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APPROXIMATION METHODS FOR SOLVING SPLIT EQUALITY OF VARIATIONAL INEQUALITY AND $f, g-$FIXED POINT PROBLEMS IN REFLEXIVE BANACH SPACES

Yirga Abebe Belay[1], Habtu Zegeye[2] and Oganeditse A. Boikanyo[3]

[1]Department of Mathematics and Statistical Sciences, Faculty of Sciences,
Botswana International University of Science and Technology,
Private Bag 0016, Palapye, Botswana
e-mail: yirga2006@gmail.com

[2]Department of Mathematics and Statistical Sciences, Faculty of Sciences,
Botswana International University of Science and Technology,
Private Bag 0016, Palapye, Botswana
e-mail: habtuzh@yahoo.com

[3]Department of Mathematics and Statistical Sciences, Faculty of Sciences,
Botswana International University of Science and Technology,
Private Bag 0016, Palapye, Botswana
e-mail: boikanyoa@gmail.com

Abstract. The purpose of this paper is to introduce and study a method for solving the split equality of variational inequality and f, g-fixed point problems in reflexive real Banach spaces, where the variational inequality problems are for uniformly continuous pseudomonotone mappings and the fixed point problems are for Bregman relatively f, g-nonexpansive mappings. A strong convergence theorem is proved under some mild conditions. Finally, a numerical example is provided to demonstrate the effectiveness of the algorithm.

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[0]Corresponding author: Habtu Zegeye(habtuzh@yahoo.com).

1. Introduction

Let E be a real Banach space with dual E^*. Let $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$ be the generalized duality pairing and the induced norm, respectively. If C is a nonempty, closed and convex subset of E, then the mapping A: $C \to E^*$ is said to be

(a) $\alpha-$strongly monotone on C if there exists $\alpha > 0$ such that
$$\langle Ax - Ay, x - y \rangle \geq \alpha ||x - y||^2,$$
for all $x, y \in C$;

(b) monotone on C if
$$\langle Ax - Ay, x - y \rangle \geq 0,$$
for all $x, y \in C$;

(c) $\alpha-$ inverse strongly monotone if there exists $\alpha > 0$ such that
$$\langle Ax - Ay, x - y \rangle \geq \alpha ||Ax - Ay||^2,$$
for all $x, y \in C$;

(d) $\alpha-$strongly pseudomonotone on C if there exists $\alpha > 0$ such that
$$\langle Ax, y - x \rangle \geq 0 \implies \langle Ay, y - x \rangle \geq \alpha ||x - y||^2,$$
for all $x, y \in C$;

(e) pseudomonotone on C if
$$\langle Ax, y - x \rangle \geq 0 \implies \langle Ay, y - x \rangle \geq 0,$$
for all $x, y \in C$;

(f) $L-$ Lipschitz continuous on C if there exists a constant $L > 0$, called the Lipschitz constant, such that
$$||Ax - Ay|| \leq L||x - y||,$$
for all $x, y \in C$;

(g) If $L < 1$, then A is called a *contraction* and if $L = 1$, then A is said to be *nonexpansive*;

(h) The mapping A is said to be *sequentially weakly continuous* if $\{Ax_n\}$ converges weakly to Ax whenever $\{x_n\}$ is a sequence that converges weakly to x.

A point $x \in C$ is called a *fixed point* of the mapping G: $C \to E$ if $Gx = x$. The set of fixed points of G is denoted by $F(G)$.

Remark 1.1. Every $\alpha-$strongly monotone mapping is monotone and hence pseudomonotone. It is also easy to see that every $\alpha-$strongly monotone mapping is $\alpha-$strongly pseudomonotone.

The variational inequality problem (VIP, in short) is defined as finding a point $x \in C$ such that
$$\langle Ax, \ y - x \rangle \geq 0 \quad \text{for all } y \in C, \tag{1.1}$$

where C is a nonempty, closed and convex subset of E and $A\colon C \to E^*$ is a mapping. The solution set of the variational inequality problem VIP(1.1) is denoted by $VI(C, A)$. The concept of variational inequality problem was initially introduced by Hartman and Stampacchia [13] as a natural generalization of boundary value problems. Such problems are applicable in a wide range of applied sciences and mathematics. It helps us to solve new problems that emerge from the fields of applied mathematics, engineering, physics, mechanics, convex programming and the theory of control.

Many authors have studied and proposed different methods for solving VIP(1.1) in different settings (see, for instance, [2, 8, 10, 11, 14, 22, 29, 30, 36, 37] and the references therein).

If H is a real Hilbert space and $A\colon H \to H$ is a Lipschitz continuous and strongly monotone, then the projected gradient method introduced by Goldstein [12] is the simplest method to solve (1.1).

In 2020, Thong *et al.* [28] proposed the following projection type algorithm to solve (1.1) in a Hilbert space setting. Given $l \in (0, 1), \mu > 0, \beta \in \left(0, \frac{1}{\mu}\right)$ and $x_1 \in C$, compute:

$$\begin{cases} s_n = P_C \left[x_n - \beta A x_n\right], \\ w_n = x_n - \gamma_n r_\beta(x_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) P_{C_n}(x_n), \end{cases} \tag{1.2}$$

where P_C is the metric projection from H onto C; A is uniformly continuous pseudomonotone mapping that is sequentially weakly continuous on bounded subsets of C; $f\colon C \to C$ is a contraction mapping with a coefficient $\delta \in [0, 1)$; $r_\beta(x_n) = x_n - s_n$; $\{\alpha_n\}$ is a sequence of real numbers in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$; $\gamma_n = l^{j_n}$, where j_n is the smallest nonnegative integer j that satisfies

$$\langle Ax_n - A\left(x_n - l^j r_\beta(x_n)\right), r_\beta(x_n)\rangle \leq \mu ||r_\beta(x_n)||^2,$$

$C_n = \{x \in C : h_n(x) \leq 0\}$ and $h_n(x) = \langle Aw_n, x - w_n\rangle$. They proved that if $VI(C, A) \neq \emptyset$, then the sequence generated by (1.2) converges strongly to some $x \in VI(C, A)$, where $x = P_{VI(C,A)} f(x)$.

In Banach spaces, more general than Hilbert spaces, Jolaoso and Shehu [16] introduced the following single Bregman projection method to solve variational inequality problems (see [33]): Let C be a nonempty, closed and convex subset of E. Given $u_1 \in E, \lambda_1 > 0$ and $\rho \in (0, \alpha)$, compute

$$u_{n+1} = \nabla f^* \left(\nabla f(z_n) - \lambda_n (Az_n - Au_n)\right), \tag{1.3}$$

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \dfrac{\rho||u_n - z_n||}{||Au_n - Az_n||}\right\}, & if \ \ Au_n \neq Az_n \\ \lambda_n & \text{otherwise,} \end{cases}$$

where $z_n = P_C^f(\nabla f^*(\nabla f(u_n) - \lambda_n Au_n))$, A: $E \to E^*$ is a pseudomonotone, sequentially weakly continuous and Lipschitz continuous mapping and $f : E \to \mathbb{R}$ is a proper, lower semi-continuous, uniformly Fréchet differentiable, $\alpha-$strongly convex, strongly coercive and Legendre function which is bounded. They proved that the sequence generated by (1.3) converges weakly to some point in $VI(C, A)$ in a reflexive real Banach space E provided that $VI(C, A) \neq \emptyset$. They also proved the strong convergence of the algorithm to a point of $VI(C, A)$ if, in addition, A is strongly pseudomonotone.

Besides these, several authors have proposed and studied different schemes for finding a common point of the solution set of variational inequality and fixed point problems (see, for example, [24, 27, 31, 34]). This method became very important in optimization theory because it is applicable in mathematical models whose constraints can be modeled as both problems.

Takahashi and Toyoda [27] introduced an iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for an inverse strongly monotone mapping in Hilbert spaces. Given a nonempty, closed and convex subset D of H and $x_0 \in D$. Let $\{x_n\}$ be the sequence generated by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) GP_D\left(x_n - \eta_n Ax_n\right), \tag{1.4}$$

where n is a nonnegative integer, A: $D \to H$ is $\beta-$ inverse strongly monotone mapping, G: $D \to H$ is a nonexpansive mapping, $\{\alpha_n\} \subset (0, 1)$ and $\{\eta_n\} \subset (0, 2\beta)$. They proved that if $VI(D, A) \cap F(G) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.4) converges weakly to an element $p \in VI(D, A) \cap F(G)$.

In 2021, Wega and Zegeye [34] studied a method of approximating a common element of the set of f-fixed points of a Bregman relatively $f-$nonexpansive mapping and the set of solutions of a variational inequality problem for a Lipschitz monotone mapping in a reflexive real Banach space. They introduced the following algorithm to find a point in $VI(K, B) \cap F_f(G)$. For a reflexive Banach space E with its dual E^*, let g: $E \to (-\infty, \infty]$ be a strongly coercive, bounded, $\lambda-$ strongly convex on bounded subsets of E, uniformly Fréchet differentiable Legendre function. Let K be a nonempty, closed and convex subset of E. Let B: $K \to E^*$ be a Lipschitz monotone mapping with Lipschitz constant L. Let G: $K \to E^*$ be a Bregman relatively $f-$nonexpansive mapping with $\Gamma = VI(K, B) \cap F_f(G) \neq \emptyset$. Given $x_0, x \in K$, let the sequence

$\{x_n\}$ be generated by

$$\begin{cases} s_n = P_K^g \nabla g^* \left(\nabla g x_n - \eta_n B x_n\right), \\ t_n = \nabla g^* \left[\xi_n \nabla g x_n + \beta_n G x_n + \tau_n \nabla g w_n\right], \\ x_{n+1} = P_K^g \nabla g^* \left(\alpha_n \nabla g(x) + (1-\alpha_n) \nabla g t_n\right), \end{cases} \tag{1.5}$$

where $P_K^g(x)$ is the Bregman projection of $x \in$ *int(dom g)* onto K, $w_n = P_K^g \nabla g^* \left(\nabla g x_n - \eta_n B s_n\right)$, $0 < \underline{\eta} \leq \eta_n \leq \bar{\eta} < \frac{\lambda}{L}$, $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{\xi_n\}, \{\beta_n\}, \{\tau_n\} \subset [\delta, 1) \subset (0,1)$ such that $\xi_n + \beta_n + \tau_n = 1$. They proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to some element $\bar{x}$, where $\bar{x} = P_\Gamma^g(x)$.

One of the generalizations of the variational inequality problem is the split equality variational inequality problem (SEVIP) which is defined as finding a point

$$(\bar{x}, \bar{u}) \in VI(C, A) \times VI(D, B) : T\bar{x} = S\bar{u}; \tag{1.6}$$

where C and D are nonempty, closed and convex subsets of real Banach spaces E_1 and E_2, respectively, A: $E_1 \to E_1^*$ and B: $E_2 \to E_2^*$ are nonlinear mappings, T: $E_1 \to E_3$ and S: $E_2 \to E_3$ are bounded linear mappings with adjoints T^*: $E_3^* \to E_1^*$ and S^*: $E_3^* \to E_2^*$, respectively, where E_3 is another Banach space. Some of the special cases of SEVIP are common solutions of variational inequality problem (CSVIP), split variational inequality problem (SVIP), split equality feasibility problem (SEFP) introduced by Moudafi [21], split feasibility problem (SFP) introduced by Censor and Elfving [9] and the split equality null point problem (SENPP).

In 2021, Kwelegano *et al.* [17] introduced an iterative algorithm which solves SEVIP for uniformly continuous and pseudomonotone mappings that are sequentially weakly continuous in Hilbert spaces and proved a strong convergence of the algorithm under certain conditions.

In 2021, Boikanyo and Zegeye [5] introduced a new algorithm which approximates SEVIP for uniformly continuous pseudomonotone mappings that are sequentially weakly continuous in real Banach spaces and they proved strong convergence results.

Let C be a nonempty, closed and convex subset of a real Banach space E and let f: $E \to (-\infty, +\infty]$ be a proper, lower semi-continuous, Gâteaux differentiable and convex function. Let G: $C \to E^*$ be any mapping. A point $p \in C$ is called

(1) an *$f-$fixed point* of G if $Gp = \nabla f p$. The set of $f-$ fixed points of G is denoted by $F_f(G)$.

(2) an $f-$*asymptotic fixed point* of G if there exists a sequence $\{u_n\}$ in C such that $u_n \rightharpoonup p$ and

$$\lim_{n\to\infty} ||\nabla f u_n - G u_n|| = 0.$$

The set of $f-$ asymptotic fixed points of G is denoted by $\widehat{F_f(G)}$.

The mapping G is called *Bregman relatively* $f-$*nonexpansive* if the following three properties hold:

(i) $F_f(G)$ is nonempty;
(ii) $D_f(p, \nabla f^* Gz) \leq D_f(p, z),$ for all $z \in C, p \in \widehat{F_f(G)}$;
(iii) $\widehat{F_f(G)} = F_f(G)$.

All the results discussed above deal with either of the following: solutions of VIPs; solutions of SEVIPs; finding a common solution of VIPs and fixed point problems. Based on these results, we raise the following important question:

Question 1.2. *Can we obtain a method for approximating a solution of split equality of variational inequality and* $f, g-$*fixed point problems in reflexive real Banach spaces, where the variational inequality problems are for uniformly continuous pseudomonotone mappings and the* $f, g-$*fixed point problems are for Bregman relatively* $f, g-$*nonexpansive mappings?*

The split equality of variational inequality and $f, g-$fixed point problem is defined as follows: Let G: $C \to E_1^*$ and K: $D \to E_2^*$ be Bregman relatively $f, g-$ nonexpansive mappings with $f-$fixed points $F_f(G)$ and $g-$fixed points $F_g(K)$, respectively. The split equality of variational inequality and $f, g-$ fixed point problems is defined as finding a point

$$(\bar{x}, \bar{u}) \in (VI(C, A) \cap F_f(G)) \times (VI(D, B) \cap F_g(K)) : T\bar{x} = S\bar{u}, \qquad (1.7)$$

where E_1 and E_2 are reflexive real Banach spaces, A: $E_1 \to E_1^*$ and B: $E_2 \to E_2^*$ are any mappings, f: $E_1 \to \mathbb{R}$ and g: $E_2 \to \mathbb{R}$ are proper, convex, lower semicontinuous, uniformly Fréchet differentiable, strongly convex, strongly coercive Legendre functions which are bounded.

Motivated and inspired by the aforementioned results, in this paper we introduce an algorithm for finding a solution of split equality of variational inequality and $f, g-$fixed point problems, where the variational inequality problems are for uniformly continuous pseudomonotone mappings and the fixed point problems are for Bregman relatively $f, g-$nonexpansive mappings in real Banach spaces. We prove a strong convergence theorem for the algorithm proposed. Finally, we provide a numerical example to demonstrate the effectiveness of the algorithm.

2. Preliminaries

Under this section, we give definitions and some important results that will be used in the subsequent analysis.

Let $\{x_n\}$ be a sequence in a reflexive real Banach space E. The strong and weak convergence of $\{x_n\}$ to a point $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let $U = \{x \in E : ||x|| = 1\}$. E is said to be *strictly convex* if $\frac{||x+y||}{2} < 1$ for all $x, y \in U$ with $x \neq y$. If the limit

$$\lim_{t \to 0} \frac{||y+tz|| - ||y||}{t} \tag{2.1}$$

exists for $y, z \in U$, then we say that E is *smooth*.

Let $f\colon E \to \mathbb{R}$ be a convex function. The domain of f, denoted by $dom f$, is defined as $dom f = \{x \in E : f(x) < +\infty\}$. The function f is said to be *proper* if $dom f \neq \emptyset$. If a function is proper, convex and lower semi-continuous, then it is continuous (see, [4]). The *Fenchel conjugate* of f is the function $f^*\colon E^* \to \mathbb{R}$, defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}$$

for any $x^* \in E^*$. The directional derivative of f at $x \in$ *int(dom f)* in the direction of y is defined as

$$f^o(x, y) = \lim_{t \downarrow 0} \frac{f(x+ty) - f(x)}{t} \tag{2.2}$$

provided that this limit exists. We say that f is *Gâteaux differentiable* at x if the limit in (2.2) exists for every $y \in E$. In this case, we define the gradient of f at x to be the linear function $\langle \nabla f(x), y\rangle = f^o(x, y)$ for all $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in$ *int(dom f)*. If the limit in (2.2) is attained uniformly for any $y \in U$, then we say that f is *uniformly Fréchet differentiable* at x.

A function $f\colon E \to \mathbb{R}$ is said to be a *Legendre function* if and only if it satisfies the following conditions:

(A) *int(dom f)*$\neq \emptyset$, f is Gâteaux differentiable and *dom* $\nabla f =$*int(dom f)*;
(B) *int(dom* f^**)*$\neq \emptyset$, f^* is Gâteaux differentiable and *dom*$\nabla f^* =$*int(dom* f^**)*.

If E is a smooth and strictly convex Banach space, then the function $f(x) = \frac{1}{p}||x||^p$ $(1 < p < \infty)$ is a proper, lower semi-continuous Legendre function with Fenchel conjugate $f^*(x^*) = \frac{1}{q}||x^*||^q$ $(1 < q < \infty)$, (see, for instance, [3]), where

$\frac{1}{p}+\frac{1}{q}=1$. In this case, the gradient of f is equal to the *generalized duality mapping*, J_p, of E. That is, $\nabla f = J_p$, where $J_p : E \to 2^{E^*}$ is defined as

$$J_p(x) = \left\{x^* \in E^* : \langle x^*, x\rangle = ||x||^p,\ ||x^*|| = ||x||^{p-1}\right\}.$$

If $p = 2$, then we write $J_p = J$ and we call it the *normalized duality mapping* and if, in addition, $E = H$, where H is a real Hilbert space, then $J = I$, where I is the identity mapping on H. If f: $E \to (-\infty, +\infty]$ is a Legendre function and E is a reflexive Banach space, then $\nabla f^* = (\nabla f)^{-1}$(see, [6]). We also have that f is a Legendre function if and only if f^* is a Legendre function (see, [3]).

Lemma 2.1. ([26]) *If E is a real Banach space and J_E is the normalized duality mapping on E, then*

$$||x + y||^2 \le ||x||^2 + 2\langle j_E(x + y), y\rangle$$

for all $x, y \in E$ and all $j_E(x + y) \in J_E(x + y)$.

Definition 2.2. The function f: $E \to \mathbb{R}\cup\{+\infty\}$ is said to be strongly coercive if $\frac{f(x)}{||x||} \to +\infty$ as $||x|| \to +\infty$.

Definition 2.3. Let E be a Banach space and f: $E \to \mathbb{R}\cup\{+\infty\}$ be a Gâteaux differentiable convex function. The function D_f: *dom* f × *int*(*dom* f) $\to$ $[0, +\infty)$ defined by

$$D_f(y, x) = f(y) - f(x) - \langle\nabla f(x), y - x\rangle$$

is called the *Bregman distance* with respect to f.

The Bregman distance has the following important properties:
For any $w, x, y, z \in E$,

(i) *Three point identity*:

$$D_f(w, x) + D_f(x, y) - D_f(w, y) = \langle\nabla f(x) - \nabla f(y), x - w\rangle. \tag{2.3}$$

(ii) *Four point identity*:

$$\begin{aligned} &D_f(x, z) + D_f(w, y) - D_f(x, y) - D_f(w, z) \\ &\quad = \langle\nabla f(y) - \nabla f(z), x - w\rangle. \end{aligned} \tag{2.4}$$

Definition 2.4. A Gâteaux differentiable function f: $E \to \mathbb{R}\cup\{+\infty\}$ defined on a reflexive real Banach space E is said to be *strongly convex* if there exists a constant $\beta > 0$ such that

$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq \beta ||x - y||^2,$$

for all $x, y \in domf$, or equivalently

$$f(y) \geq f(x) + \langle \nabla f(x), y - x\rangle + \frac{\beta}{2}||x - y||^2.$$

If E is a smooth and strictly convex Banach space, then $f(x) = \frac{1}{2}||x||^2$ is a strongly coercive, bounded, uniformly Fréchet differentiable and strongly convex function with strong convexity constant $\beta \in (0, 1]$ and Fenchel conjugate $f^*(x^*) = \frac{1}{2}||x^*||^2$.

It can be easily shown that if f is a strongly convex function with constant $\beta > 0$, then for all $y \in domf$ and $x \in int(domf)$, we have

$$D_f(y, x) \geq \frac{\beta}{2}||x - y||^2. \tag{2.5}$$

Definition 2.5. Let $C \subseteq int(dom\ f)$ be a nonempty, closed and convex subset of real Banach space E, where $f\colon E \to \mathbb{R} \cup \{+\infty\}$ is a convex and Gâteaux differentiable function. The Bregman projection of $x \in int(dom\ f)$ onto C is the unique vector $P_C^f(x)$ of C with the property

$$D_f(P_C^f(x), x) = inf\left\{D_f(y, x) : y \in C\right\}.$$

The Bregman projection also satisfies the following properties:

$$z = P_C^f(x) \text{ if and only if } \langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \text{ for all } y \in C, \tag{2.6}$$

and

$$D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \text{ for all } x \in E, y \in C. \tag{2.7}$$

Lemma 2.6. ([1]) *Let E_1 and E_2 be reflexive Banach spaces. Then, $E = E_1 \times E_2$ is also a reflexive Banach space with dual $E^* = E_1^* \times E_2^*$ and duality pairing*

$$\langle (x_1, y_1), (x_2, y_2)\rangle = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle$$

for all $(x_1, y_1) \in E$, $(x_2, y_2) \in E^*$ and $(x_n, y_n) \rightharpoonup (x, y)$ implies $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$.

If C is a nonempty, closed and convex subset of E, $f\colon E_1 \to \mathbb{R} \cup \{+\infty\}$, $g : E_2 \to \mathbb{R} \cup \{+\infty\}$, $(x, y) \in E$, $(x^*, y^*) = P_C^h(x, y)$, where $h = (f, g)$ and $\nabla h = (\nabla f, \nabla g)$, then

$$\langle (u, v) - (x^*, y^*), (\nabla f(x), \nabla g(y)) - (\nabla f(x^*), \nabla g(y^*))\rangle \leq 0 \tag{2.8}$$

for all $(u, v) \in C$.

Lemma 2.7. ([25]) *If E is a reflexive real Banach space and $f\colon E \to (-\infty, +\infty]$ is a proper, lower semi-continuous, convex and Gâteaux differentiable function, then $f^*\colon E^* \to (-\infty, +\infty]$ is a proper, weak* lower semi-continuous and convex function. Thus, for all $x \in E$, we have*

$$D_f\left(x, \nabla f^*\left(\sum_{i=1}^{N} s_i \nabla f(y_i)\right)\right) \leq \sum_{i=1}^{N} s_i D_f(x, y_i),$$

where $\{y_i\}_{i=1}^N \subseteq E$ and $\{s_i\}_{i=1}^N \subseteq (0,1)$ with $\sum_{i=1}^N s_i = 1$.

We say that a function f is *uniformly convex* with modulus ϕ if for all $x, y \in dom\ f$ and $\gamma \in [0,1]$, we have

$$f(\gamma x + (1-\gamma)y) \leq \gamma f(x) + (1-\gamma) f(y) - \gamma(1-\gamma)\phi(||x-y||),$$

where ϕ is an increasing function and $\phi(x) = 0$ only for $x = 0$.

The subdifferential ∂f of f at x is defined by

$$\partial f(x) = \{x^* \in E^* : \langle x^*, y - x\rangle \leq f(y) - f(x),\ \forall\, y \in E\} \text{ (see, [15]).}$$

Lemma 2.8. ([35]) *Let f be a convex and lower semi-continuous function on a Banach space E. The following assertions are equivalent:*

(i) *f is uniformly convex;*

(ii) *there exists modulus ϕ, for all $(x, x^*), (y, y^*) \in Gph(\partial f)$ such that*

$$f(y) \geq f(x) + \langle x^*, y - x\rangle + \phi(||x-y||);$$

(iii) *$dom\ f^* = E^*$, f^* is Fréchet differentiable and ∇f^* is uniformly continuous.*

Note that a strongly convex function is uniformly convex with $\phi(x) = \frac{\beta}{2}||x||^2$ and hence the class of uniformly convex functions contains the class of strongly convex functions.

Lemma 2.9. ([23]) *Let E be a Banach space and $f\colon E \to (-\infty, +\infty]$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}$ and $\{u_n\}$ be bounded sequences in E. Then, $\lim_{n\to\infty} D_f(x_n, u_n) = 0$ if and only if $\lim_{n\to\infty}(x_n - u_n) = 0$.*

Let $f\colon E \to \mathbb{R}$ be a Gâteaux differentiable Legendre function. The nonnegative real-valued function $V_f : E \times E^* \to [0, +\infty)$ defined by

$$V_f(x, x^*) = f(x) - \langle x^*, x\rangle + f^*(x^*) \text{ for all } x \in E,\ \ x^* \in E^*, \tag{2.9}$$

satisfies the properties

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)), \tag{2.10}$$

and

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*) \tag{2.11}$$

for all $x \in E$, $x^*, y^* \in E^*$

Lemma 2.10. ([20]) *Let C be a nonempty, closed and convex subset of a reflexive real Banach space E. If $A\colon C \to E^*$ is a continuous pseudomonotone mapping, then $VI(C, A)$ is closed and convex. Moreover, $\langle Ap, q - p\rangle \geq 0$ for all $q \in C$ if and only if $\langle Aq, q - p\rangle \geq 0$ for all $q \in C$.*

Lemma 2.11. ([5]) *If $\{c_n\}$ is a sequence of nonnegative real numbers such that*

$$c_{n+1} \leq (1 - \alpha_n)\, c_n + \alpha_n d_n,$$

where $\{\alpha_n\} \subset (0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{d_n\}$ is a sequence of real numbers with $\limsup_{n\to\infty} d_n \leq 0$, then $\lim_{n\to\infty} c_n = 0$.

Lemma 2.12. ([18]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers. If $\{a_{n_i}\}$ is a subsequence of $\{a_n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$, then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, m_k=$\max\{n \leq k : a_n < a_{n+1}\}$.

The modulus of total convexity of a Gâteaux differentiable function f is the function v_f: $int(domf) \times [0, \infty) \to [0, \infty)$ defined by

$$v_f(x, t) = \inf\{D_f(y, x) : y \in domf, ||y - x|| = t\}.$$

In this case we say that f is *totally convex* at a point $x \in int(dom\ f)$ if $v_f(x, t) > 0$ whenever $t > 0$. The function f is said to be totally convex if it is totally convex at every point $x \in int(dom\ f)$.

On bounded subsets of E, the concepts of uniform convexity and total convexity are the same (see, [7]).

Lemma 2.13. ([19]) *Let E be a reflexive real Banach space and $f\colon E \to \mathbb{R}$ be a totally convex function. If $\{D_f(x_n, x_0)\}$ is bounded for any $x_0 \in E$, then $\{x_n\}$ is bounded.*

Lemma 2.14. ([32]) *Let f be a continuous, convex and strongly coercive real valued function defined on a reflexive real Banach space E. Then the following are equivalent:*

(i) *f is uniformly smooth and bounded on bounded subsets of E;*
(ii) *$f^*\colon E^* \to \mathbb{R}$ is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*;*

(iii) *f^* is strongly coercive, uniformly convex on bounded subsets of E^* and $dom f^* = E^*$.*

Lemma 2.15. *Let C be a subset of a reflexive real Banach space E and $A\colon C \to E^*$ be any mapping. Let $f\colon E \to \mathbb{R}$ be a strongly convex function with constant η and has Lipschitz continuous gradient with constant γ. If $\gamma \leq \eta$, then for any $x \in E$ and $\alpha \geq \beta > 0$, the following inequality holds:*

$$\frac{||x - P_C^f \nabla f^*(\nabla f(x) - \alpha Ax)||}{\alpha} \leq \frac{||x - P_C^f \nabla f^*(\nabla f(x) - \beta Ax)||}{\beta}.$$

Proof. Let $x_\alpha = P_C^f \nabla f^*(\nabla f(x) - \alpha Ax)$ and $x_\beta = P_C^f \nabla f^*(\nabla f(x) - \beta Ax)$. Then it follows from (2.6) that

$$\langle \nabla f(x) - \alpha Ax - \nabla f(x_\alpha), y - x_\alpha \rangle \leq 0$$

and

$$\langle \nabla f(x) - \beta Ax - \nabla f(x_\beta), y - x_\beta \rangle \leq 0$$

for all $y \in C$, which implies that

$$\Big\langle \frac{\nabla f(x_\alpha) - \nabla f(x)}{\alpha} + Ax, x_\beta - x_\alpha \Big\rangle \geq 0$$

and

$$\Big\langle \frac{\nabla f(x_\beta) - \nabla f(x)}{\beta} + Ax, x_\alpha - x_\beta \Big\rangle \geq 0.$$

Adding both inequalities and using the Cauchy-Schwarz inequality, strong convexity of f and Lipschitz continuity of ∇f, we get

$$\begin{aligned}
0 \leq & \Big\langle \frac{\nabla f(x) - \nabla f(x_\alpha)}{\alpha} - \frac{\nabla f(x) - \nabla f(x_\beta)}{\beta}, x_\alpha - x_\beta \Big\rangle \\
= & \Big\langle \frac{\nabla f(x) - \nabla f(x_\alpha)}{\alpha} - \frac{\nabla f(x) - \nabla f(x_\beta)}{\beta}, (x - x_\beta) - (x - x_\alpha) \Big\rangle \\
= & -\Big\langle \frac{\nabla f(x) - \nabla f(x_\alpha)}{\alpha}, x - x_\alpha \Big\rangle + \Big\langle \frac{\nabla f(x) - \nabla f(x_\alpha)}{\alpha}, x - x_\beta \Big\rangle \\
& -\Big\langle \frac{\nabla f(x) - \nabla f(x_\beta)}{\beta}, x - x_\beta \Big\rangle + \Big\langle \frac{\nabla f(x) - \nabla f(x_\beta)}{\beta}, x - x_\alpha \Big\rangle \\
\leq & -\frac{1}{\alpha}\langle \nabla f(x) - \nabla f(x_\alpha), x - x_\alpha \rangle - \frac{1}{\beta}\langle \nabla f(x) - \nabla f(x_\beta), x - x_\beta \rangle \\
& +\frac{1}{\alpha}||\nabla f(x) - \nabla f(x_\alpha)||\,||x - x_\beta|| + \frac{1}{\beta}||\nabla f(x) - \nabla f(x_\beta)||\,||x - x_\alpha||
\end{aligned}$$

$$\begin{aligned}
&\leq -\frac{\eta}{\alpha}||x-x_\alpha||^2 - \frac{\eta}{\beta}||x-x_\beta||^2 + \frac{\gamma}{\alpha}||x-x_\alpha||\,||x-x_\beta|| \\
&+ \frac{\gamma}{\beta}||x-x_\beta||\,||x-x_\alpha|| \\
&\leq \frac{\eta}{\alpha}||x-x_\alpha||^2 - \frac{\eta}{\beta}||x-x_\beta||^2 + \frac{\eta}{\alpha}||x-x_\alpha||\,||x-x_\beta|| \\
&+ \frac{\eta}{\beta}||x-x_\beta||\,||x-x_\alpha||. \qquad (2.12)
\end{aligned}$$

Thus, from (2.12), we obtain

$$0 \;\geq\; (||x-x_\alpha|| - ||x-x_\beta||)\left(\frac{||x-x_\alpha||}{\alpha} - \frac{||x-x_\beta||}{\beta}\right). \qquad (2.13)$$

Now, suppose on the contrary that we have

$$\frac{||x-x_\alpha||}{\alpha} > \frac{||x-x_\beta||}{\beta}, \qquad (2.14)$$

which implies that

$$||x-x_\alpha|| > ||x-x_\beta||. \qquad (2.15)$$

Combining (2.14) and (2.15), we obtain

$$0 \;<\; (||x-x_\alpha|| - ||x-x_\beta||)\left(\frac{||x-x_\alpha||}{\alpha} - \frac{||x-x_\beta||}{\beta}\right),$$

which is a contradiction to (2.13) hence the proof is completed. □

Examples of functions that satisfy conditions of the hypothesis in Lemma 2.15 are functions of the type $f(x) = k||x||^2$, for $k > 0$. One can show that f is strongly convex with strong convexity constant $2k$ and ∇f is Lipschitz continuous with Lipschitz constant $2k$.

3. Main results

The following assumptions will be used in the sequel.

Condition 3.1.

($A1$) Let C and D be nonempty, closed and convex subsets of the smooth, strictly convex and reflexive real Banach spaces E_1 and E_2, respectively.

(A2) Let f: $E_1 \to \mathbb{R}$ and g: $E_2 \to \mathbb{R}$ be proper, lower semi-continuous, strongly coercive, uniformly Fréchet differentiable, strongly convex Legendre functions which are bounded on bounded subsets of E_1 and E_2, respectively. Let f and g have Lipschitz continuous gradients with the strong convexity constant of f (respectively, g) greater than or equal to the Lipschitz constant of ∇f (respectively, ∇g).

Condition 3.2.

($B1$) Let A: $C \to E_1^*$ and B: $D \to E_2^*$ be uniformly continuous, pseudomonotone and sequentially weakly continuous mappings;

($B2$) Let G: $E_1 \to E_1^*$ and K: $E_2 \to E_2^*$ be Bregman relatively $f-$nonexpansive and Bregman relatively $g-$nonexpansive mappings, respectively;

($B3$) Let T: $E_1 \to E_3$ and S: $E_2 \to E_3$ be bounded linear mappings with adjoints T^*: $E_3^* \to E_1^*$ and S^*: $E_3^* \to E_2^*$, respectively, where E_3 is another smooth, strictly convex real reflexive Banach space;

($B4$) Let the set of solutions of (1.7), denoted by Υ, be nonempty, that is, $\Upsilon = \{(x,u) \in (VI(C,A) \cap F_f(G)) \times (VI(D,B) \cap F_g(K)) : Tx = Su\} \neq \emptyset$.

Condition 3.3.

($C1$) Let $\beta = min\{\beta_1, \beta_2\}$, where β_1 and β_2 are the strong convexity constants of f and g, respectively;

($C2$) Let $\{\alpha_n\} \subseteq (0,1)$ be such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Algorithm A.
Initialization: Choose $(x_1, u_1) \in E_1 \times E_2$, $\mu \in (0, \beta)$, $l, \gamma, \tau \in (0,1)$. For $x \in C, u \in D$ define the algorithm as follows:

Step 1: Given the current iterates x_n and u_n, compute

$$\begin{cases} z_n = P_C^f \left[\nabla f^* \left(\nabla f(x_n) - \gamma_n T^* J_{E_3} \left(Tx_n - Su_n\right)\right)\right], \\ w_n = P_D^g \left[\nabla g^* \left(\nabla g(u_n) - \gamma_n S^* J_{E_3} \left(Su_n - Tx_n\right)\right)\right], \end{cases} \tag{3.1}$$

where $0 < \rho \leq \gamma_n \leq \rho_n$ for $n \in \{m \in \mathbb{N} : Tx_m - Su_m \neq 0\}$, otherwise $\gamma_n = \rho$, for some $\rho > 0$, and

$$\rho_n = \min\left\{\rho + 1, \frac{\beta||Tx_n - Su_n||^2}{2[||T^* J_{E_3}(Tx_n - Su_n)||^2 + ||S^* J_{E_3}(Su_n - Tx_n)||^2]}\right\}.$$

Step 2: Compute

$$\begin{cases} y_n = P_C^f \left(\nabla f^* \left(\nabla f(z_n) - \lambda_n A z_n\right)\right), \\ v_n = P_D^g \left(\nabla g^* \left(\nabla g(w_n) - \eta_n B w_n\right)\right), \end{cases}$$

where $\lambda_n = \gamma l^{j_m}$, for j_m is the smallest nonnegative integer j satisfying

$$\gamma l^j ||Ay_n - Az_n|| \leq \mu ||y_n - z_n||, \tag{3.2}$$

and $\eta_n = \gamma l^{k_m}$, for k_m is the smallest nonnegative integer k satisfying

$$\gamma l^k ||Bv_n - Bw_n|| \leq \mu ||v_n - w_n||. \tag{3.3}$$

Step 3: Compute

$$a_n = \nabla f^* \left(\nabla f(y_n) - \lambda_n (Ay_n - Az_n)\right),$$

$$b_n = \nabla g^* \left(\nabla g(v_n) - \eta_n(Bv_n - Bw_n)\right),$$

$$\begin{cases} x_{n+1} = \nabla f^* \left(\alpha_n \nabla f(x) + (1-\alpha_n)\left[\tau \nabla f(a_n) + (1-\tau)G(a_n)\right]\right), \\ u_{n+1} = \nabla g^* \left(\alpha_n \nabla g(u) + (1-\alpha_n)\left[\tau \nabla g(b_n) + (1-\tau)K(b_n)\right]\right). \end{cases} \tag{3.4}$$

Set $n := n+1$ and go to **Step 1**.

Hereunder, we present some results that are fundamental to the convergence analysis of the sequences generated by Algorithm A. We begin by proving that the proposed algorithm is well-defined.

Lemma 3.1. *Assume that Conditions* $(A1)-(A2), (B1)-(B4)$ *and* $(C1)-(C2)$ *hold. Then the Armijo line-search rules* (3.2) *and* (3.3) *are well-defined.*

Proof. If $z_n \in VI(C, A)$, then $z_n = P_C^f \nabla f^*(\nabla f(z_n) - \lambda_n A z_n)$. In this case, we have $z_n = y_n$ and hence (3.2) holds for $j = 0$. Now, we consider the case when $z_n \notin VI(C, A)$ and assume on the contrary that for all $j \geq 0$ we have

$$\gamma l^j ||Ay_n - Az_n|| > \mu ||y_n - z_n||.$$

That is,

$$\begin{aligned} &||AP_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n) - Az_n|| \\ &\quad > \frac{\mu}{\gamma l^j} ||P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n) - z_n||. \end{aligned} \tag{3.5}$$

Since P_C^f and ∇f^* are continuous, we have that

$$\lim_{j\to\infty} ||z_n - P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n)|| = 0. \tag{3.6}$$

By the uniform continuity of the mapping A on bounded subsets of C, we obtain

$$\lim_{j\to\infty} ||AP_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n) - Az_n|| = 0. \tag{3.7}$$

From (3.5) and (3.7), we have

$$\lim_{j\to\infty} \frac{||z_n - P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n)||}{\gamma l^j} = 0. \tag{3.8}$$

Since ∇f is Lipschitz continuous, there exists a real number $R > 0$ such that

$$\begin{aligned} 0 &\leq \lim_{j\to\infty} \frac{||\nabla f(z_n) - \nabla f\left(P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n)\right)||}{\gamma l^j} \\ &\leq R \lim_{j\to\infty} \frac{||z_n - P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j Az_n)||}{\gamma l^j}, \end{aligned}$$

which implies from (3.8) that

$$\lim_{j\to\infty} \frac{||\nabla f(z_n) - \nabla f P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j A z_n)||}{\gamma l^j} = 0. \tag{3.9}$$

Let $k_j = P_C^f \nabla f^*(\nabla f(z_n) - \gamma l^j A z_n)$. Then, by (2.6), we get

$$\langle \nabla f(k_j) - \nabla f(z_n) + \gamma l^j A z_n, y - k_j \rangle \geq 0, \quad \forall y \in C,$$

which implies that

$$\left\langle \frac{\nabla f(k_j) - \nabla f(z_n)}{\gamma l^j}, y - k_j \right\rangle + \langle A z_n, z_n - k_j \rangle + \langle A z_n, y - z_n \rangle \geq 0, \forall y \in C. \tag{3.10}$$

Taking the limit as $j \to \infty$ in (3.10) and using (3.6) and (3.9), we obtain

$$\langle A z_n, y - z_n \rangle \geq 0 \text{ for all } y \in C \tag{3.11}$$

and this implies that $z_n \in VI(C, A)$, which is a contradiction. Hence, (3.2) holds. Similarly, one can show that (3.3) holds and hence the proof is complete. □

Theorem 3.2. *Assume that Conditions* $(A1)-(A2)$, $(B1)-(B4)$ *and* $(C1)-(C2)$ *hold. Then the sequences* $\{x_n\}$ *and* $\{u_n\}$ *generated by Algorithm A are bounded.*

Proof. Denote

$$q_n = \nabla f^* \left(\nabla f(x_n) - \gamma_n T^* J_{E_3}(T x_n - S u_n) \right)$$

and

$$t_n = \nabla g^* \left(\nabla g(u_n) - \gamma_n S^* J_{E_3}(S u_n - T x_n) \right).$$

Let $(\bar{x}, \bar{u}) \in \Upsilon$. Then by (3.4), Lemma 2.7 and Bregman relatively $f-$non-expansiveness of G, we have

$$\begin{aligned} D_f(\bar{x}, x_{n+1}) &= D_f\big(\bar{x}, \nabla f^* \left(\alpha_n \nabla f(x) + (1-\alpha_n) \left[\tau \nabla f(a_n) + (1-\tau) G(a_n) \right] \right)\big) \\ &\leq \alpha_n D_f(\bar{x}, x) + (1-\alpha_n) D_f\big(\bar{x}, \nabla f^* \left[\tau \nabla f(a_n) + (1-\tau) G(a_n) \right]\big) \\ &\leq \alpha_n D_f(\bar{x}, x) + (1-\alpha_n)\tau D_f(\bar{x}, a_n) \\ &\quad + (1-\alpha_n)(1-\tau) D_f(\bar{x}, \nabla f^*(G(a_n))) \\ &\leq \alpha_n D_f(\bar{x}, x) + (1-\alpha_n)\tau D_f(\bar{x}, a_n) \\ &\quad + (1-\alpha_n)(1-\tau) D_f(\bar{x}, a_n) \\ &= \alpha_n D_f(\bar{x}, x) + (1-\alpha_n) D_f(\bar{x}, a_n). \end{aligned} \tag{3.12}$$

From the definition of a_n, we have

$$\begin{aligned} D_f(\bar{x}, a_n) &= D_f\big(\bar{x}, \nabla f^*(\nabla f(y_n) - \lambda_n(Ay_n - Az_n))\big) \\ &= f(\bar{x}) - \langle \nabla f(y_n) - \lambda_n(Ay_n - Az_n), \bar{x} - a_n\rangle - f(a_n) \\ &= f(\bar{x}) + \langle \nabla f(y_n), a_n - \bar{x}\rangle + \langle \lambda_n(Ay_n - Az_n), \bar{x} - a_n\rangle - f(a_n) \\ &= f(\bar{x}) - \langle \nabla f(y_n), \bar{x} - y_n\rangle - f(y_n) + \langle \nabla f(y_n), \bar{x} - y_n\rangle + f(y_n) \\ &\quad + \langle \nabla f(y_n), a_n - \bar{x}\rangle + \langle \lambda_n(Ay_n - Az_n), \bar{x} - a_n\rangle - f(a_n) \\ &= D_f(\bar{x}, y_n) + \langle \nabla f(y_n), a_n - y_n\rangle \\ &\quad + f(y_n) - f(a_n) + \langle \lambda_n(Ay_n - Az_n), \bar{x} - a_n\rangle \\ &= D_f(\bar{x}, y_n) - D_f(a_n, y_n) + \langle \lambda_n(Ay_n - Az_n), \bar{x} - a_n\rangle. \end{aligned} \tag{3.13}$$

Using (2.4), we get

$$D_f(\bar{x}, y_n) - D_f(a_n, y_n) = D_f(\bar{x}, z_n) - D_f(a_n, z_n) + \langle \nabla f(z_n) - \nabla f(y_n), \bar{x} - a_n\rangle.$$

Thus, (3.13) becomes

$$\begin{aligned} D_f(\bar{x}, a_n) = D_f(\bar{x}, z_n) - D_f(a_n, z_n) + \langle \nabla f(z_n) - \nabla f(y_n), \bar{x} - a_n\rangle \\ + \langle \lambda_n(Ay_n - Az_n), \bar{x} - a_n\rangle. \end{aligned} \tag{3.14}$$

Furthermore, from (2.3), we obtain

$$D_f(a_n, z_n) = D_f(a_n, y_n) + D_f(y_n, z_n) - \langle \nabla f(y_n) - \nabla f(z_n), y_n - a_n\rangle. \tag{3.15}$$

Therefore, from (3.14) and (3.15) we obtain

$$\begin{aligned} D_f(\bar{x}, a_n) &= D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) \\ &\quad + \langle \nabla f(z_n) - \nabla f(y_n), \bar{x} - a_n\rangle \\ &\quad + \langle \nabla f(y_n) - \nabla f(z_n), y_n - a_n\rangle + \langle \lambda_n(Ay_n - Az_n), \bar{x} - a_n\rangle \\ &= D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) \\ &\quad + \langle \nabla f(z_n) - \nabla f(y_n), \bar{x} - y_n\rangle \\ &\quad + \langle \lambda_n(Ay_n - Az_n), \bar{x} - a_n\rangle \\ &= D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) \\ &\quad + \langle \nabla f(z_n) - \nabla f(y_n), \bar{x} - y_n\rangle \\ &\quad + \langle \lambda_n(Ay_n - Az_n), \bar{x} - y_n + y_n - a_n\rangle \\ &= D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) \\ &\quad + \langle \nabla f(z_n) - \nabla f(y_n), \bar{x} - y_n\rangle \\ &\quad + \langle \lambda_n(Ay_n - Az_n), \bar{x} - y_n\rangle + \langle \lambda_n(Ay_n - Az_n), y_n - a_n\rangle \end{aligned}$$

$$
\begin{aligned}
&= D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) + \langle \lambda_n (Ay_n - Az_n) \\
&\quad + \nabla f(z_n) - \nabla f(y_n), \bar{x} - y_n \rangle + \langle \lambda_n (Ay_n - Az_n), y_n - a_n \rangle \\
&= D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) \\
&\quad - \langle \lambda_n (Ay_n - Az_n) - (\nabla f(y_n) - \nabla f(z_n)), y_n - \bar{x} \rangle \\
&\quad + \langle \lambda_n (Ay_n - Az_n), y_n - a_n \rangle.
\end{aligned} \tag{3.16}
$$

Since $y_n = P_C^f [\nabla f^* (\nabla f(z_n) - \lambda_n A z_n)]$, by (2.6) we get

$$
\langle \nabla f(z_n) - \lambda_n A z_n - \nabla f(y_n), y_n - \bar{x} \rangle \geq 0. \tag{3.17}
$$

Since $\bar{x} \in VI(C, A)$ and $y_n \in C$, we have $\langle A\bar{x}, y_n - \bar{x} \rangle \geq 0$. Moreover, the fact that A is pseudomonotone implies that $\langle Ay_n, y_n - \bar{x} \rangle \geq 0$, and thus

$$
\langle \lambda_n A y_n, y_n - \bar{x} \rangle \geq 0. \tag{3.18}
$$

Combining (3.17) and (3.18), we get

$$
\langle \lambda_n (Ay_n - Az_n) - (\nabla f(y_n) - \nabla f(z_n)), y_n - \bar{x} \rangle \geq 0. \tag{3.19}
$$

Thus, from (3.16) and (3.19), we obtain

$$
\begin{aligned}
D_f(\bar{x}, a_n) \leq{} & D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) \\
& + \langle \lambda_n (Ay_n - Az_n), y_n - a_n \rangle.
\end{aligned} \tag{3.20}
$$

Furthermore, from (3.20), Cauchy Schwarz inequality, (3.2) and (2.5), we get

$$
\begin{aligned}
D_f(\bar{x}, a_n) &\leq D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) \\
&\quad + \lambda_n ||y_n - a_n|| \, ||Ay_n - Az_n|| \\
&\leq D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) \\
&\quad + \mu ||y_n - a_n|| \, ||y_n - z_n|| \\
&\leq D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) \\
&\quad + \frac{\mu}{2} \big(||y_n - a_n||^2 + ||y_n - z_n||^2 \big) \\
&\leq D_f(\bar{x}, z_n) - D_f(a_n, y_n) - D_f(y_n, z_n) \\
&\quad + \frac{\mu}{\beta} \big(D_f(a_n, y_n) + D_f(y_n, z_n) \big) \\
&= D_f(\bar{x}, z_n) - \left(1 - \frac{\mu}{\beta} \right) \big(D_f(a_n, y_n) + D_f(y_n, z_n) \big).
\end{aligned} \tag{3.21}
$$

Using (2.7), (2.10) and (2.11), we get

$$\begin{aligned}
D_f(\bar{x}, z_n) &\leq D_f\big(\bar{x}, \nabla f^*\left(\nabla f(x_n) - \gamma_n T^* J_{E_3}(Tx_n - Su_n)\right)\big) - D_f(z_n, q_n)\\
&\leq D_f\big(\bar{x}, \nabla f^*\left(\nabla f(x_n) - \gamma_n T^* J_{E_3}(Tx_n - Su_n)\right)\big)\\
&= V_f\big(\bar{x}, \nabla f(x_n) - \gamma_n T^* J_{E_3}(Tx_n - Su_n)\big)\\
&\leq V_f\left(\bar{x}, \nabla f(x_n)\right)\\
&- \Big\langle \gamma_n T^* J_{E_3}(Tx_n - Su_n),\\
&\quad \nabla f^*\left(\nabla f(x_n) - \gamma_n T^* J_{E_3}(Tx_n - Su_n)\right) - \bar{x}\Big\rangle\\
&= D_f(\bar{x}, x_n) - \gamma_n \big\langle T^* J_{E_3}(Tx_n - Su_n), q_n - \bar{x}\big\rangle\\
&= D_f(\bar{x}, x_n) - \gamma_n \big\langle J_{E_3}(Tx_n - Su_n), Tq_n - T\bar{x}\big\rangle.
\end{aligned} \tag{3.22}$$

Substituting (3.22) into (3.21), we obtain

$$\begin{aligned}
D_f(\bar{x}, a_n) \leq D_f(\bar{x}, x_n) &- \left(1 - \frac{\mu}{\beta}\right)\Big(D_f(a_n, y_n) + D_f(y_n, z_n)\Big)\\
&- \gamma_n \langle J_{E_3}(Tx_n - Su_n), Tq_n - T\bar{x}\rangle.
\end{aligned} \tag{3.23}$$

Thus, from (3.23) and (3.12), we get

$$\begin{aligned}
D_f(\bar{x}, x_{n+1}) \leq\ & \alpha_n D_f(\bar{x}, x) + (1-\alpha_n) D_f(\bar{x}, x_n)\\
&- (1-\alpha_n)\left(1 - \frac{\mu}{\beta}\right)\Big(D_f(a_n, y_n) + D_f(y_n, z_n)\Big)\\
&- (1-\alpha_n)\gamma_n \langle J_{E_3}(Tx_n - Su_n), Tq_n - T\bar{x}\rangle.
\end{aligned} \tag{3.24}$$

Similarly, we have

$$\begin{aligned}
D_g(\bar{u}, u_{n+1}) \leq\ & \alpha_n D_g(\bar{u}, u) + (1-\alpha_n) D_g(\bar{u}, u_n)\\
&- (1-\alpha_n)\left(1 - \frac{\mu}{\beta}\right)\Big(D_g(b_n, v_n) + D_g(v_n, w_n)\Big)\\
&- (1-\alpha_n)\gamma_n \langle J_{E_3}(Su_n - Tx_n), St_n - S\bar{u}\rangle.
\end{aligned} \tag{3.25}$$

Now, denote

$$\Omega_n = D_f(\bar{x}, x_n) + D_g(\bar{u}, u_n)$$

and

$$\Sigma = D_f(\bar{x}, x) + D_g(\bar{u}, u).$$

Since $\mu \in (0, \beta)$ and $\beta > 0$, we have that $1 > \dfrac{\mu}{\beta} > 0$. Thus, $1 - \dfrac{\mu}{\beta} > 0$. Then, combining (3.24) and (3.25) and using the fact that $T\bar{x} = S\bar{u}$, we get

$$\Omega_{n+1} \leq \alpha_n \Sigma + (1-\alpha_n)\Omega_n - (1-\alpha_n)\gamma_n \langle Tq_n - St_n, J_{E_3}(Tx_n - Su_n)\rangle. \tag{3.26}$$

But, we have by the Cauchy Schwarz inequality that

$$\begin{aligned}
-\langle Tq_n - St_n, J_{E_3}(Tx_n - Su_n)\rangle &= -\langle Tx_n - Su_n, J_{E_3}(Tx_n - Su_n)\rangle \\
&\quad - \langle Tq_n - Tx_n, J_{E_3}(Tx_n - Su_n)\rangle \\
&\quad - \langle Su_n - St_n, J_{E_3}(Tx_n - Su_n)\rangle \\
&= -||Tx_n - Su_n||^2 \\
&\quad - \langle q_n - x_n, T^* J_{E_3}(Tx_n - Su_n)\rangle \qquad (3.27) \\
&\quad - \langle u_n - t_n, S^* J_{E_3}(Tx_n - Su_n)\rangle \\
&\leq -||Tx_n - Su_n||^2 \\
&\quad + ||q_n - x_n||\, ||T^* J_{E_3}(Tx_n - Su_n)|| \\
&\quad + ||u_n - t_n||\, ||S^* J_{E_3}(Tx_n - Su_n)||.
\end{aligned}$$

From the strong convexity of f and the definition of q_n, we have

$$\begin{aligned}
||q_n - x_n|| &= ||\nabla f^* \left(\nabla f(x_n) - \gamma_n T^* J_{E_3}(Tx_n - Su_n)\right) - \nabla f^* \left(\nabla f(x_n)\right)|| \\
&\leq \frac{1}{\beta_1} ||\gamma_n T^* J_{E_3}(Tx_n - Su_n)|| \\
&\leq \frac{\gamma_n}{\beta} ||T^* J_{E_3}(Tx_n - Su_n)||.
\end{aligned} \qquad (3.28)$$

Similarly, the strong convexity of g and the definition of t_n gives

$$||t_n - u_n|| \leq \frac{\gamma_n}{\beta} ||S^* J_{E_3}(Su_n - Tx_n)||. \qquad (3.29)$$

Substituting (3.29) and (3.28) into (3.27) and applying the property of γ_n, we get

$$\begin{aligned}
-\gamma_n \langle Tq_n - St_n, J_{E_3}(Tx_n - Su_n)\rangle &\leq -\gamma_n ||Tx_n - Su_n||^2 \\
&\quad + \frac{\gamma_n^2}{\beta} ||T^* J_{E_3}(Tx_n - Su_n)||^2 \\
&\quad + \frac{\gamma_n^2}{\beta} ||S^* J_{E_3}(Su_n - Tx_n)||^2 \\
&\leq -\frac{\rho}{2} ||Tx_n - Su_n||^2 - \frac{\gamma_n}{2} ||Tx_n - Su_n||^2 \\
&\quad + \frac{\gamma_n^2}{\beta} ||T^* J_{E_3}(Tx_n - Su_n)||^2 \\
&\quad + \frac{\gamma_n^2}{\beta} ||S^* J_{E_3}(Su_n - Tx_n)||^2 \\
&\leq -\frac{\rho}{2} ||Tx_n - Su_n||^2, \qquad (3.30)
\end{aligned}$$

and substituting (3.30) into (3.26), we obtain

$$\begin{aligned}\Omega_{n+1} &\leq \alpha_n\Sigma + (1-\alpha_n)\Omega_n - (1-\alpha_n)\frac{\rho}{2}||Tx_n - Su_n||^2\\ &\leq \alpha_n\Sigma + (1-\alpha_n)\Omega_n,\end{aligned}$$

which implies by the mathematical induction that $\Omega_n \leq \max\{\Omega_1, \Sigma\}$. Hence we have that $\{D_f(\bar{x}, x_n) + D_g(\bar{u}, u_n)\}$ is bounded which implies that the sequences $\{D_f(\bar{x}, x_n)\}$ and $\{D_g(\bar{u}, u_n)\}$ are bounded. By Lemma 2.13, we have that $\{x_n\}$ and $\{u_n\}$ are bounded. □

Lemma 3.3. *Assume that Conditions* $(A1)-(A2)$, $(B1)-(B4)$ *and* $(C1)-(C2)$ *hold. Let* $\{z_n\}$, $\{y_n\}$, $\{w_n\}$ *and* $\{v_n\}$ *be as defined in Algorithm A. Then we have the following statements:*

(1) *If there exist subsequences* $\{z_{n_k}\}$ *and* $\{y_{n_k}\}$ *of* $\{z_n\}$ *and* $\{y_n\}$, *respectively, such that* $z_{n_k} \rightharpoonup p \in C$ *and* $||z_{n_k} - y_{n_k}|| \to 0$ *as* $k \to \infty$, *then*
   - (i) $0 \leq \liminf_{k\to\infty}\langle Az_{n_k}, z - z_{n_k}\rangle$ *for all* $z \in C$;
   - (ii) $p \in VI(C, A)$.

(2) *If there exist subsequences* $\{w_{n_k}\}$ *and* $\{v_{n_k}\}$ *of* $\{w_n\}$ *and* $\{v_n\}$, *respectively, such that* $w_{n_k} \rightharpoonup q \in D$ *and* $||w_{n_k} - v_{n_k}|| \to 0$ *as* $k \to \infty$, *then*
   - (i) $0 \leq \liminf_{k\to\infty}\langle Bw_{n_k}, w - w_{n_k}\rangle$ *for all* $w \in D$;
   - (ii) $q \in VI(D, B)$.

Proof. (1) Let the hypotheses be satisfied.

(i) Put $s_{n_k} = P_C^f \nabla f^*(\nabla f z_{n_k} - \lambda_{n_k} l^{-1} Az_{n_k})$. By Lemma 2.15 and (3.6) we have

$$||z_{n_k} - s_{n_k}|| \leq \frac{1}{l}||z_{n_k} - y_{n_k}|| \to 0, \text{ as } k \to \infty. \tag{3.31}$$

Therefore, $s_{n_k} \rightharpoonup p \in C$. Thus, we have that $\{s_{n_k}\}$ is bounded. Since A is uniformly continuous on bounded subsets of E_1, we have

$$||Az_{n_k} - As_{n_k}|| \to 0, \text{ as } k \to \infty. \tag{3.32}$$

By the Armijo line-search rule (3.2), we have

$$\begin{aligned}&\lambda_{n_k} l^{-1}||AP_C^f\nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1}Az_{n_k}) - Az_{n_k}||\\ &\quad > \mu||z_{n_k} - P_C^f\nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1}Az_{n_k})||,\end{aligned}$$

which implies that

$$\begin{aligned}&\frac{1}{\mu}||AP_C^f\nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1}Az_{n_k}) - Az_{n_k}||\\ &\quad > \frac{||z_{n_k} - P_C^f\nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1}Az_{n_k})||}{\lambda_{n_k} l^{-1}}.\end{aligned} \tag{3.33}$$

From (3.32) and (3.33), we have

$$\lim_{k\to\infty} \frac{||z_{n_k} - P_C^f \nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1} A z_{n_k})||}{\lambda_{n_k} l^{-1}} = 0. \tag{3.34}$$

Since ∇f is Lipschitz continuous, we have

$$\begin{aligned} 0 &\le \lim_{k\to\infty} \frac{||\nabla f z_{n_k} - \nabla f\left(P_C^f \nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1} A z_{n_k})\right)||}{\lambda_{n_k} l^{-1}} \\ &\le L \lim_{k\to\infty} \frac{||z_{n_k} - P_C^f \nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1} A z_{n_k})||}{\lambda_{n_k} l^{-1}} \end{aligned} \tag{3.35}$$

for some $L > 0$. Thus, we obtain from (3.34) and (3.35) that

$$\lim_{k\to\infty} \frac{||\nabla f z_{n_k} - \nabla f\left(P_C^f \nabla f^*(\nabla f(z_{n_k}) - \lambda_{n_k} l^{-1} A z_{n_k})\right)||}{\lambda_{n_k} l^{-1}} = 0. \tag{3.36}$$

From the definition of s_{n_k} and (2.6), we obtain

$$\langle \nabla f(z_{n_k}) - \lambda_{n_k} l^{-1} A z_{n_k} - \nabla f(s_{n_k}), z - s_{n_k}\rangle \le 0 \text{ for all } z \in C.$$

This implies that

$$\begin{aligned} &\left\langle \frac{\nabla f(z_{n_k}) - \nabla f(s_{n_k})}{\lambda_{n_k} l^{-1}}, z - s_{n_k} \right\rangle + \langle A z_{n_k}, s_{n_k} - z_{n_k}\rangle \\ &\qquad \le \langle A z_{n_k}, z - z_{n_k}\rangle \text{ for all } z \in C. \end{aligned} \tag{3.37}$$

Taking the limit on both sides of (3.37) as $k \to \infty$ and using (3.36), (3.31), uniform continuity of A and the boundedness of the sequences $\{z_{n_k}\}$ and $\{s_{n_k}\}$, we obtain

$$\liminf_{k\to\infty} \langle A z_{n_k}, z - z_{n_k}\rangle \ge 0 \text{ for all } z \in C. \tag{3.38}$$

(ii) Let $\{\varepsilon_k\}$ be a sequence of decreasing nonnegative numbers such that $\varepsilon_k \to 0$ as $k \to \infty$. For each ε_k, we choose N_k to be the smallest positive integer such that

$$\langle A z_{n_k}, z - z_{n_k}\rangle + \varepsilon_k \ge 0 \text{ for all } k \ge N_k \tag{3.39}$$

where the existence of N_k follows from (3.38). Since $\{\varepsilon_k\}$ is decreasing, $\{N_k\}$ is increasing. If there exists $N > 0$ such that $A z_{N_k} = 0$ for all $k \ge N$, then

$$\langle A z_{N_k}, z - z_{N_k}\rangle \ge 0$$

for all $k \ge N$ and $z \in C$. Since A is pseudomonotone, we have that

$$\langle A z, z - z_{N_k}\rangle \ge 0 \quad \text{for all } k \ge N \text{ and } z \in C. \tag{3.40}$$

Taking the limit on both sides of (3.40) as $k \to \infty$, we obtain

$$\langle A z, z - p\rangle \ge 0 \text{ for all } k \ge N \text{ and } z \in C.$$

By Lemma 2.10, we conclude that $p \in VI(C, A)$. If there exists a subsequence $\{N_{k_i}\}$ of $\{N_k\}$, again denoted by $\{N_k\}$, such that $A_{z_{N_k}} \neq 0$ for all $k \in \mathbb{N}$, then setting $t_{N_k} = \dfrac{J_{E_1}^{-1} A z_{N_k}}{||Az_{N_k}||^2}$, we get $\langle Az_{N_k},\ t_{N_k}\rangle = 1$ for each k. Therefore, from (3.39)

$$\langle Az_{N_k},\ z + \varepsilon_k t_{N_k} - z_{N_k}\rangle \geq 0.$$

Since A is pseudomonotone, we have that

$$\langle A(z + \varepsilon_k t_{N_k}),\ z + \varepsilon_k t_{N_k} - z_{N_k}\rangle \geq 0. \tag{3.41}$$

Since $\{z_{n_k}\}$ converges weakly to p as $k \to \infty$ and A is sequentially weakly continuous, we have that $\{Az_{n_k}\}$ converges weakly to Ap. Suppose $Ap \neq 0$ (otherwise, $p \in VI(C, A)$). Then by the sequentially weakly lower semi-continuity of the norm, we get

$$0 < ||Ap|| \leq \liminf_{k\to\infty} ||Az_{n_k}||.$$

Since $\{z_{N_k}\} \subset \{z_{n_k}\}$ and $\varepsilon_k \to 0$ as $k \to \infty$, we get that

$$\begin{aligned} 0 \leq \limsup_{k\to\infty} ||\varepsilon_k t_{N_k}|| &= \limsup_{k\to\infty} \left(\frac{\varepsilon_k}{||Az_{n_k}||}\right) \\ &\leq \frac{\limsup_{k\to\infty} \varepsilon_k}{\liminf_{k\to\infty} ||Az_{n_k}||} \leq \frac{0}{||Ap||} = 0. \end{aligned}$$

Hence, $\limsup_{k\to\infty} ||\varepsilon_k t_{N_k}|| = 0$. So, taking the limit on both sides of (3.41) as $k \to \infty$, we get

$$\langle Az, z - p\rangle \geq 0 \quad \text{for all } z \in C.$$

Therefore, by Lemma 2.10, we have $p \in VI(C, A)$.

(2) Part (2) of the lemma can be proved in a similar way. □

Theorem 3.4. *Suppose that Conditions* $(A1)-(A2)$, $(B1)-(B4)$ *and* $(C1)-(C2)$ *hold. Then the sequence* $\{(x_n, u_n)\}$ *generated by Algorithm A converges strongly to* $(\bar{x}, \bar{u}) \in \Upsilon$, *where* $(\bar{x}, \bar{u}) = P_\Upsilon^h(x, u)$, *for* $h = (f, g)$.

Proof. Let $(\bar{x}, \bar{u}) = P_\Upsilon^h(x, u)$. Denote $C_n = \tau \nabla f(a_n) + (1 - \tau) G(a_n)$ and

$$\begin{aligned} \Delta_n &= \langle \nabla f(x) - \nabla f(\bar{x}), x_n - \bar{x}\rangle + \langle \nabla g(u) - \nabla g(\bar{u}), u_n - \bar{u}\rangle \\ &= \langle (\nabla f(x), \nabla g(u)) - (\nabla f(\bar{x}), \nabla g(\bar{u})), (x_n, u_n) - (\bar{x}, \bar{u})\rangle. \end{aligned}$$

Then, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \Delta_{n+1} &= \langle \nabla f(x) - \nabla f(\bar{x}), x_n - \bar{x}\rangle + \langle \nabla g(u) - \nabla g(\bar{u}), u_n - \bar{u}\rangle \\ &\quad + \langle \nabla f(x) - \nabla f(\bar{x}), x_{n+1} - x_n\rangle \\ &\quad + \langle \nabla g(u) - \nabla g(\bar{u}), u_{n+1} - u_n\rangle \\ &\leq \Delta_n + \Lambda\big[||x_{n+1} - x_n|| + ||u_{n+1} - u_n||\big], \end{aligned}$$

for some constant $\Lambda > 0$. From (3.4), (2.10) and (2.11) we obtain

$$\begin{aligned}
&D_f(\bar{x}, x_{n+1}) \\
&\quad = D_f\Big(\bar{x}, \nabla f^*\left(\alpha_n \nabla f(x) + (1-\alpha_n)C_n\right)\Big) \\
&\quad = V_f\Big(\bar{x}, \alpha_n \nabla f(x) + (1-\alpha_n)\nabla f\left(\nabla f^*(C_n)\right)\Big) \\
&\quad \leq V_f\Big(\bar{x}, \alpha_n \nabla f(x) + (1-\alpha_n)\nabla f\left(\nabla f^*(C_n)\right) - \alpha_n\left(\nabla f(x) - \nabla f(\bar{x})\right)\Big) \\
&\qquad - \Big\langle -\alpha_n\left(\nabla f(x) - \nabla f(\bar{x})\right), \\
&\qquad\qquad \nabla f^*\left(\alpha_n \nabla f(x) + (1-\alpha_n)\nabla f\left(\nabla f^*(C_n)\right)\right) - \bar{x}\Big\rangle \\
&\quad = V_f\Big(\bar{x}, \alpha_n \nabla f(\bar{x}) + (1-\alpha_n)\nabla f\left(\nabla f^*(C_n)\right)\Big) \\
&\qquad + \alpha_n\langle \nabla f(x) - \nabla f(\bar{x}),\ x_{n+1} - \bar{x}\rangle \\
&\quad = D_f(\bar{x}, \nabla f^*\left(\alpha_n \nabla f(\bar{x}) + (1-\alpha_n)\nabla f\left(\nabla f^*(C_n)\right)\right)) \\
&\qquad + \alpha_n\langle \nabla f(x) - \nabla f(\bar{x}),\ x_{n+1} - \bar{x}\rangle \\
&\quad \leq \alpha_n D_f(\bar{x}, \bar{x}) + (1-\alpha_n) D_f(\bar{x}, \nabla f^*(C_n)) \\
&\qquad + \alpha_n\langle \nabla f(x) - \nabla f(\bar{x}),\ x_{n+1} - \bar{x}\rangle,
\end{aligned}$$

which implies by Lemma 2.7 that

$$\begin{aligned}
D_f(\bar{x}, x_{n+1}) \leq\ & (1-\alpha_n) D_f(\bar{x}, \nabla f^*(C_n)) \\
& + \alpha_n\langle \nabla f(x) - \nabla f(\bar{x}),\ x_{n+1} - \bar{x}\rangle.
\end{aligned} \tag{3.42}$$

Furthermore, from (2.9), (2.10), part (iii) of Lemma 2.14 and Bregman relatively $f-$nonexpansiveness of G, we have

$$\begin{aligned}
D_f(\bar{x}, \nabla f^*(C_n)) &= D_f(\bar{x}, \nabla f^*(\tau \nabla f(a_n) + (1-\tau)G(a_n))) \\
&= V_f(\bar{x}, \tau \nabla f(a_n) + (1-\tau)G(a_n)) \\
&= f(\bar{x}) + f^*\left(\tau \nabla f(a_n) + (1-\tau)G(a_n)\right) \\
&\quad - \langle \tau \nabla f(a_n) + (1-\tau)G(a_n), \bar{x}\rangle \\
&\leq f(\bar{x}) + \tau f^*\left(\nabla f(a_n)\right) + (1-\tau) f^*(G(a_n)) \\
&\quad - \tau(1-\tau)\phi_1\left(||\nabla f(a_n) - G(a_n)||\right) \\
&\quad - \tau\langle \nabla f(a_n), \bar{x}\rangle - (1-\tau)\langle G(a_n), \bar{x}\rangle \\
&= \tau V_f(\bar{x}, \nabla f(a_n)) + (1-\tau) V_f(\bar{x}, G(a_n)) \\
&\quad - \tau(1-\tau)\phi_1\left(||\nabla f(a_n) - G(a_n)||\right)
\end{aligned}$$

$$
\begin{aligned}
&= \tau D_f(\bar{x}, a_n) + (1-\tau) D_f(\bar{x}, \nabla f^* G(a_n)) \\
&\quad - \tau(1-\tau)\phi_1 \left(||\nabla f(a_n) - G(a_n)||\right) \\
&\leq \tau D_f(\bar{x}, a_n) + (1-\tau) D_f(\bar{x}, a_n) \\
&\quad - \tau(1-\tau)\phi_1 \left(||\nabla f(a_n) - G(a_n)||\right) \\
&= D_f(\bar{x}, a_n) - \tau(1-\tau)\phi_1 \left(||\nabla f(a_n) - G(a_n)||\right),
\end{aligned}
\tag{3.43}
$$

where ϕ_1 is the modulus of uniform convexity of f^*.

Substituting (3.43) into (3.42), we get

$$
\begin{aligned}
D_f(\bar{x}, x_{n+1}) \leq (1-\alpha_n) & D_f(\bar{x}, a_n) \\
& - (1-\alpha_n)\tau(1-\tau)\phi_1 \left(||\nabla f(a_n) - G(a_n)||\right) \\
& + \alpha_n \langle \nabla f(x) - \nabla f(\bar{x}),\ x_{n+1} - \bar{x} \rangle.
\end{aligned}
\tag{3.44}
$$

Again using (3.23), we obtain

$$
\begin{aligned}
D_f(\bar{x}, x_{n+1}) \leq (1-\alpha_n) & D_f(\bar{x}, x_n) \\
& - (1-\alpha_n)\left(1 - \frac{\mu}{\beta}\right)\Big(D_f(a_n, y_n) + D_f(y_n, z_n)\Big) \\
& - (1-\alpha_n)\tau(1-\tau)\phi_1 \left(||\nabla f(a_n) - G(a_n)||\right) \\
& - (1-\alpha_n)\gamma_n \langle J_{E_3}(Tx_n - Su_n), Tq_n - T\bar{x} \rangle \\
& + \alpha_n \langle \nabla f(x) - \nabla f(\bar{x}), x_{n+1} - \bar{x} \rangle.
\end{aligned}
\tag{3.45}
$$

Similarly, we get

$$
\begin{aligned}
D_g(\bar{u}, u_{n+1}) \leq (1-\alpha_n) & D_g(\bar{u}, u_n) \\
& - (1-\alpha_n)\left(1 - \frac{\mu}{\beta}\right)\Big(D_g(b_n, v_n) + D_g(v_n, w_n)\Big) \\
& - (1-\alpha_n)\tau(1-\tau)\phi_2 \left(||\nabla g(b_n) - K(b_n)||\right) \\
& - (1-\alpha_n)\gamma_n \langle J_{E_3}(Su_n - Tx_n), St_n - S\bar{u} \rangle \\
& + \alpha_n \langle \nabla g(u) - \nabla g(\bar{u}), u_{n+1} - \bar{u} \rangle,
\end{aligned}
\tag{3.46}
$$

where ϕ_2 is the modulus of uniform convexity of g^*.

Let $\Theta_n = D_f(\bar{x}, x_n) + D_g(\bar{u}, u_n)$. Then, combining (3.45) and (3.46) and using the relation in (3.30), we obtain

$$
\begin{aligned}
\Theta_{n+1} \leq\ & (1-\alpha_n)\Theta_n - (1-\alpha_n)\left(1-\frac{\mu}{\beta}\right) D_f(a_n, y_n) \\
& - (1-\alpha_n)\left(1-\frac{\mu}{\beta}\right) D_f(y_n, z_n) - (1-\alpha_n)\left(1-\frac{\mu}{\beta}\right) D_g(b_n, v_n) \\
& - (1-\alpha_n)\left(1-\frac{\mu}{\beta}\right) D_g(v_n, w_n) \\
& - (1-\alpha_n)\tau(1-\tau)\phi_1\left(||\nabla f(a_n) - G(a_n)||\right) \\
& - (1-\alpha_n)\tau(1-\tau)\phi_2\left(||\nabla g(b_n) - K(b_n)||\right) \\
& - (1-\alpha_n)\gamma_n\langle J_{E_3}(Tx_n - Su_n), Tq_n - T\bar{x}\rangle \\
& - (1-\alpha_n)\gamma_n\langle J_{E_3}(Su_n - Tx_n), St_n - S\bar{u}\rangle \\
& + \alpha_n\langle \nabla f(x) - \nabla f(\bar{x}), x_{n+1} - \bar{x}\rangle + \alpha_n\langle \nabla g(u) - \nabla g(\bar{u}), u_{n+1} - \bar{u}\rangle \\
\leq\ & (1-\alpha_n)\Theta_n - (1-\alpha_n)\left(1-\frac{\mu}{\beta}\right) D_f(a_n, y_n) \\
& - (1-\alpha_n)\left(1-\frac{\mu}{\beta}\right) D_f(y_n, z_n) - (1-\alpha_n)\left(1-\frac{\mu}{\beta}\right) D_g(b_n, v_n) \\
& - (1-\alpha_n)\left(1-\frac{\mu}{\beta}\right) D_g(v_n, w_n) \\
& - \tau(1-\tau)(1-\alpha_n)\phi_1\left(||\nabla f(a_n) - G(a_n)||\right) \\
& - \tau(1-\tau)(1-\alpha_n)\phi_2\left(||\nabla g(b_n) - K(b_n)||\right) \\
& - (1-\alpha_n)\frac{\rho}{2}||Tx_n - Su_n||^2 + \alpha_n\langle \nabla f(x) - \nabla f(\bar{x}), x_{n+1} - \bar{x}\rangle \\
& + \alpha_n\langle \nabla g(u) - \nabla g(\bar{u}), u_{n+1} - \bar{u}\rangle.
\end{aligned} \tag{3.47}
$$

This implies that

$$
\begin{aligned}
& \left(1-\frac{\mu}{\beta}\right)\Big[D_f(a_n, y_n) + D_f(y_n, z_n) + D_g(b_n, v_n) + D_g(v_n, w_n)\Big] \\
& \quad + \frac{\rho}{2}||Tx_n - Su_n||^2 + \tau(1-\tau)\phi_1\big(||\nabla f(a_n) - G(a_n)||\big) \\
& \quad + \tau(1-\tau)\phi_2\big(||\nabla g(b_n) - K(b_n)||\big) \\
& \leq \Theta_n - \Theta_{n+1} + \alpha_n M,
\end{aligned} \tag{3.48}
$$

for some $M > 0$, where the existence of such M is guaranteed by the boundedness of $\{x_n\}$ and $\{u_n\}$.

Now, we show that the sequence $\{\Theta_n\}$ of real numbers, converges strongly to zero by considering two cases:

Case I. If there exists a natural number n_0 such that $\Theta_{n+1} \leq \Theta_n$ for all $n \geq n_0$, then $\{\Theta_n\}$ converges. Taking the limit as $n \to \infty$ in (3.48), we get

$$\lim_{n\to\infty} ||Tx_n - Su_n||^2 = 0 \tag{3.49}$$

and

$$\lim_{n\to\infty} \phi_1 \left(||\nabla f(a_n) - G(a_n)||\right) = 0 = \lim_{n\to\infty} \phi_2 \left(||\nabla g(b_n) - K(b_n)||\right),$$

which implies that

$$\lim_{n\to\infty} ||\nabla f(a_n) - G(a_n)|| = 0 = \lim_{n\to\infty} ||\nabla g(b_n) - K(b_n)||. \tag{3.50}$$

From the definition of z_n, (2.10), (2.11), property of the Bregman projection, and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&D_f(x_n, z_n)\\
&\quad \leq D_f\Big(x_n, \nabla f^* \left(\nabla f(x_n) - \gamma_n T^* J_{E_3} \left(Tx_n - Su_n\right)\right)\Big)\\
&\quad = V_f(x_n, \nabla f(x_n) - \gamma_n T^* J_{E_3} \left(Tx_n - Su_n\right))\\
&\quad \leq V_f(x_n, \nabla f(x_n))\\
&\qquad - \left\langle \gamma_n T^* J_{E_3}(Tx_n - Su_n), \nabla f^* \left(\nabla f(x_n) - \gamma_n T^* J_{E_3} \left(Tx_n - Su_n\right)\right) - x_n \right\rangle\\
&\quad = D_f(x_n, x_n) - \gamma_n \langle T^* J_{E_3}(Tx_n - Su_n), q_n - x_n\rangle\\
&\quad \leq \gamma_n ||T^* J_{E_3}(Tx_n - Su_n)||\ ||q_n - x_n||.
\end{aligned} \tag{3.51}$$

Substituting (3.28) into (3.51), then taking the limit on both sides and using (3.49), we obtain

$$\begin{aligned}
0 &\leq \lim_{n\to\infty} D_f(x_n, z_n) \leq \lim_{n\to\infty} \left(\frac{\gamma_n^2}{\beta} ||T||^2\ ||J_{E_3}(Tx_n - Su_n)||^2\right)\\
&= \lim_{n\to\infty} \left(\frac{\gamma_n^2}{\beta} ||T||^2\ ||Tx_n - Su_n||^2\right) = 0.
\end{aligned}$$

This implies that $\lim_{n\to\infty} D_f(x_n, z_n) = 0$ and hence by Lemma 2.9 we obtain

$$\lim_{n\to\infty} ||x_n - z_n|| = 0. \tag{3.52}$$

Similarly, one can show that

$$\lim_{n\to\infty} ||u_n - w_n|| = 0. \tag{3.53}$$

Moreover, by taking the limit as $n \to \infty$ on both sides of (3.48), we obtain

$$\begin{aligned}
&\lim_{n\to\infty} D_f(a_n, y_n) = 0, \quad \lim_{n\to\infty} D_f(y_n, z_n) = 0,\\
&\lim_{n\to\infty} D_g(b_n, v_n) = 0, \quad \lim_{n\to\infty} D_g(v_n, w_n) = 0,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} ||a_n - y_n|| = 0, \quad \lim_{n\to\infty} ||y_n - z_n|| = 0 \tag{3.54}$$

and

$$\lim_{n\to\infty} ||b_n - v_n|| = 0, \quad \lim_{n\to\infty} ||v_n - w_n|| = 0. \tag{3.55}$$

From (3.52) and (3.54), we have

$$\lim_{n\to\infty} ||x_n - y_n|| \leq \lim_{n\to\infty} ||x_n - z_n|| + \lim_{n\to\infty} ||z_n - y_n|| = 0 \tag{3.56}$$

and from (3.53) and (3.55), we obtain

$$\lim_{n\to\infty} ||u_n - v_n|| \leq \lim_{n\to\infty} ||u_n - w_n|| + \lim_{n\to\infty} ||w_n - v_n|| = 0. \tag{3.57}$$

From (3.54) and (3.56), we have

$$\lim_{n\to\infty} ||a_n - x_n|| \leq \lim_{n\to\infty} ||a_n - y_n|| + \lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{3.58}$$

From (3.4) and (3.50), we have

$$\begin{aligned}\lim_{n\to\infty} ||\nabla f(x_{n+1}) - \nabla f(a_n)|| &\leq \lim_{n\to\infty} (\alpha_n ||\nabla f(x) - \tau \nabla f(a_n)||) \\ &\quad + (1-\tau) \lim_{n\to\infty} ((1-\alpha_n)||G(a_n) - \nabla f(a_n)||) \\ &= 0.\end{aligned} \tag{3.59}$$

From (3.59) and part (iii) of Lemma 2.8, we have

$$\lim_{n\to\infty} ||x_{n+1} - a_n|| = 0. \tag{3.60}$$

Therefore, from (3.60) and (3.58), we obtain

$$\begin{aligned}\lim_{n\to\infty} ||x_{n+1} - x_n|| &\leq \lim_{n\to\infty} ||x_{n+1} - a_n|| + \lim_{n\to\infty} ||a_n - x_n|| \\ &= 0.\end{aligned} \tag{3.61}$$

Similarly, we have

$$\lim_{n\to\infty} ||u_{n+1} - u_n|| = 0. \tag{3.62}$$

Since $\{(x_n, u_n)\}$ is bounded and $E_1 \times E_2$ is reflexive (Lemma 2.6), there exists a subsequence $\{(x_{n_k}, u_{n_k})\}$ of $\{(x_n, u_n)\}$ which converges weakly to some $(x^*, u^*) \in E_1 \times E_2$ and

$$\limsup_{n\to\infty} \Delta_n = \lim_{k\to\infty} \Delta_{n_k}. \tag{3.63}$$

Consequently, we have $x_{n_k} \rightharpoonup x^*$ and $u_{n_k} \rightharpoonup u^*$. From (3.52) and (3.53), we have $z_{n_k} \rightharpoonup x^*$ and $w_{n_k} \rightharpoonup u^*$, respectively. So, from (3.54), the fact that $z_{n_k} \rightharpoonup x^*$ and Lemma 3.3, we obtain $x^* \in VI(C, A)$. Similarly, one can show that $u^* \in VI(D, B)$. From (3.58), we have $a_n \rightharpoonup x^*$. Thus, with the help of (3.50) and the definition of $f-$asymptotic fixed points, we conclude that

$x^* \in \widehat{F_f(G)}$. From the Bregman relatively $f-$nonexpansivity of G, we have $x^* \in F_f(G)$. So, $x^* \in VI(C,A) \cap F_f(G)$.

Similarly, we can show that $u^* \in VI(D,B) \cap F_g(K)$.

Moreover, by Lemma 2.1 we have

$$\begin{aligned} ||Tx^* - Su^*||^2 &= ||Tx_{n_k} - Su_{n_k} + Tx^* - Tx_{n_k} + Su_{n_k} - Su^*||^2 \\ &\leq ||Tx_{n_k} - Su_{n_k}||^2 \\ &\quad + 2\langle J_{E_3}(Tx^* - Su^*),\ Tx^* - Tx_{n_k} + Su_{n_k} - Su^*\rangle. \end{aligned} \tag{3.64}$$

Since T and S are sequentially weakly continuous, we have that $x_{n_k} \rightharpoonup x^*$ implies $Tx_{n_k} \rightharpoonup Tx^*$, and $u_{n_k} \rightharpoonup u^*$ implies $Su_{n_k} \rightharpoonup Su^*$. Thus, we obtain using (3.49) that $Tx^* = Su^*$. Therefore, $(x^*, u^*) \in \Upsilon$.

Furthermore, from the definition of Δ_{n+1}, (3.61), (3.62), (3.63) and (2.8) we have

$$\begin{aligned} \limsup_{n\to\infty} \Delta_{n+1} &\leq \limsup_{n\to\infty} \Delta_n + \Lambda \limsup_{n\to\infty} [||x_{n+1} - x_n|| + ||u_{n+1} - u_n||] \\ &= \lim_{k\to\infty} \Delta_{n_k} + \Lambda \lim_{k\to\infty} [||x_{n_k+1} - x_{n_k}|| + ||u_{n_k+1} - u_{n_k}||] \\ &= \lim_{k\to\infty} \langle (\nabla f(x), \nabla g(u)) - (\nabla f(\bar{x}), \nabla g(\bar{u})), (x_{n_k}, u_{n_k}) - (\bar{x}, \bar{u})\rangle \\ &= \langle (\nabla f(x), \nabla g(u)) - (\nabla f(\bar{x}), \nabla g(\bar{u})), (x^*, u^*) - (\bar{x}, \bar{u})\rangle \\ &\leq 0. \end{aligned} \tag{3.65}$$

From (3.47) we have

$$\Theta_{n+1} \leq (1 - \alpha_n)\Theta_n + \alpha_n \Delta_{n+1}. \tag{3.66}$$

So, (3.65), (3.66), Lemma 2.11 and the condition on α_n give that

$$\lim_{n\to\infty} \Theta_n = 0,$$

which implies that

$$\lim_{n\to\infty} D_f(\bar{x}, x_n) = 0$$

and

$$\lim_{n\to\infty} D_g(\bar{u}, u_n) = 0.$$

Thus, by Lemma 2.9 we obtain $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{n\to\infty} u_n = \bar{u}$.

Case II. If there exists a subsequence $\{\Theta_{n_i}\}$ of $\{\Theta_n\}$ with $\Theta_{n_i} < \Theta_{n_i+1}$ for all $i \geq 0$, then by Lemma 2.12, we can find a nondecreasing sequence $\{m_k\}$ of positive integers such that $\lim_{k\to\infty} m_k = \infty$ and

$$\Theta_{m_k} \leq \Theta_{m_k+1} \quad \text{and} \quad \Theta_k \leq \Theta_{m_k+1}, \tag{3.67}$$

for all positive integers k. Thus, (3.48) becomes

$$\begin{aligned}\left(1-\frac{\mu}{\beta}\right)&\Big[D_f(a_{m_k},y_{m_k})+D_f(y_{m_k},z_{m_k})+D_g(b_{m_k},v_{m_k})+D_g(v_{m_k},w_{m_k})\Big]\\&+\tau(1-\tau)\Big[\phi_1\left(||\nabla f(a_{m_k})-G(a_{m_k})||\right)+\phi_2\left(||\nabla g(b_{m_k})-K(b_{m_k})||\right)\Big]\\&+\frac{\rho}{2}||Tx_{m_k}-Su_{m_k}||^2\\&\leq \Theta_{m_k}-\Theta_{m_k+1}+\alpha_{m_k}M.\end{aligned} \tag{3.68}$$

Taking the limit as $k\to\infty$ to both sides of (3.68), we derive

$$\lim_{k\to\infty}||Tx_{m_k}-Su_{m_k}||^2=0,$$

$$\lim_{k\to\infty}||\nabla f(a_{m_k})-G(a_{m_k})||=\lim_{k\to\infty}||\nabla g(b_{m_k})-K(b_{m_k})||=0.$$

Following the method used in Case I, we get

$$\begin{aligned}\lim_{k\to\infty}||x_{m_k}-z_{m_k}||&=\lim_{k\to\infty}||u_{m_k}-w_{m_k}||=0,\\\lim_{k\to\infty}||y_{m_k}-z_{m_k}||&=\lim_{k\to\infty}||v_{m_k}-w_{m_k}||=0,\\\lim_{k\to\infty}||x_{m_k}-y_{m_k}||&=\lim_{k\to\infty}||u_{m_k}-v_{m_k}||=0.\end{aligned}$$

Furthermore, following similar steps as in Case I, we obtain

$$\lim_{k\to\infty}||x_{m_k+1}-x_{m_k}||=0,\quad \lim_{k\to\infty}||u_{m_k+1}-u_{m_k}||=0$$

and

$$\limsup_{k\to\infty}\Delta_{m_k+1}\leq 0.$$

Thus, from (3.47) and (3.67), we have

$$\alpha_{m_k}\Theta_{m_k}\leq\Theta_{m_k}-\Theta_{m_k+1}+\alpha_{m_k}\Delta_{m_k+1}\leq\alpha_{m_k}\Delta_{m_k+1},$$

which implies that

$$\Theta_{m_k}\leq\Delta_{m_k+1}. \tag{3.69}$$

Taking the limit on both sides of (3.69) as $k\to\infty$ and using the fact that $\limsup_{k\to\infty}\Delta_{m_k+1}\leq 0$, we get that the sequence $\Theta_{m_k}\to 0$ as $k\to\infty$. It follows from (3.66) that $\Theta_{m_k+1}\to 0$ as $k\to\infty$. Since $\Theta_k\leq\Theta_{m_k+1}$ for all $k\geq 0$, we have that $\Theta_k\to 0$ as $k\to\infty$. Thus, we have $\lim_{k\to\infty}D_f(\bar{x},x_k)=0$ and $\lim_{k\to\infty}D_g(\bar{u},u_k)=0$, which implies by Lemma 2.9 that $\lim_{k\to\infty}x_k=\bar{x}$ and $\lim_{k\to\infty}u_k=\bar{u}$.

Thus, we have shown, in Cases I and II, that the sequence $\{(x_n,u_n)\}$ generated by Algorithm A, converges strongly to $(\bar{x},\bar{u})=P^h_{\Upsilon}(x,u)$, and this completes the proof. □

If A and B are uniformly continuous and monotone mappings, then the assumption that A and B are sequentially weakly continuous is not required and hence the following corollary follows.

Corollary 3.5. *Assume that $A\colon C \to E_1^*$ and $B\colon D \to E_2^*$ are uniformly continuous and monotone mappings. If Conditions $(A1)-(A2), (B2)-(B4)$ and $(C1)-(C2)$ are satisfied, then the sequence $\{(x_n, u_n)\}$ generated by Algorithm A, converges strongly to $(\bar{x}, \bar{u}) \in \Upsilon$, where $(\bar{x}, \bar{u}) = P_\Upsilon^h(x, u)$.*

If $x = 0 = u$, then Algorithm A can be used to locate an element of the solution with the minimum norm and hence we have the following corollary.

Corollary 3.6. *Suppose that the Conditions $(A1)-(A2)$, $(B1)-(B4)$ and $(C1)-(C2)$ are satisfied. Then the sequence $\{(x_n, u_n)\}$ generated by Algorithm A with $x = 0$ and $u = 0$, converges strongly to $(\bar{x}, \bar{u}) \in \Upsilon$, where $(\bar{x}, \bar{u}) = P_\Upsilon^h(0, 0)$.*

4. Applications

Condition 4.1.

This section deals with applications of the main result to some specific cases. The following are the assumptions that will be used in these cases.

- (A3) Let C and D be nonempty, closed and convex subsets of a smooth, strictly convex, reflexive real Banach space E with dual E^*;
- (A4) Let $f, g\colon E \to \mathbb{R}$ be proper, lower semi-continuous, uniformly Fréchet differentiable, strongly convex, strongly coercive, Legendre functions which are bounded on bounded subsets. Let f and g have Lipschitz continuous gradients with the strong convexity constant of f (respectively, g) greater than or equal to the Lipschitz constant of ∇f (respectively, ∇g);
- (B5) Let $A, B\colon E \to E^*$ be uniformly continuous, pseudomonotone and sequentially weakly continuous on bounded subsets of E;
- (B6) Let $G, K\colon E \to E^*$ be Bregman relatively $f-$nonexpansive and Bregman relatively $g-$nonexpansive mappings, respectively.

4.1. **Common Solutions of Variational Inequality and $f, g-$Fixed Point Problems.** Let $E = E_1 = E_2 = E_3$, $T = S = I$ and let f and g be as in Condition 4.1 (A4). Then the split equality of variational inequality and $f, g-$ fixed point problems reduces to finding a common solution of two variational inequality and $f, g-$fixed point problems. This problem can be expressed as:

find $\bar{x} \in (VI(C, A) \cap F_f(G))$ and $\bar{u} \in (VI(D, B) \cap F_g(K))$ such that $\bar{x} = \bar{u}$.

Denote $\Sigma = \{(\bar{x}, \bar{u}) \in (VI(C, A) \cap F_f(G)) \times (VI(D, B) \cap F_g(K)) : \bar{x} = \bar{u}\}$.

In this case, we have the following corollaries.

Corollary 4.1. *Assume that* $\Sigma \neq \emptyset$. *If Conditions* $(A3)-(A4),(B5)-(B6)$, $(C1)-(C2)$ *are satisfied, then the sequence* $\{(x_n, u_n)\}$ *generated by Algorithm A,with* $E = E_1 = E_2 = E_3$ *and* $T = S = I$, *converges strongly to* $(\bar{x}, \bar{u}) \in \Sigma$, *where* $(\bar{x}, \bar{u}) = P^h_\Sigma(x, u)$.

Corollary 4.2. *Assume that* $\Sigma \neq \emptyset$ *and let* $A\colon E \to E^*$ *and* $B\colon E \to E^*$ *be uniformly continuous monotone mappings. If Conditions* $(A3)-(A4),(B6)$, $(C1)-(C2)$ *are satisfied, then the sequence* $\{(x_n, u_n)\}$ *generated by Algorithm A, with* $E = E_1 = E_2 = E_3$ *and* $T = S = I$, *converges strongly to* $(\bar{x}, \bar{u}) \in \Sigma$, *where* $(\bar{x}, \bar{u}) = P^h_\Sigma(x, u)$.

4.2. **Split Equality of Null Point and $f, g-$Fixed Point Problems.** Let f and g be as in $(A2)$. If $C = E_1$ and $D = E_2$, then the split equality of variational inequality and $f, g-$fixed point problem reduces to the split equality of null point and $f, g-$fixed point problem which can be described as finding a point $(\bar{x}, \bar{u})$ with the property

$$(\bar{x}, \bar{u}) \in \big(A^{-1}(0) \cap F_f(G)\big) \times \big(B^{-1}(0) \cap F_g(K)\big) : T\bar{x} = S\bar{u},$$

where $A^{-1}(0) = \{x \in E_1 : 0 \in Ax\}$ and $B^{-1}(0) = \{u \in E_2 : 0 \in Bu\}$.

Denote

$$\Upsilon^* = \big\{(\bar{x}, \bar{u}) \in \big(A^{-1}(0) \cap F_f(G)\big) \times \big(B^{-1}(0) \cap F_g(K)\big) : T\bar{x} = S\bar{u}\big\}.$$

In this case, we have the following results:

Corollary 4.3. *Assume that* $\Upsilon^* \neq \emptyset$. *If Conditions* $(A1)-(A2),(B1)-(B3)$, *with* $C = E_1, D = E_2$, $(C1)-(C2)$ *are satisfied, then the sequence* $\{(x_n, u_n)\}$ *generated by Algorithm A, converges strongly to* $(\bar{x}, \bar{u}) \in \Upsilon^*$, *where* $(\bar{x}, \bar{u}) = P^h_{\Upsilon^*}(x, u)$.

Corollary 4.4. *Let* $A\colon E_1 \to E_1^*$ *and* $B\colon E_2 \to E_2^*$ *be uniformly continuous monotone mappings and assume that* $\Upsilon^* \neq \emptyset$. *If Conditions* $(A1)-(A2),(B2)-(B3)$ *with* $C = E_1$ *and* $D = E_2$, $(C1)-(C2)$ *are satisfied, then the sequence* $\{(x_n, u_n)\}$ *generated by Algorithm A, converges strongly to* $(\bar{x}, \bar{u}) \in \Upsilon^*$, *where* $(\bar{x}, \bar{u}) = P^h_{\Upsilon^*}(x, u)$.

4.3. **Split Equality Variational Inequality Problem.** Let f and g be as in $(A2)$. If, in Condition 3.2, $Gx = \nabla f(x)$ for all $x \in C$ and $Ku = \nabla g(u)$ for all $u \in D$, then the split equality of variational inequality and $f, g-$fixed point problems reduces to the split equality variational inequality problem which seeks to

$$\text{find } \bar{x} \in VI(C, A) \text{ and } \bar{u} \in VI(D, B) \text{ such that } T\bar{x} = S\bar{u}.$$

Denote $\Gamma = \{(\bar{x}, \bar{u}) \in VI(C, A) \times VI(D, B) : T\bar{x} = S\bar{u}\}$.

One can easily show that ∇f and ∇g are Bregman $f-$relatively nonexpansive and Bregman $g-$relatively nonexpansive, respectively, and hence we have the following results.

Corollary 4.5. *Assume that* $\Gamma \neq \emptyset$. *If Conditions* $(A1)-(A2),(B1)-(B3),(C1)-(C2)$ *are satisfied with* $Gx = \nabla f(x)$ *for all* $x \in C$ *and* $Ku = \nabla g(u)$ *for all* $u \in D$, *then the sequence* $\{(x_n, u_n)\}$ *generated by Algorithm A, converges strongly to* $(\bar{x}, \bar{u}) \in \Gamma$, *where* $(\bar{x}, \bar{u}) = P^h_\Gamma(x, u)$.

Corollary 4.6. *Let* A: $E_1 \to E_1^*$ *and* B: $E_2 \to E_2^*$ *be uniformly continuous monotone mappings. Assume that* $\Gamma \neq \emptyset$. *If Conditions* $(A1)-(A2)$, $(B2)-(B3)$, $(C1)-(C2)$ *are satisfied with* $Gx = \nabla f(x)$ *for all* $x \in C$ *and* $Ku = \nabla g(u)$ *for all* $u \in D$, *then the sequence* $\{(x_n, u_n)\}$ *generated by Algorithm A, converges strongly to* $(\bar{x}, \bar{u}) \in \Gamma$, *where* $(\bar{x}, \bar{u}) = P^h_\Gamma(x, u)$.

4.4. **Split Equality $f-$Fixed Point Problem.** If we have $A = 0$ and $B = 0$, in Algorithm A, then the split equality of variational inequality and $f, g-$fixed point problem reduces to the split equality $f, g-$fixed point problem which seeks to

$$\text{find } \bar{x} \in F_f(G) \text{ and } \bar{u} \in F_g(K) \text{ such that } T\bar{x} = S\bar{u}.$$

Denote $\Gamma^* = \{(\bar{x}, \bar{u}) \in F_f(G) \times F_g(K) : T\bar{x} = S\bar{u}\}$.

Corollary 4.7. *Assume that* $\Gamma^* \neq \emptyset$. *If Conditions* $(A1)-(A2),(B2)-(B3),(C1)-(C2)$ *are satisfied. then the sequence* $\{(x_n, u_n)\}$ *generated by Algorithm A, with* $A = B = 0$, *converges strongly to* $(\bar{x}, \bar{u}) \in \Gamma^*$, *where* $(\bar{x}, \bar{u}) = P_{\Gamma^*}(x, u)$.

5. Numerical Example

Under this section, a numerical example is given to demonstrate the convergence of the sequence generated by Algorithm A.

Example 5.1. Given $E_1 = E_2 = E_3 = \mathbb{R}^2$. Let the norm and inner product on $\mathbb{R}^2$ be, respectively, given by $||x|| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^2 |x_i|^2}$ and $\langle x, u\rangle = \sum_{i=1}^2 x_i u_i$ for all $x, u \in \mathbb{R}^2$. Define the mappings A: $\mathbb{R}^2 \to \mathbb{R}^2$ and B: $\mathbb{R}^2 \to \mathbb{R}^2$ by

$$Ax = A(x_1, x_2) = \left(\frac{5}{2} + \sqrt{x_1^2 + x_2^2}\right)(x_1, x_2 - 1),$$
$$Bu = B(u_1, u_2) = \sqrt{u_1^2 + u_2^2}\,(u_1, u_2)$$

and let C and D be given by $C = \{x \in \mathbb{R}^2 : ||x|| \leq 2\}$, $D = \{u \in \mathbb{R}^2 : ||u|| \leq 2\}$. Clearly, C and D are nonempty, closed and convex subsets of $\mathbb{R}^2$, and the mappings A and B are uniformly continuous and sequentially weakly continuous on C and D, respectively. It can be easily shown that B is pseudomonotone on subsets of D. We only show here that A is pseudomonotone on $\mathbb{R}^2$. To this end, let $\langle Ax, y-x\rangle \geq 0$. Then we have,

$$\left(\frac{5}{2}+\sqrt{x_1^2+x_2^2}\right)(x_1,x_2-1),(y_1-x_1,y_2-x_2)\Big\rangle \geq 0,$$

this implies that

$$\left(\frac{5}{2}+\sqrt{x_1^2+x_2^2}\right)\Big\langle(x_1,x_2-1),(y_1-x_1,y_2-x_2)\Big\rangle \geq 0.$$

Since $\frac{5}{2}+\sqrt{x_1^2+x_2^2} > 0$, we conclude that $\langle(x_1,x_2-1),(y_1-x_1,y_2-x_2)\rangle \geq 0$.

Now, for $(y_1,y_2)\in\mathbb{R}^2$ we have

$$\begin{aligned}\langle Ay, y-x\rangle &= \left(\frac{5}{2}+\sqrt{y_1^2+y_2^2}\right)\Big\langle(y_1,y_2-1),(y_1-x_1,y_2-x_2)\Big\rangle\\ &\geq \left(\frac{5}{2}+\sqrt{y_1^2+y_2^2}\right)\Big[\Big\langle(y_1,y_2-1),(y_1-x_1,y_2-x_2)\Big\rangle\Big]\\ &- \left(\frac{5}{2}+\sqrt{y_1^2+y_2^2}\right)\Big[\Big\langle(x_1,x_2-1),(y_1-x_1,y_2-x_2)\Big\rangle\Big]\\ &= \left(\frac{5}{2}+\sqrt{y_1^2+y_2^2}\right)\left(|y_1-x_1|^2+|y_2-x_2|^2\right)\\ &\geq \frac{5}{2}||y-x||^2\\ &\geq 0.\end{aligned}$$

Therefore, A is pseudomonotone on $\mathbb{R}^2$.

Let us define T, S: $\mathbb{R}^2\to\mathbb{R}^2$ by

$$Tx = T(x_1,x_2) = (5x_1,0) \text{ and } Su = S(u_1,u_2) = (2u_1,3u_2),$$

where $(x_1,x_2),(u_1,u_2)\in\mathbb{R}^2$. Then T and S are bounded linear maps on $\mathbb{R}^2$ with adjoints $T^*x=(5x_1,0)$ and $S^*u=(2u_1,3u_2)$, respectively. Now, we have $\langle A(0,1),(x_1,x_2)-(0,1)\rangle\geq 0$ for all $(x_1,x_2)\in C$, $\langle B(0,0),(u_1,u_2)-(0,0)\rangle\geq 0$ for all $(u_1,u_2)\in D$. So, $\bar{x}\in VI(C,A)$ and $\bar{u}\in VI(D,B)$, where $\bar{x}=(0,1)$ and $\bar{u}=(0,0)$. Let us define G: $C\to\mathbb{R}^2$ and K: $D\to\mathbb{R}^2$ by

$$Gx = G(x_1,x_2) = (0,x_2) \text{ and } Ku = K(u_1,u_2) = (u_2,u_1),$$

and f,g: $\mathbb{R}^2\to\mathbb{R}$ by $f(x)=\frac{1}{2}||x||^2$ and $g(u)=\frac{1}{2}||u||^2$. Then, we have $\nabla f(x)=x$, $\nabla g(u)=u$ and $J_E=I$, where I is the identity mapping on $\mathbb{R}^2$. One can easily show that G is Bregman relatively $f-$nonexpansive and K is Bregman relatively $g-$nonexpansive. Moreover,

$$G\bar{x}=G(0,1)=(0,1)=\nabla f(0,1) \text{ and } K\bar{u}=K(0,0)=(0,0)=\nabla g(0,0).$$

From this, we conclude that $\bar{x} \in F_f(G)$ and $\bar{u} \in F_g(K)$. Moreover, $T(0,1) = (0,0) = S(0,0)$. Thus,

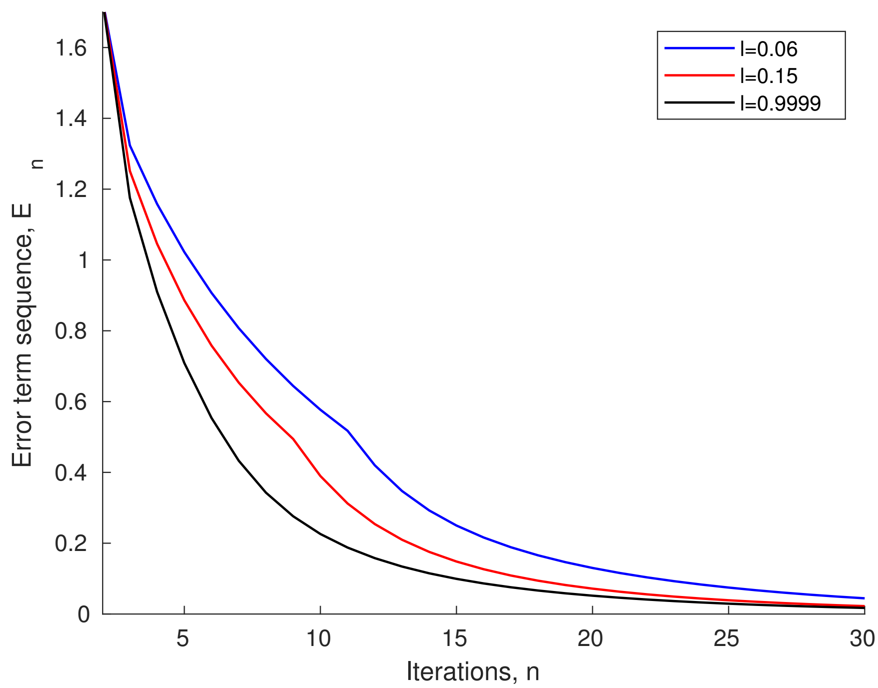
$$(\bar{x}, \bar{u}) \in (VI(C,A) \cap F_f(G)) \times (VI(D,B) \cap F_g(K)) \text{ with } T\bar{x} = S\bar{u}.$$

Now, taking $\alpha_n = \dfrac{1}{n + 100000}$ for $n \geq 1$ and

$$\gamma_n = \begin{cases} \left(\dfrac{2}{5}\right) \dfrac{(5x_{1n} - 2u_{1n})^2 + (3u_{2n})^2}{(25x_{1n} - 10u_{1n})^2 + (4u_{1n} - 10x_{1n})^2 + (9u_{2n})^2} & \text{if } n \in \Omega, \\ \dfrac{1}{1000000} \text{ if } n \notin \Omega. \end{cases}$$

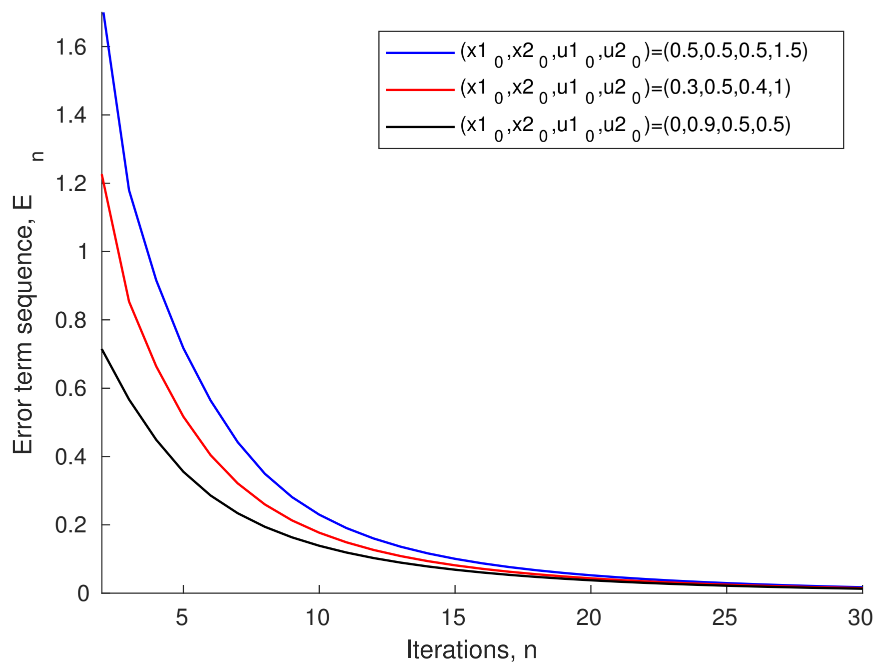
then the Conditions $(A1) - (A2), (B1) - (B4)$, and $(C1) - (C2)$ are satisfied.

The figures below show that the error term sequence $\{E_n\} = \{(x_n, u_n) - (\bar{x}, \bar{u})\}$ converges strongly to zero for different choices of the parameter l and different initial points.



$$((x_{1_0}, x_{2_0}), (u_{1_0}, u_{2_0})) = ((0.5, 0.5), (0.5, 1.5)), \quad \beta = 0.5, \quad \mu = 0.4, \quad \gamma = 0.5$$

FIGURE 1. Illustration of convergence rate of the sequence for different values of the parameter l.



Parameters $l = 0.9, \quad \beta = 0.5, \quad \mu = 0.4, \quad \gamma = 0.5$

Figure 2. Illustration of convergence rate of the sequence for different initial values.

The numerical experiments were carried out using MATLAB version R2020a and all programs were run on a 64-bit OS PC with Intel(R) Core(TM) i7-8550U CPU@1.80GHz 1.99GHz and 16GB RAM.

Remark 5.2. Figure 1 shows that the convergence of the sequence generated by Algorithm A gets faster as l gets closer to 1. From Figure 2, we observe that for any choice of initial point, the sequence $\{(x_n, u_n)\}$ converges to a solution of the split equality of variational inequality and $f, g-$fixed point problem. That is, the ultimate convergence of the sequence does not depend on the choice of initial points.

6. Conclusions

In this paper, we have proposed a method for finding a solution of split equality of variational inequality and $f, g-$fixed point problems, where the variational inequality problems are for uniformly continuous pseudomonotone mappings and the $f, g-$fixed point problems are for Bregman relatively

$f, g-$nonexpansive mappings in reflexive Banach spaces. We have proved strong convergence of the algorithm using the Bregman distance approach. Finally, a numerical example is provided to show the applicability of the proposed algorithm. The results in this paper extend most of the results which are discussed in the literature in one or the other way. Specifically, the results of our method improve the result obtained by Wega and Zegeye [34] in the sense that it extends the result from finding common solution of variational inequality and $f-$fixed point problems to finding a solution of split equality of variational inequality and $f, g-$fixed point problems. Our result also extends the results obtained by Boikanyo and Zegeye [5] in the sense that it addresses $f, g-$fixed point problems on top of the split equality variational inequality problems.

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