big|f'\big(a^{1/m}\big)\big|^m|f'|_{b,a}^{\beta(\alpha;|f'|_{b,a})}G\bigg(1,\frac{b}{a}|f'|_{b,a}^{\alpha}\bigg)\bigg\},
\end{aligned}\tag{3.3}$$

where $|f'|_{a,b}$, $\beta(\alpha;\xi)$, and $G(u,v)$ are respectively defined as in (2.1) *and* (2.3).

Proof. Using the (α,m)-geometric convexity of $|f'(x)|$ on $\big[0,\max\{b,b^{1/m}\}\big]$ yields

$$\begin{aligned}
\big|f'\big(a^{t/2}b^{1-t/2}\big)\big| &\le |f'(a)|^{(t/2)^\alpha}\big|f'\big(b^{1/m}\big)\big|^{m[1-(t/2)^\alpha]}\\
&= \big|f'\big(b^{1/m}\big)\big|^m|f'|_{a,b}^{(t/2)^\alpha}
\end{aligned}$$

$$\leq \left|f'\left(b^{1/m}\right)\right|^m |f'|_{a,b}^{\beta(\alpha;|f'|_{a,b})+\alpha t/2} \tag{3.4}$$

and

$$\begin{aligned}\left|f'\left(a^{1-t/2}b^{t/2}\right)\right| &\leq \left|f'\left(a^{1/m}\right)\right|^{m[1-(t/2)^\alpha]}|f'(b)|^{(t/2)^\alpha} \\ &= \left|f'\left(b^{1/m}\right)\right|^m |f'|_{b,a}^{(t/2)^\alpha} \\ &\leq \left|f'\left(a^{1/m}\right)\right|^m |f'|_{b,a}^{\beta(\alpha;|f'|_{b,a})+\alpha t/2} \end{aligned} \tag{3.5}$$

for all $t \in [0,1]$.

By Lemmas 2.1 and 2.2 and inequalities (3) and (3.5), we obtain

$$\begin{aligned}&\left|\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x - f\left(\sqrt{ab}\,\right)\right| \\ &\leq \frac{\ln b - \ln a}{4}\int_0^1 t\left[a^{t/2}b^{1-t/2}\left|f'\left(a^{t/2}b^{1-t/2}\right)\right| \right.\\ &\quad\left. + a^{1-t/2}b^{t/2}\left|f'\left(a^{1-t/2}b^{t/2}\right)\right|\right]\mathrm{d}t \\ &\leq \frac{\ln b - \ln a}{4}\int_0^1 t\left\{b\left|f'\left(b^{1/m}\right)\right|^m |f'|_{a,b}^{\beta(\alpha;|f'|_{a,b})}\left(\frac{a}{b}|f'|_{a,b}^\alpha\right)^{t/2}\right.\\ &\quad\left. + a\left|f'\left(a^{1/m}\right)\right|^m |f'|_{b,a}^{\beta(\alpha;|f'|_{b,a})}\left(\frac{b}{a}|f'|_{b,a}^\alpha\right)^{t/2}\right\}\mathrm{d}t \\ &= \frac{\ln b - \ln a}{4}\left\{b\left|f'\left(b^{1/m}\right)\right|^m |f'|_{a,b}^{\beta(\alpha;|f'|_{a,b})} G\left(1, \frac{a}{b}|f'|_{a,b}^\alpha\right)\right.\\ &\quad\left. + a\left|f'\left(a^{1/m}\right)\right|^m |f'|_{b,a}^{\beta(\alpha;|f'|_{b,a})} G\left(1, \frac{b}{a}|f'|_{b,a}^\alpha\right)\right\}.\end{aligned}$$

The proof of Theorem 3.3 is complete. □

Corollary 3.4. *Under the conditions of Theorem* 3.3*,*

(1) *if* $\alpha = 1$*, then*

$$\begin{aligned}&\left|\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x - f\left(\sqrt{ab}\,\right)\right| \\ &\qquad\leq \frac{\ln b - \ln a}{4}\left\{b\left|f'\left(b^{1/m}\right)\right|^m G\left(1, \frac{a}{b}|f'|_{a,b}\right)\right.\\ &\qquad\quad\left. + a\left|f'\left(a^{1/m}\right)\right|^m G\left(1, \frac{b}{a}|f'|_{b,a}\right)\right\};\end{aligned}$$

(2) *if* $m=1$, *then*

$$\left|\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}x-f\big(\sqrt{ab}\,\big)\right| \le \frac{\ln b-\ln a}{4}\biggl\{b|f'(b)||f'|_{a,b}^{\beta(\alpha;|f'|_{a,b})}G\biggl(1,\frac{a}{b}|f'|_{a,b}^{\alpha}\biggr) + a|f'(a)||f'|_{b,a}^{\beta(\alpha;|f'|_{b,a})}G\biggl(1,\frac{b}{a}|f'|_{b,a}^{\alpha}\biggr)\biggr\};$$

(3) *if* $\alpha=m=1$, *then*

$$\left|\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}x-f\big(\sqrt{ab}\,\big)\right| \le \frac{\ln b-\ln a}{4}\big\{L\big([a|f'(a)|]^{1/2},[b|f'(b)|]^{1/2}\big)\big\}^2.$$

Corollary 3.5. *Under the conditions of Theorem* 3.3*,*

(1) *if* $|f'(a)|\le\big|f'\big(b^{1/m}\big)\big|^m$ *and* $|f'(b)|\le\big|f'\big(a^{1/m}\big)\big|^m$, *then*

$$\left|\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}x-f\big(\sqrt{ab}\,\big)\right| \le \frac{\ln b-\ln a}{4}\biggl\{b\big|f'\big(b^{1/m}\big)\big|^mG\biggl(1,\frac{a}{b}|f'|_{a,b}^{\alpha}\biggr) + a\big|f'\big(a^{1/m}\big)\big|^mG\biggl(1,\frac{b}{a}|f'|_{b,a}^{\alpha}\biggr)\biggr\};$$

(2) *if* $|f'(a)|\le\big|f'\big(b^{1/m}\big)\big|^m$ *and* $|f'(b)|\ge\big|f'\big(a^{1/m}\big)\big|^m$, *then*

$$\left|\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}x-f\big(\sqrt{ab}\,\big)\right| \le \frac{\ln b-\ln a}{4}\biggl\{b\big|f'\big(b^{1/m}\big)\big|^mG\biggl(1,\frac{a}{b}|f'|_{a,b}^{\alpha}\biggr) + a\big|f'\big(a^{1/m}\big)\big|^m|f'|_{b,a}^{1-\alpha}G\biggl(1,\frac{b}{a}|f'|_{b,a}^{\alpha}\biggr)\biggr\};$$

(3) *if* $|f'(a)|\ge\big|f'\big(b^{1/m}\big)\big|^m$ *and* $|f'(b)|\le\big|f'\big(a^{1/m}\big)\big|^m$, *then*

$$\left|\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}x-f\big(\sqrt{ab}\,\big)\right| \le \frac{\ln b-\ln a}{4}\biggl\{b\big|f'\big(b^{1/m}\big)\big|^m|f'|_{a,b}^{1-\alpha}G\biggl(1,\frac{a}{b}|f'|_{a,b}^{\alpha}\biggr)$$

$$+ a\big|f'\big(a^{1/m}\big)\big|^m G\bigg(1, \frac{b}{a}|f'|_{b,a}^{\alpha}\bigg)\bigg\};$$

(4) *if* $|f'(a)| \ge \big|f'\big(b^{1/m}\big)\big|^m$ *and* $|f'(b)| \ge \big|f'\big(a^{1/m}\big)\big|^m$, *then*

$$\begin{aligned}\bigg|\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x - f\big(\sqrt{ab}\big)\bigg| \\ \le \frac{\ln b - \ln a}{4}\bigg\{b\big|f'\big(b^{1/m}\big)\big|^m \times |f'|_{a,b}^{1-\alpha} G\bigg(1, \frac{a}{b}|f'|_{a,b}^{\alpha}\bigg) \\ + a\big|f'\big(a^{1/m}\big)\big|^m |f'|_{b,a}^{1-\alpha} G\bigg(1, \frac{b}{a}|f'|_{b,a}^{\alpha}\bigg)\bigg\}.\end{aligned}$$

Theorem 3.6. *Let* $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ *be a differentiable function on* I° *and* $f' \in L([a,b])$ *for* $a, b \in I^\circ$ *with* $0 < a < b$. *If* $|f'(x)|$ *is* s*-geometrically convex on* $[a,b]$ *for some* $s \in (0,1]$, *then*

$$\begin{aligned}\bigg|\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x - f\big(\sqrt{ab}\big)\bigg| \\ \le \frac{\ln b - \ln a}{4}|f'(a)|^{\beta(s;|f'(a)|)}|f'(b)|^{\beta(s;|f'(b)|)} \\ \times \big\{L\big(\big[a|f'(a)|^s\big]^{1/2}, \big[b|f'(b)|^s\big]^{1/2}\big)\big\}^2,\end{aligned} \tag{3.6}$$

where $\beta(\alpha;\xi)$, *and* $L(u,v)$ *are respectively defined as in* (2.1) *and* (2.2).

Proof. Using the s-geometric convexity of $|f'(x)|$ on $[a,b]$ yields

$$\begin{aligned}&\big|f'\big(a^{t/2}b^{1-t/2}\big)\big| \\ &\le |f'(a)|^{(t/2)^s}|f'(b)|^{(1-t/2)^s} \\ &\le |f'(a)|^{\beta(s;|f'(a)|)+\alpha t/2}|f'(b)|^{\beta(s;|f'(b)|)+\alpha(1-t/2)} \\ &= |f'(a)|^{\beta(s;|f'(a)|)}\big[|f'(a)|^s\big]^{t/2}|f'(b)|^{\beta(s;|f'(b)|)}\big[|f'(b)|^s\big]^{1-t/2}\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\big|f'\big(a^{1-t/2}b^{t/2}\big)\big| \\ &\le |f'(a)|^{(1-t/2)^s}|f'(b)|^{(t/2)^s} \\ &\le |f'(a)|^{\beta(s;|f'(a)|)+\alpha(1-t/2)}|f'(b)|^{\beta(s;|f'(b)|)+\alpha t/2} \\ &= |f'(a)|^{\beta(s;|f'(a)|)}\big[|f'(a)|^s\big]^{1-t/2}|f'(b)|^{\beta(s;|f'(b)|)}\big[|f'(b)|^s\big]^{t/2}\end{aligned} \tag{3.8}$$

for all $t \in [0,1]$.

Making use of Lemmas 2.1 and 2.2 and inequalities (3.7) and (3.8) leads to

$$\begin{aligned}
&\left|\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x - f\big(\sqrt{ab}\,\big)\right| \\
&\le \frac{\ln b-\ln a}{4}\int_0^1 t\big[a^{t/2}b^{1-t/2}\big|f'\big(a^{t/2}b^{1-t/2}\big)\big| + a^{1-t/2}b^{t/2}\big|f'\big(a^{1-t/2}b^{t/2}\big)\big|\big]\,\mathrm{d}t \\
&\le \frac{\ln b-\ln a}{4}|f'(a)|^{\beta(s;|f'(a)|)}|f'(b)|^{\beta(s;|f'(b)|)} \\
&\quad\times\int_0^1 t\Big\{\big[a|f'(a)|^s\big]^{t/2}\big[b|f'(b)|^s\big]^{1-t/2} + \big[a|f'(a)|^s\big]^{1-t/2}\big[b|f'(b)|^s\big]^{t/2}\Big\}\,\mathrm{d}t \\
&= \frac{\ln b-\ln a}{4}|f'(a)|^{\beta(s;|f'(a)|)}|f'(b)|^{\beta(s;|f'(b)|)}\big\{L\big(\big[a|f'(a)|^s\big]^{1/2},\big[b|f'(b)|^s\big]^{1/2}\big)\big\}^2.
\end{aligned}$$

The proof of Theorem 3.6 is complete. □

Corollary 3.7. *Under the conditions of Theorem* 3.6,

(1) *if* $|f'(a)|\le 1$ *and* $|f'(b)|\le 1$, *then*

$$\begin{aligned}
&\left|\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x - f\big(\sqrt{ab}\,\big)\right| \\
&\qquad\le \frac{\ln b-\ln a}{4}\big\{L\big([a|f'(a)|^s]^{1/2},[b|f'(b)|^s]^{1/2}\big)\big\}^2;
\end{aligned}\tag{3.9}$$

(2) *if* $|f'(a)|\le 1\le |f'(b)|$, *then*

$$\begin{aligned}
&\left|\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x - f\big(\sqrt{ab}\,\big)\right| \\
&\qquad\le \frac{\ln b-\ln a}{4}|f'(b)|^{1-s}\big\{L\big([a|f'(a)|^s]^{1/2},[b|f'(b)|^s]^{1/2}\big)\big\}^2;
\end{aligned}\tag{3.10}$$

(3) *if* $|f'(b)|\le 1\le |f'(a)|$, *then*

$$\begin{aligned}
&\left|\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x - f\big(\sqrt{ab}\,\big)\right| \\
&\qquad\le \frac{\ln b-\ln a}{4}|f'(a)|^{1-s}\big\{L\big([a|f'(a)|^s]^{1/2},[b|f'(b)|^s]^{1/2}\big)\big\}^2;
\end{aligned}\tag{3.11}$$

(4) *if* $|f'(a)|\ge 1$ *and* $|f'(b)|\ge 1$, *then*

$$\begin{aligned}
&\left|\frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x - f\big(\sqrt{ab}\,\big)\right| \\
&\quad\le \frac{\ln b-\ln a}{4}|f'(a)f'(b)|^{1-s}\big\{L\big([a|f'(a)|^s]^{1/2},[b|f'(b)|^s]^{1/2}\big)\big\}^2;
\end{aligned}\tag{3.12}$$

(5) *if* $s = 1$, *then*

$$\left|\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x - f\big(\sqrt{ab}\big)\right| \le \frac{\ln b - \ln a}{4}\big\{L\big([a|f'(a)|]^{1/2}, [b|f'(b)|]^{1/2}\big)\big\}^2. \tag{3.13}$$

Theorem 3.8. *Let* $f : \mathbb{R}_0 \to \mathbb{R}_+$ *be* (α, m)*-geometrically convex on the closed interval* $\big[0, \max\{b, b^{1/m}\}\big]$ *and* $f \in L([a, b])$ *for* $0 < a < b$ *and* $(\alpha, m) \in (0, 1]^2$. *Then*

$$\begin{aligned}\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x &\le \min\Big\{\big[f\big(a^{1/m}\big)\big]^{m(1-\alpha)} f_{b,a}^{\beta(\alpha;f_{b,a})} L\big(\big[f\big(a^{1/m}\big)\big]^{\alpha m}, [f(b)]^{\alpha}\big),\\ &\qquad \big[f\big(b^{1/m}\big)\big]^{m(1-\alpha)} f_{a,b}^{\beta(\alpha;f_{a,b})} L\big([f(a)]^{\alpha}, \big[f\big(b^{1/m}\big)\big]^{\alpha m}\big)\Big\},\end{aligned} \tag{3.14}$$

where $f_{a,b}$, $\beta(\alpha; \xi)$, *and* $L(u, v)$ *are respectively defined as in* (2.1) *and* (2.2).

Proof. Putting $x = a^{1-t}b^t$ for $0 \le t \le 1$ and using the (α, m)-geometric convexity of $f(x)$ on $\big[0, \max\{b, b^{1/m}\}\big]$ reveal

$$\begin{aligned}\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x &= \int_0^1 f\big(a^{1-t}b^t\big)\,\mathrm{d}t\\ &\le \int_0^1 \big[f\big(a^{1/m}\big)\big]^{m(1-t^{\alpha})}[f(b)]^{t^{\alpha}}\,\mathrm{d}t\\ &\le \big[f\big(a^{1/m}\big)\big]^{m}\int_0^1 f_{b,a}^{\beta(\alpha;f_{b,a})+\alpha t}\,\mathrm{d}t\\ &= \big[f\big(a^{1/m}\big)\big]^{m(1-\alpha)} f_{b,a}^{\beta(\alpha;f_{b,a})} L\big(\big[f\big(a^{1/m}\big)\big]^{\alpha m}, [f(b)]^{\alpha}\big).\end{aligned}$$

The proof of Theorem 3.8 is complete. □

Corollary 3.9. *Under the conditions of Theorem* 3.8,

(1) *if* $\alpha = 1$, *then*

$$\begin{aligned}&\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x\\ &\le \min\big\{L\big(\big[f\big(a^{1/m}\big)\big]^m, f(b)\big), L\big(f(a), \big[f\big(b^{1/m}\big)\big]^m\big)\big\};\end{aligned}$$

(2) *if* $m = 1$, *then*

$$\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}x \le \min\Big\{\big[f(a)\big]^{1-\alpha}f_{b,a}^{\beta(\alpha;f_{b,a})}, \big[f(b)\big]^{1-\alpha}f_{a,b}^{\beta(\alpha;f_{a,b})}\Big\}L\big(f^{\alpha}(a),f^{\alpha}(b)\big);$$

(3) *if* $\alpha=m=1$, *then*

$$\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}x\le L(f(a),f(b)).$$

Theorem 3.10. *Let* $f,g:\mathbb{R}_0\to\mathbb{R}_+$ *be* (α,m)*-geometrically convex functions on the closed interval* $\big[0,\max\{0,b^{1/m}\}\big]$ *and* $fg\in L([a,b])$ *for* $(\alpha,m)\in(0,1]^2$ *and* $0<a<b$. *Then*

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)g(x)}{x}\,\mathrm{d}x &\le \min\Big\{(fg)_{b,a}^{\beta(\alpha;(fg)_{b,a})}\big[f\big(a^{1/m}\big)g\big(a^{1/m}\big)\big]^{m(1-\alpha)}\\ &\quad\times L\big(\big[f\big(a^{1/m}\big)g\big(a^{1/m}\big)\big]^{\alpha m},[f(b)g(b)]^{\alpha}\big),\\ &\quad (fg)_{a,b}^{\beta(\alpha;(fg)_{a,b})}\big[f\big(b^{1/m}\big)g\big(b^{1/m}\big)\big]^{m(1-\alpha)}\\ &\quad\times L\big([f(a)g(a)]^{\alpha},\big[f\big(b^{1/m}\big)g\big(b^{1/m}\big)\big]^{\alpha m}\big)\Big\},\end{aligned}\tag{3.15}$$

where $f_{a,b}$, $\beta(\alpha;\xi)$,*and* $L(u,v)$ *are respectively defined as in* (2.1) *and* (2.2).

Proof. The proof is similar to that of Theorem 3.8. □

Corollary 3.11. *Under the conditions of Theorem* 3.10,

(1) *if* $\alpha=1$, *then*

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)g(x)}{x}\,\mathrm{d}x &\le \min\Big\{L\big(\big[f\big(a^{1/m}\big)g\big(a^{1/m}\big)\big]^{m},f(b)g(b)\big),\\ &\quad L\big(f(a)g(a),\big[f\big(b^{1/m}\big)g\big(b^{1/m}\big)\big]^{m}\big)\Big\};\end{aligned}$$

(2) *if* $m=1$, *then*

$$\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)g(x)}{x}\,\mathrm{d}x \le \min\Big\{(fg)_{b,a}^{\beta(\alpha;(fg)_{b,a})}\big[f(a)g(a)\big]^{1-\alpha},$$

$$
(fg)_{a,b}^{\beta(\alpha;(fg)_{a,b})}\big[f(b)g(b)\big]^{1-\alpha}\Big\}
\times L\big([f(a)g(a)]^{\alpha},\big[f(b)g(b)\big]^{\alpha}\big);
$$

(3) *if* $\alpha = m = 1$, *then*

$$
\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)g(x)}{x}\,\mathrm{d}x \le L(f(a)g(a), f(b)g(b)).
$$

Theorem 3.12. *Let* $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}_+$ *be a* s*-geometrically convex function on* I° *for some* $s \in (0,1]$, $a, b \in I^\circ$ *with* $0 < a < b$ *and* $f \in L([a,b])$. *Then*

$$
\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x \le [f(a)]^{\beta(s,f(a))}[f(b)]^{\beta(s,f(b))}L\big(f^s(a), f^s(b)\big), \tag{3.16}
$$

where $\beta(\alpha;\xi)$ *and* $L(u,v)$ *are respectively defined as in* (2.1) *and* (2.2).

Proof. Letting $x = a^{1-t}b^t$ for $0 \le t \le 1$ and utilizing the s-geometric convexity of $f(x)$ on $[a,b]$ give

$$
\begin{aligned}
\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x &= \int_0^1 f\big(a^{1-t}b^t\big)\,\mathrm{d}t \\
&\le \int_0^1 [f(a)]^{(1-t)^s}[f(b)]^{t^s}\,\mathrm{d}t \\
&\le \int_0^1 [f(a)]^{\beta(s,f(a))+s(1-t)}[f(b)]^{\beta(s,f(b))+st}\,\mathrm{d}t \\
&= [f(a)]^{\beta(s,f(a))}[f(b)]^{\beta(s,f(b))}L\big(f^s(a), f^s(b)\big).
\end{aligned}
$$

The proof of Theorem 3.12 is complete. □

Theorem 3.13. *Let* $f, g : I \subseteq \mathbb{R}_0 \to \mathbb{R}_+$ *be* s*-geometrically convex functions on* I° *for* $s \in (0,1]$, $a, b \in I^\circ$ *with* $0 < a < b$ *and* $fg \in L([a,b])$. *Then*

$$
\begin{aligned}
\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)g(x)}{x}\,\mathrm{d}x &\le [f(a)g(a)]^{\beta(s,f(a)g(a))}[f(b)g(b)]^{\beta(s,f(b)g(b))} \\
&\quad \times L\big([f(a)g(a)]^s, [f(b)g(b)]^s\big).
\end{aligned} \tag{3.17}
$$

where $\beta(\alpha;\xi)$ *and* $L(u,v)$ *are respectively defined as in* (2.1) *and* (2.2).

Proof. The proof is similar to that of Theorem 3.12. □

Acknowledgements. The authors would like to thank Dr. Mevlüt Tunç for his reminding us of some errors in the original version of this manuscript.

This work was partially supported by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region under Grant No. NJZY13159, China.

References

[1] R.-F. Bai, F. Qi, and B.-Y. Xi, *Hermite-Hadamard type inequalities for the m- and (α, m)-logarithmically convex functions*, Filomat **27** (2013), no. 1, 1–7.

[2] S.-P. Bai, S.-H. Wang, and F. Qi, *Some Hermite-Hadamard type inequalities for n-time differentiable (α, m)-convex functions*, J. Inequal. Appl. 2012, **2012**:267, 11 pages.

[3] L. Chun and F. Qi, *Integral inequalities of Hermite-Hadamard type for functions whose 3rd derivatives are s-convex*, Appl. Math. **3** (2012), no. 11, 1680–1685.

[4] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (1998), no. 5, 91–95.

[5] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Type Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[6] S. S. Dragomir and T. M. Rassias, *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, 2002.

[7] W.-D. Jiang, D.-W. Niu, Y. Hua, and F. Qi, *Generalizations of Hermite-Hadamard inequality to n-time differentiable functions which are s-convex in the second sense*, Analysis (Munich) **32** (2012), no. 3, 209–220.

[8] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl. Math. Comput. **147** (2004), no. 1, 137–146.

[9] C. P. Niculescu and L.-E. Persson, *Convex Functions and their Applications*, CMS Books in Mathematics, Springer-Verlag, 2005.

[10] C. E. M. Pearce and J. Pečarić, *Inequalities for differentiable mappings with application to special means and quadrature formulae*, Appl. Math. Lett. **13** (2000), no. 2, 51–55.

[11] F. Qi, Z.-L. Wei, and Q. Yang, *Generalizations and refinements of Hermite-Hadamard's inequality*, Rocky Mountain J. Math. **35** (2005), no. 1, 235–251.

[12] Y. Shuang, H.-P. Yin, and F. Qi, *Hermite-Hadamard type integral inequalities for geometric-arithmetically s-convex functions*, Analysis (Munich) **33** (2013), in press.

[13] S.-H. Wang, B.-Y. Xi, and F. Qi, *On Hermite-Hadamard type inequalities for (α, m)-convex functions*, Int. J. Open Probl. Comput. Sci. Math. **5** (2012), no. 4, 47–56.

[14] S.-H. Wang, B.-Y. Xi, and F. Qi, *Some new inequalities of Hermite-Hadamard type for n-time differentiable functions which are m-convex*, Analysis (Munich) **32** (2012), no. 3, 247–262.

[15] B.-Y. Xi, R.-F. Bai, and F. Qi, *Hermite-Hadamard type inequalities for the m- and (α, m)-geometrically convex functions*, Aequationes Math. **84** (2012), no. 3, 261–269.

[16] B.-Y. Xi, J. Hua, and F. Qi, *Hermite-Hadamard type inequalities for extended s-convex functions on the co-ordinates in a rectangle*, J. Appl. Anal. **20** (2014), no. 1, in press.

[17] B.-Y. Xi and F. Qi, *Some Hermite-Hadamard type inequalities for differentiable convex functions and applications*, Hacet. J. Math. Stat. **42** (2013), in press.

[18] B.-Y. Xi and F. Qi, *Some inequalities of Hermite-Hadamard type for h-convex functions*, Adv. Inequal. Appl. **2** (2013), no. 1, 1–15.

[19] B.-Y. Xi and F. Qi, *Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means*, J. Funct. Spaces Appl. **2012** (2012), Article ID 980438, 14 pages.

[20] B.-Y. Xi, S.-H. Wang, and F. Qi, *Some inequalities of Hermite-Hadamard type for functions whose* 3*rd derivatives are* P*-convex*, Appl. Math. **3** (2012), no. 12, 1898–1902.

[21] T.-Y. Zhang, A.-P. Ji, and F. Qi, *On integral inequalities of Hermite-Hadamard type for* s*-geometrically convex functions*, Abstr. Appl. Anal. **2012** (2012), Article ID 560586, 14 pages.

[22] T.-Y. Zhang, A.-P. Ji, and F. Qi, *Some inequalities of Hermite-Hadamard type for GA-convex functions with applications to means*, Matematiche (Catania) **68** (2013), no. 1, 229–239.