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REMARKS ON THE KKM STRUCTURES OF KHANH AND QUAN

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Abstract. Since Knaster, Kuratowski, and Mazurkiewicz established their KKM theorem in 1929, it was first applied to topological vector spaces mainly by Fan and Granas. Later it was extended to convex spaces by Lassonde and to extensions of c-spaces by Horvath. In 1992, such study was called the KKM theory by ourselves. Then the theory was extended to generalized convex spaces or G-convex spaces. Motivated by such spaces, there have appeared several particular types of artificial spaces. In 2006 we introduced abstract convex spaces which contain all existing spaces appeared in the KKM theory. Later in 2014-2020, Khahn and Quan introduced "topologically based existence theorems" and the so-called KKM structure. In the present paper, we show that their structure is a particular type of already known KKM spaces.

1. Introduction

Since Knaster, Kuratowski, and Mazurkiewicz established their KKM theorem in 1929, it was first applied to topological vector spaces mainly by Fan and Granas. Later it was extended to convex spaces by Lassonde and to extensions of c-spaces by Horvath. In 1992, such study was first called the KKM theory by ourselves. Then the theory was extended to generalized convex spaces or

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G-convex spaces. Motivated by such spaces, there have appeared several particular types of artificial spaces. In order to unify them, in 2006, we introduced abstract convex spaces which contain all existing spaces appeared in the KKM theory. For such history, see [19].

Later in 2014-2020, Khahn and Quan [2, 3, 4] introduced "topologically based existence theorems" and the so-called KKM structure. In the present article, we show that their structure is a particular type of already known KKM spaces and their works can be extended to more general spaces.

This article is organized as follows: Section 2 is a preliminary on our abstract convex spaces based mainly on Lu-Zhang-Li [7] in 2021. In Section 3, we recall some subclasses of abstract convex spaces such as G-convex spaces and ϕ_A-spaces, which are renamed G-spaces and F-spaces, respectively, in this article. Section 4 deals with basic KKM theorems for abstract convex spaces. They extend most of known generalizations of the original KKM theorem in 1929. In Section 5, we are concerned with the so-called KKM structure of Khanh-Quan [2, 3, 4] appeared in 2014-2020. We show that their KKM structure is nothing but the F-space. Section 6 deals with our previous critical works on modifications, imitations, or extensions of G-spaces. Finally, Section 7 is for the conclusion with some additional remarks.

2. Abstract convex spaces

In this section, we introduce some basic definitions related to abstract convex spaces according to Lu-Zhang-Li [7] based our earlier works. For more details, the reader may refer to our works mentioned in [19].

Throughout this paper, $\langle D \rangle$ denotes the set of all nonempty finite subsets of a set D.

Definition 2.1. ([9]) If E is a topological space, D is a nonempty set, and $\Gamma : \langle D \rangle \multimap E$ is a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for every $A \in \langle D \rangle$, then the triple $(E, D; \Gamma)$ is called an *abstract convex space*. When $E = D$, we denote $(E, E; \Gamma)$ by $(E; \Gamma)$.

Definition 2.2. ([9]) Given an abstract convex space $(E, D; \Gamma)$ and a nonempty subset D' of D, we define the Γ-convex hull of D' by

$$co_\Gamma(D') = \bigcup \{\Gamma_A : A \in \langle D' \rangle\}.$$

Definition 2.3. ([9]) Let $(E, D; \Gamma)$ be an abstract convex space. A nonempty subset E' of E is called a Γ-convex subset of $(E, D; \Gamma)$ relative to a nonempty subset D' of D if we have $\Gamma_N \subset E'$ for every $N \in \langle D' \rangle$, that is, $\mathrm{co}_\Gamma(D') \subset E'$. In case $E = D$, a nonempty subset E' of E is said to be Γ-convex if $\mathrm{co}_\Gamma(E') \subset E'$, that is, E' is Γ-convex relative to itself.

Remark 2.4. Given an abstract convex space $(E, D; \Gamma)$, by Definition 2.3, we can see that if a nonempty subset E' of E is a Γ-convex subset of $(E, D; \Gamma)$ relative to a nonempty subset D' of D, then $(E', D'; \Gamma|_{\langle D' \rangle})$ itself is an abstract convex space which is called a *subspace* of $(E, D; \Gamma)$.

Definition 2.5. ([9]) Let $(E, D; \Gamma)$ be an abstract convex space and Z be a set. For a multimap $H : E \multimap Z$ with nonempty values, if a multmap $G : D \multimap Z$ satisfies $H(\Gamma_A) \subset G(A)$ for every $A \in \langle D \rangle$, then G is called a KKM map with respect to H. A KKM map $G : D \multimap E$ is a KKM map with respect to the identity mapping 1_E.

Definition 2.6. ([19]) Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \quad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp. a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp. open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp. $F \in \mathfrak{KO}(E, D, Z)$].

3. Subclasses of abstract convex spaces

We give some classical definitions of subclasses of abstract convex spaces; see [18, 19]:

Definition 3.1. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\Delta_n = \mathrm{co}\{e_i\}_{i=0}^n$ is the standard n-simplex, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$, that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}$. We may write $(X, \Gamma) = (X, X; \Gamma)$.

Definition 3.2. A *space having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

In the present article, G-convex spaces and ϕ_A-spaces will be called *G-spaces* and *F-spaces*, resp., for simplicity.

Definition 3.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

Now we have the following diagram for subclasses of abstract convex spaces $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ Horvath space $\Longrightarrow$ G-space $\Longrightarrow$ F-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

Each subclass has a large number of concrete examples; see [19, 20]. For the relatively new Horvath spaces, see [21].

4. Basic KKM theorems for abstract convex spaces

We defined a subspace of an abstract convex space and $\mathfrak{KC}$-maps on subspaces in Section 2. The following generalize [9, Proposition 2], resp., with a slightly modified proof:

Lemma 4.1. *Let $(E, D; \Gamma)$ be an abstract convex space, $(X, D'; \Gamma')$ a subspace, and Z a topological space. If $F \in \mathfrak{K}(E, Z)$, then $F|_X \in \mathfrak{K}(X, Z)$, where $\mathfrak{K}$ denotes $\mathfrak{KC}$ or $\mathfrak{KO}$.*

In the present section, we consider further properties of partial KKM spaces. The following equivalent form of [13, Theorem 3] is basic:

Theorem 4.2. (*Generalized partial KKM principle*) *Let $(E, D; \Gamma)$ be a partial KKM space and $G : D \multimap E$ a map such that*

(1) *G is closed-valued;*
(2) *G is a KKM map (that is, $\Gamma_A \subset G(A)$ for all $A \in \langle D \rangle$); and*
(3) *there exists a nonempty compact subset K of E such that one of the following holds:*
  (i) *$K = E$;*

(ii) $K = \bigcap\{G(z) \mid z \in M\}$ *for some* $M \in \langle D\rangle$*; or*

(iii) *for each* $N \in \langle D\rangle$*, there exists a compact* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$L_N \cap \bigcap_{z\in D'} G(z) \subset K.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Proof. Case (i): In this case every $G(y)$ is compact. Hence Case (i) reduces to (ii).

Case (ii): Since $\{G(z) \mid z \in D\}$ has the finite intersection property, so does $\{K \cap G(z) \mid z \in D\}$ in the compact set K. Hence it has the whole intersection property.

Case (iii): Suppose that $K \cap \bigcap\{G(z) \mid z \in D\} =$; that is, $K \subset \bigcup\{X \setminus G(z) \mid z \in N\}$ for some $N \in \langle D\rangle$. Let L_N be the compact Γ-convex subset of E in (iii). Define $G' : D' \multimap L_N$ by $G'(z) := G(z) \cap L_N$ for $z \in D'$. Then $A \in \langle D'\rangle$ implies $\Gamma'_A := \Gamma_A \cap L_N \subset G(A) \cap L_N = G'(A)$ by (2); and hence $G' : D' \multimap L_N$ is a KKM map on $(L_N, D'; \Gamma')$ with closed values. Since $(X, D; \Gamma)$ satisfies the partial KKM principle, so does $(L_N, D'; \Gamma')$ by Lemma 4.1. Hence, $\{G'(z) \mid z \in D'\}$ has the finite intersection property. Since L_N is compact, and $\bigcap\{G'(z) \mid z \in D'\} \neq$ by Case (i). For any $y \in \bigcap\{G'(z) \mid z \in D'\}$, we have $y \in K$ by (ii). However, since $y \in K \subset \bigcup\{X \setminus G(z) \mid z \in N\}$, we have $y \notin G(z)$ for some $z \in N \subset D'$. This is a contradiction. Therefore, we must have $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$. □

Recall that conditions (i)-(iii) in Theorem 4.2 are usually called the *compactness conditions* or the *coercivity conditions*. More formally we define as follows:

Definition 4.3. For an abstract convex space $(E, D; \Gamma)$ and a closed-valued map $G : D \multimap E$, a *coercivity condition* for G is the one guaranteeing the whole intersection property of the family $\{G(y)\}_{y\in D}$ whenever it has the finite intersection property.

Example 4.4. Theorem 4.2 shows that each of (i)-(iii) is a coercivity condition for any $(E, D; \Gamma)$ and any G. There appeared several hundred particular cases of the condition (iii).

For particular spaces and particular maps, we may have another coercivity conditions; for example, see [13].

Consider the following related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y\in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y\in D}\overline{G(y)}=\overline{\bigcap_{y\in D}G(y)}$ (G is *intersectionally closed-valued*).
(c) $\bigcap_{y\in D}\overline{G(y)}=\bigcap_{y\in D}G(y)$ (G is *transfer closed-valued*).
(d) G is closed-valued.

Luc et al. in 2010 noted that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and gave examples of multimaps satisfying (b) but not (c).

The following is Theorem C in [16], which is equivalent to Theorem 4.2:

Theorem 4.5. *Let $(E,D;\Gamma)$ be an abstract convex space, Z a topological space, $F\in\mathfrak{KC}(E,D,Z)$, and $G:D\multimap Z$ a map such that*

(1) *$\overline{G}$ is a KKM map w.r.t. F; and*
(2) *there exists a nonempty compact subset K of Z such that either*
  (i) *$\bigcap\{\overline{G(y)}\mid y\in M\}\subset K$ for some $M\in\langle D\rangle$; or*
  (ii) *for each $N\in\langle D\rangle$, there exists a Γ-convex subset L_N of E relative to some $D'\subset D$ such that $N\subset D'$, $\overline{F(L_N)}$ is compact, and*

$$\overline{F(L_N)}\cap\bigcap_{y\in D'}\overline{G(y)}\subset K.$$

Then we have

$$\overline{F(E)}\cap K\cap\bigcap_{y\in D}\overline{G(y)}\neq\emptyset.$$

Furthermore,

(a) *if G is transfer closed-valued, then*

$$\overline{F(E)}\cap K\cap\bigcap\{G(y)\mid y\in D\}\neq\emptyset:$$

(b) *if G is intersectionally closed-valued, then*

$$\bigcap\{G(y)\mid y\in D\}\neq\emptyset.$$

Note that Theorem 4.5 has hundreds of particular forms.

5. The KKM structure of Khanh and Quan

Recently, Khanh and Quan [4] introduced the following:

Let E be a topological vector space with a topology $\mathcal{T}_E$. Then, to any finite subset $N=\{x_0,\ldots,x_n\}$ of E, we associate the following continuous map $\sigma_N:\Delta_{|N|}\to E$

$$\sigma_N(e)=x:=\sum_{i=0}^{n}\lambda_i x_i\quad\text{for all } e=\sum_{i=0}^{n}\lambda_i e_i\in\Delta_{|N|}.$$

Let $\Sigma_E=\{\sigma_N|N\in\langle E\rangle\}$. Then, the pair $(\Sigma_E,\mathcal{T}_E)$ defines the usual convexity structure on E in the sense that a subset K of E is convex if and only if for all $\sigma_N \in \Sigma_E$ and $M \subset N \cap K$, $\sigma_N(\Delta_M) \subset K$. Thus we can view $(\Sigma_E, \mathcal{T}_E)$ as the usual convexity structure on K. Generalizing this for any sets leads to the following definition.

Definition 5.1. ([2, 3, 4]) A pair $\mathcal{F} := (\Sigma_X, \mathcal{T}_Y)$ is called a *KKM structure* of the pair of sets (X, Y) if $\mathcal{T}_Y$ is a topology on Y and $\Sigma_X := \{\sigma_N : \Delta_{|N|} \to Y \mid N \in \langle X\rangle\}$ is a family of maps such that each $\sigma_N \in \Sigma_X$ is $\mathcal{T}_Y$-continuous. In the special case $X = Y$, such a $\mathcal{F}$ is termed a KKM structure of X. If $\mathcal{T}_Y$ is compact, that is, Y is $\mathcal{T}_Y$-compact, $(\Sigma_X, \mathcal{T}_Y)$ is called a compact KKM structure.

After giving some examples, the authors noted that:

Note that in some spaces, introduced by many authors, such as a convex space, H-space, G-convex space, FC-space, GFC-space and so on, there is a KKM structure implicitly. However, in applications building such spaces may be much more difficult than using a KKM structure. Moreover, we can flexibly choose suitable KKM structures depending on situations. This also is crucial for getting full characterizations (not merely sufficient conditions) for the existence of important points in mathematical analysis and optimization-related problems.

In this paper, we try to provide topologically based existence theorems, unifying both the mentioned directions. Moreover, most of the above-encountered results are only sufficient conditions for existence. Inspired by the papers [5] and [6], we focus on necessary and sufficient conditions. Furthermore, for the purpose of unification, we pay attention to the equivalence of the existence of different kinds of points in both KKM structures and connectedness structures.

Now we reformulate Definition 5.1 as follows:

Definition 5.2. A pair $\mathcal{F} := (\mathcal{T}_E, \Sigma_D)$ is called a *KKM structure* of the pair of sets (E, D) if $\mathcal{T}_E$ is a topology on E and $\Sigma_D := \{\sigma_N : \Delta_{|N|} \to E \mid N \in \langle D\rangle\}$ is a family of maps such that each $\sigma_N \in \Sigma_D$ is $\mathcal{T}_E$-continuous. In the special case $E = D$, such a $\mathcal{F}$ is termed a KKM structure of E. If $\mathcal{T}_E$ is compact, that is, E is $\mathcal{T}_E$-compact, $(\Sigma_D, \mathcal{T}_E)$ is called a compact KKM structure.

Note that (1) each KKM structure leads to an abstract convex space, (2) we gave already a large number of examples of abstract convex spaces, and (3) some of them can be examples of KKM structures.

In order to simplify the KKM structure, we have the following:

Lemma 5.3. *A pair (E, D) has a KKM structure $\mathcal{F} := (\mathcal{T}_E, \Sigma_D)$ if and only if we have an F-space $(E, D; \Sigma_D)$.*

Proof. Note that Σ_D is a family of continuous functions $\phi_A : \Delta_n \to X$ for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$. □

Consequently, the study on the KKM structures is the one for corresponding F-spaces or GFC-spaces.

Theorem 5.4. *A multimap $T : D \multimap E$ between two nonempty sets D and E has intersection points, that is, $\bigcap_{x\in D} T(x) \neq \emptyset$, if and only if there exists a partial KKM space $(E, D; \Gamma)$ such that*

(1) *T is a KKM map (that is, $\Gamma_N \subset T(N)$ for each $N \in \langle D\rangle$);*
(2) *the values of T are nonempty and closed;*
(3) *$\bigcap_{x\in N_0} T(x)$ is compact for some $N_0 \in \langle X\rangle$.*

Proof. ($\Longrightarrow$): Assume that $\bigcap_{x\in D} T(x) \neq \emptyset$. Then there exists $\bar{y} \in \bigcup_{x\in D} T(x)$. Let $\mathcal{F} := (\Sigma_X, \mathcal{T}_Y)$ be defined by $\Sigma_X = \{\sigma_N : \Delta_{|N|} \to Y | \sigma_N(e) = \bar{y}$ for all $e \in \Delta_{|N|},\ N \in \langle X\rangle\}$ and $\mathcal{T}_Y = \{U + Y | \bar{y} \notin U\} \cup \{Y\}$. It is not hard to check that (1)-(3) in Theorem 5.4 hold.

($\Longleftarrow$): A consequence of Case (ii) of our Theorem 4.2 (*Generalized partial KKM principle*). □

The proof of *Only if* part is just Khanh-Quan's proof of the following Theorem 2.6 of [4]:

Corollary 5.5. ([4]) *A multimap $T : X \multimap Y$ between two nonempty sets X and Y has intersection points, that is, $\bigcap_{x\in X} T(x) \neq \emptyset$, if and only if there exists a KKM structure $\mathcal{F} := (\Sigma_X, \mathcal{T}_Y)$ of (X, Y) such that*

(1) *for all $\sigma_N \in \Sigma_X$ and $M \subset N$, $\sigma_N(\Delta_M) \subset \bigcup_{x\in M} T(x)$;*
(2) *the values of T are nonempty and $\mathcal{T}_Y$-closed;*
(3) *$\bigcap_{x\in N_0} T(x)$ is $\mathcal{T}_Y$-compact for some $N_0 \in \langle X\rangle$.*

The well-known Knaster-Kuratowski-Mazurkiewicz (KKM) Theorem in 1929 was extended by Ky Fan in 1961 [1] as the following consequence of Theorem 5.4:

The 1961 KKM Lemma. (Fan [1]) *Let X be an arbitrary set in a Hausdorff topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *convex hull of any finite subset $\{x_1, \cdots, x_n\}$ of X is contained in $\bigcup_{i=1}^n F(x_i)$.*
(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x\in X} F(x) \neq \emptyset$.

This is usually known as the Fan-KKM lemma or the Fan-KKM theorem or the KKMF theorem. Fan assumed the Hausdorffness of Y, which was known to be superfluous later. Fan and his followers applied his KKM lemma to various problems in many fields in mathematics; see [1, 19].

6. Our previous studies on F-spaces and others

Since we began to study the KKM theory, we have been trying to improve the theory in several occasions. Sometimes we criticized other authors' inadequate works. Especially, we published several papers related F-spaces, that is, ϕ_A-spaces of ourselves and GFC-spaces of Khanh et al.

In the following, we list the abstracts of our works on such matters in the chronological order for the reader's convenience:

(I) ϕ_A-spaces [10] in 2008: Basic results in the KKM theory on abstract convex spaces and the KKM maps are applied to ϕ_A-spaces which unify various imitations of G-convex spaces. We show that basic theorems on ϕ_A-spaces can be applied to correct and improve results on the so-called R-KKM maps on the so-called L-convex spaces.

(II) FC-spaces [11] in 2009: We show that FC-spaces due to Ding are particular types of L-spaces due to Ben-El-Mechaiekh et al., and hence particular types of G-convex spaces. Some counter-examples are given and related matters are also discussed.

(III) Abstract convexity structures [12] in 2010: All results in "Some properties of abstract convexity structures on topological spaces" by S.-w. Xiang and H. Yang [Some properties of abstract convexity structures on topological spaces, Nonlinear Analysis 67 (2007) 803–808] and "A further characteristic of abstract convexity structures on topological spaces" by S.-w. Xiang and S. Xia [A further characteristic of abstract convexity structures on topological spaces, J. Math. Anal. Appl. 335 (2007) 716–723] are shown to be consequences of known ones or can be stated in more general forms.

(IV) Abstract convex spaces [13] in 2010: The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. In this paper, we clearly derive a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, this paper unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

(V) ϕ_A-spaces [15] in 2012: In our previous works, we showed that every ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ can be made into a G-convex space in several ways. In this work, we show that a ϕ_A-space can be made into a G-convex space $(X, D; \Gamma)$ iff it has a KKM map $G : D \multimap X$, and that it is a KKM space.

Moreover, we show that recent examples of GFC-spaces due to Khanh et al. and Ding are not adequate to claim that GFC-spaces or FC-spaces properly include G-convex spaces.

(VI) FWC-spaces [17] in 2013: Recently, Lu and Zhang [CAMWA 64 (2012) 570–588] introduced the concepts of FWC-spaces (short form of finite weakly convex spaces) as a unified form of many known modifications of G-convex spaces, and the better admissible class of multimaps on them. In this paper, we show that their FWC-spaces and their better admissible classes are inadequately defined and that their results can not be true.

(VII) GFC-spaces [18] in 2013: Earlier we found that our ϕ_A-spaces can be made into G-convex spaces in several ways and that GFC-spaces due to Khanh et al. are all ϕ_A-spaces. Recently, they [JOTA 151: 552–572 (2011)] gave an example of a GFC-space which is a 'trivial' G-convex space. In this paper, we show that a GFC-space can be made into a nontrivial G-convex space $(X, D; \Gamma)$ iff it has a nontrivial KKM map $G : D \multimap X$. Consequently, their example has only a trivial KKM map and is not adequate to show that GFC-spaces properly extend G-convex spaces.

(VIII) GFC-spaces [18] in 2013: In the KKM theory, G-convex spaces are extended to KKM spaces or abstract convex spaces in 2006. Various types of ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ appeared until 2007 can be made into G-convex spaces in several ways. Moreover, various types of generalized KKM maps on ϕ_A-spaces are simply KKM maps on G-convex spaces. Therefore, our G-convex space theory can be applied to various types of ϕ_A-spaces. However, Khanh et al. in 2009 introduced a disguised form of ϕ_A-spaces called GFC-spaces. In the present paper, we review their works on GFC-spaces and clarify that their basic results are consequences of known ones. Finally, further comments on each of seven papers on GFC-spaces are given.

7. Conclusion

We began to initiate the KKM theory in 1992, to study G-spaces in 1993, and to introduce abstract convex spaces in 2006. In 2021, we extended the study to a large scaled logical system called the Grand KKM Theory [22]. In such frame, we introduced various multimap classes, various types of (partial) KKM spaces or abstract convex spaces, and hundreds of theorems on them.

In the present paper, we noticed that the KKM structure is restricted to F-spaces (or GFC-spaces). Therefore facts or theorems on them can be extended to more general other useful spaces like KKM spaces, partial KKM spaces, and abstract convex spaces.

Khanh and Quan claimed that their results is useful to show equivalent conditions for the existence of certain points. However, as our Theorem 5.4 shows, such existence can be obtained by already known results. For any abstract convex spaces $(E, D; \Gamma)$, a topology $\mathcal{T}_E$ of E and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values are already given.

The references [23]-[32] are the list of the author's article appeared in NFAA and closely related to the present article.

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