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INERTIAL PROXIMAL AND CONTRACTION METHODS FOR SOLVING MONOTONE VARIATIONAL INCLUSION AND FIXED POINT PROBLEMS

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Abstract. In this paper, we study an iterative algorithm that is based on inertial proximal and contraction methods embellished with relaxation technique, for finding common solution of monotone variational inclusion, and fixed point problems of pseudocontractive mapping in real Hilbert spaces. We establish a strong convergence result of the proposed iterative method based on prediction stepsize conditions, and under some standard assumptions on the algorithm parameters. Finally, some special cases of general problem are given as applications. Our results improve and generalized some well-known and related results in literature.

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$, and associated norm $||\cdot||$. Let C be nonempty, closed and convex subset of $\mathcal{H}$. Let $\mathcal{A}:\mathcal{H}\to 2^{\mathcal{H}}$ be a maximal monotone operator and $f:\mathcal{H}\to\mathcal{H}$ be a real single-valued nonlinear mapping . Then, monotone variational inclusion problems (simply denoted by MVIP) can be formulated as follows:

$$\text{Search for } z^*\in\mathcal{H} \text{ such that } 0\in\mathcal{A}(z^*)+f(z^*). \tag{1.1}$$

We shall denote the solution set of (1.1) by Ω. Essentially, the concept of variational inclusion problem plays a cardinal role in nonlinear analysis and theory of optimization. MVIP has turned out to be a very useful tool for investigating a wide range of many related problems in mathematical and applied sciences. It is strategically found in the center of image processing (for instance see [11, 16, 28] and the references therein) and many other mathematical problems (Check; [10],[11],[21]-[23]). Most importantly, maximal monotone variational inclusion problem gives a solid foundation for the study of many optimization problems such as, convex programming problems, variational inequality problems, equilibrium problems, complementarity problem, optimal control problems and many others, (see for instance [2], [6],[14],[20],[34],[47] and references therein).

Suppose that $f=0$ in (1.1), we have a zero point problem for maximal monotone operator as follows:

$$\text{Search for } z^*\in\mathcal{H} \text{ such that } 0\in\mathcal{A}(z^*), \tag{1.2}$$

where $\mathcal{A}$ is a maximal monotone operator. The monotone inclusion problem (1.2) was first proposed and studied by Martinet [31] in 1970 and generalized in 1976 by Rockafeller [38]. Martinet proposed the classical technique of proximal point algorithm (PPA) for the solution of (1.2) as follows:

$$x_{n+1}=(I+\lambda_n\mathcal{A})^{-1}x_n \quad n\geq 1, \tag{1.3}$$

where I is an identity operator, and $\{\lambda_n\}$ is a sequence of non-negative real numbers. After the advent of PPA by Martinet, many authors have developed algorithms for solving the inclusion problem (1.2) (see for example [37],[38] and references therein). It is pertinent to note that $z^*\in\mathcal{H}$ is a solution of the MVIP (1.1) if and only if z^* solves the fixed point problem,

$$z=J_\mu^{\mathcal{A}}(z-\lambda f(z)),$$

that is, $z^*=J_\mu^{\mathcal{A}}(I-\lambda f)z^*$, where $J_\mu^{\mathcal{A}}:=(I+\mu\mathcal{A})^{-1}$ is the resolvent operator associated with $\mu>0$. This has enhanced the integration of resolvent operator into the construction of algorithms for approximating the solution of monotone inclusion problem (1.1). Consequently, numerous methods using resolvent operator have been developed and studied by many authors. Common amongst them is the classical proximal method (see Fang and Huang [21], Solodov and Svaiter [39], Tseng [42], Zeng et al. [45] and references therein for details).

Generally, several methods of approximating the solution of monotone variational inclusion problem (1.1) have been studied and extended by many authors. Famous among these methods is the forward-backward splitting method (for details of this technique see [28, 35, 42] and references therein). Moreover, the modification of these methods by researchers have not only enjoyed versatility by using relaxation techniques (check [9, 29]) but with more acceleration using the inertial techniques (see [1, 7, 8, 13, 30, 33, 41, 43] for more). Thus, in attempt to establish the solution of (1.1), various proximal gradient algorithms have been proposed and studied. Although, many of the results obtained are based on some stringent conditions imposed on the cost operator f in (1.1), and other related algorithm parameters. For example, in 1998, Haung [26] studied a class of problem (1.1) where the underlying operators, $\mathcal{A}$ is maximally monotone, f is strongly monotone and L- Lipschitz continuous. He proved existence of solutions for the completely class of general variational inclusion problems and strong convergence theorem for the sequence of iterates generated by the algorithms. Zeng et al. [45], also proposed and analyzed new iterative method for solving MVIP (1.1). They obtained strong convergence theorem under some assumptions on the algorithm parameters with strict condition that f is inverse strongly monotone. These algorithms did not significantly improve the existing results because the stringent conditions rather make the iterative methods computationally expensive. Moreover, strong monotonicity is very difficult to obtain in practical problems. Hence, the need to relax the conditions on f and other parameters to solve MVIP (1.1).

In attempt to achieve some of these relaxations, in 2001, Alvarez and Attouch in [5] introduced and studied inertial algorithm for solving monotone inclusion problem as follows:

$$\begin{cases} x_0, x_1 \in \mathcal{H}_1, \\ u_n = x_n + \theta_n \left(x_n - x_{n-1}\right), \\ x_{n+1} = J_{\lambda_n}^{\mathcal{A}} \left(u_n\right), n \geq 1, \end{cases} \tag{1.4}$$

where $\{\lambda_n\}$ is positive real sequence, $J_{\lambda_n}^{\mathcal{A}}$ is the resolvent of the maximal monotone operator $\mathcal{A}$ with respect to λ_n. They established that under standard assumptions on the control parameters, the sequence $\{x_n\}$ generated by (1.4) converges weakly to the solution of (1.1).

In 2003, Moudafi and Oliny in [33], proposed and analyzed an iterative method that is based on inertial proximal point approach. The algorithm,

$$\begin{cases} x_0, x_1 \in \mathcal{H}_1, \\ u_n = x_n + \theta_n (x_n - x_{n-1}), \\ x_{n+1} = J_{\lambda_n}^{\mathcal{A}} (x_n - \lambda_n f u_n), n \geq 1, \end{cases} \tag{1.5}$$

where, $f : \mathcal{H} \to \mathcal{H}$ is monotone, L-Lipschitz continuous and $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ is set-valued maximally monotone was studied. It was shown that under some mild conditions imposed on the algorithm parameters, the sequence $\{x_n\}$ generated by (1.5) converge weakly to the solution of MVIP (1.1).

Also, Jung in [27] proposed and studied an iterative method for approximating the common solution of monotone inclusion, variational inequality and fixed point problems when the underlying operator is pseudocontractive. He introduced iterative algorithm that generated net $\{x_t\}$ implicitly as follows:

$$x_t = \theta_t x_t + (1 - \theta_t) T_{r_t} (t\gamma V x_t + (I - t\mu G)) J_{\lambda_t}^{B} \mathcal{A}_{\nu_t} x_t, \ t \in (0, 1), \tag{1.6}$$

where $B : \mathcal{H} \to 2^{\mathcal{H}}$ is maximal monotone, $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is a continuous monotone mapping, $V : C \to C$ is ℓ-Lipschitzian and $G : C \to C$ is k-Lipschitzian and ξ-strongly monotone. Under some appropriate conditions he obtained a strong convergence in real Hilbert space.

It is worth mentioning that if $\mathcal{A}$ is the normal cone of the nonempty, closed convex subset C of the Hilbert space $\mathcal{H}$, then, the monotone variational inclusion problem (1.1) can be reformulated in terms of variational inequality problem (VIP) as follows:

$$\text{Find } z^* \in C \text{ such that } \langle f(z^*), x - z^* \rangle \geq 0, \ \forall x \in C. \tag{1.7}$$

One of the simplest and most efficient methods of approximating the solution of (1.7) is by the projection and contraction methods (PCM). Many authors have proposed and analyzed this method for solving VIP when the underlying operator is monotone (see [19, 25, 40] for details).

In 2014, Cai et al. in [12] employed the projection and contraction techniques in obtaining the the solution of monotone variational inequalities. Their results were principally based on the conditions in the definition of prediction stepsize conditions, where the 'optimal' step length,

$$\beta_n := \frac{\langle x_n - y_n, d(x_n, y_n) \rangle}{||d(x_n, y_n)||^2}$$

in correction step.

Inspired by the results of Cai et al. [12], Dong et al. in [20] modified and extended PCM for solving monotone variational inequality problems to infinite dimensional Hilbert space. They proposed and investigated the algorithm:

$$\begin{cases} z_n = P_C\left(y_n - \lambda f(y_n)\right), \\ d\left(y_n, z_n\right) = y_n - z_n - \lambda\left(f(y_n) - f(z_n)\right), \\ y_{n+1} = y_n - \gamma\beta_n d\left(y_n, z_n\right), n \geq 1, \end{cases} \tag{1.8}$$

where λ is a fixed prediction stepsize. They further modified and extended algorithm (1.8) to finding common solution set of monotone variational inequalities and fixed point set when the underlying mapping is nonexpansive. That is,

$$\begin{cases} z_n = P_C\left(y_n - \lambda f(y_n)\right), \\ d\left(y_n, z_n\right) = y_n - z_n - \lambda\left(f(y_n) - f(z_n)\right), \\ y_{n+1} = \alpha_n y_n + (1-\alpha_n)T\left(y_n - \gamma\beta_n d\left(y_n, z_n\right)\right), n \geq 1, \end{cases} \tag{1.9}$$

where $T : \mathcal{H} \to \mathcal{H}$ is a nonexpansive operator. They established that the algorithm (1.8) converges weakly to a point in the solution set of VIP (1.7), and (1.9) converges weakly to a common element in the solution set of VIP and $F(T)$.

Recently, Zhang and Wang in [46] proposed and studied proximal and contraction method for solving MVIP (1.1). The algorithm stemmed from the combination of Dong et al. projection and contraction algorithm in [19] with resolvent operator. By replacing the metric projection operator in Dong et al. [19] with resolvent operator, they successfully removed the difficulty associated with computation of projection when the feasible set has complex structure and this gave their algorithms some special properties. Recall that the resolvent operator is a special form of projection operator (projection and resolvent operators coincide in the normal cones) that enjoys many special properties that make it a central tool in monotone operator theory and its applications. The following algorithms were proposed by them:

$$\begin{cases} y_n = J^{\mathcal{A}}_{\lambda_n}\left(x_n - \lambda_n f(x_n)\right), \\ d(x_n, y_n) = x_n - y_n - \lambda_n\left(f(x_n) - f(y_n)\right), \\ x_{n+1} = x_n + \gamma\beta_n d(x_n, y_n), n \geq 1, \end{cases} \tag{1.10}$$

where $\gamma \in (0, 2)$, $\beta_n := \frac{\phi(x_n, y_n)}{||d(x_n, y_n)||^2}$ and $\phi(x_n, y_n) := \langle x_n - y_n, d(x_n, y_n)\rangle$, $f : \mathcal{H} \to \mathcal{H}$ is monotone and Lipschitz continuous, $\mathcal{A} : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone operator and

$$\begin{cases} y_n = J^{\mathcal{A}}_{\lambda_n}\left(x_n - \lambda_n f(x_n)\right), \\ d(x_n, y_n) = x_n - y_n - \lambda_n\left(f(x_n) - f(y_n)\right), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)S\left(x_n - \gamma\beta_n d(x_n, y_n)\right), n \geq 1, \end{cases} \tag{1.11}$$

where $S : \mathcal{H} \to \mathcal{H}$ is a nonexpansive mapping, $\{\lambda_n\}$ is a variable stepsize satisfying the prediction stepsize conditions and $\liminf_{n\to\infty} \lambda_n = \lambda > 0$. It was shown that under some appropriate assumptions on the algorithms parameters, algorithm (1.10) converges weakly to the solution of MVIP (1.1) and algorithm (1.11) converges weakly to common element of Ω and the set of fixed points of the nonexpansive mapping $S : \mathcal{H} \to \mathcal{H}$. It is worth mentioning generally that, strong convergence is more preferable to weak convergence in infinite dimensional Hilbert spaces (see results in Bauschke and Combettes [9]). Also, we know that stepsizes play essential roles in the convergence properties of iterative methods, since efficiency of iterative methods depend heavily on it as a result of the knowledge of operator norm or coefficient of the operator and otherwise.

Motivated by the results of Zhang and Wang [46], Jung [27] and many other results in this direction, in this paper, we proposed a proximal and contraction algorithm with inertial extrapolation and relaxation techniques for approximating the common solution of monotone variational inclusion, and fixed point problems when the underlying operator is pseudocontractive in the framework of real Hilbert spaces. We establish that the sequence generated by this method converges strongly to the solution of the problems under the prediction stepsizes (which theoretically has wider range) conditions considered by Cai et al. [12] and, Zhang and Wang [46] and other standard mild assumptions on our algorithm parameters. Thus, our proposed method does not require prior knowledge of Lipschitz constant L of the cost operator f which generally enhances its efficiency and applicability. Moreover, we apply our obtained results to convex minimization problems. Finally, the results obtained in the work generalize and improve some well-known results in literature.

2. Preliminaries

In this section we will give some definitions and present results that will help us in convergence analysis later.

Definition 2.1. ([36, 44]) Let $\mathcal{H}$ be real Hilbert space and C a nonempty, closed and convex subset of $\mathcal{H}$. Let $S : \mathcal{H} \to \mathcal{H}$ be a real single-valued operator and $F(S) := \{u \in \mathcal{H} : u = Su\}$ denotes the set of all the fixed points of S. Then, S is said to be:

(a) L-Lipschitz continuous if there exists $L > 0$ such that

$$||Su - S\nu|| \leq L||u - \nu||, \quad \forall u, \nu \in \mathcal{H};$$

(b) non-expansive if

$$||Su - S\nu|| \leq ||u - \nu||, \quad \forall u, \nu \in \mathcal{H};$$

(c) pseudocontractive if

$$\langle Su - S\nu, u - \nu \rangle \leq ||u - \nu||^2 \text{ for each } u, \nu \in \mathcal{H}.$$

Equivalently, a mapping S is said to be pseudocontractive if

$$||Su - S\nu||^2 \leq ||u - \nu||^2 + ||(I - S)u - (I - S)\nu||^2, \ \ \forall u, \nu \in C.$$

Definition 2.2. ([46]) Let C be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ be a real single-valued operator. Then, $\mathcal{A}$ is said to be:

(a) monotone on C if

$$\langle \mathcal{A}u - \mathcal{A}\nu, \ u - \nu \rangle \geq 0, \ \ \forall u, \nu \in C;$$

(b) ζ-strongly monotone on C if there exists $\zeta > 0$ such that

$$\langle \mathcal{A}u - \mathcal{A}\nu, \ u - \nu \rangle \geq \zeta ||u - \nu||^2, \ \ \forall u, \nu \in C;$$

(c) η-inverse strongly monotone (η-ism) if there exists a positive constant η such that

$$\langle \mathcal{A}u - \mathcal{A}\nu, \ u - \nu \rangle \geq \eta ||\mathcal{A}u - \mathcal{A}\nu||^2, \ \ \forall u, \nu \in C;$$

(d) firmly non-expansive on C if

$$\langle \mathcal{A}u - \mathcal{A}\nu, \ u - \nu \rangle \geq ||\mathcal{A}u - \mathcal{A}\nu||^2, \ \ \forall u, \nu \in C.$$

Definition 2.3. ([33]) Let $\mathcal{H}$ be a real Hilbert space and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued operator. Then, M is said to be:

(a) monotone if

$$\langle x - y, u - \nu \rangle \geq 0, \ \forall x, y \in \mathcal{H}, u \in M(x), \nu \in M(y);$$

(b) maximal monotone if the graph of M denoted and defined by

$$G(M) := \{(x, y) \in \mathcal{H} \times \mathcal{H} : y \in M(x)\}$$

is not properly contained in the graph of any other monotone operator. In general M is maximal monotone if and only if $(x, u) \in \mathcal{H} \times \mathcal{H}$, $\langle x - y, u - \nu \rangle > 0$ for all $(y, \nu) \in G(M)$ implies that $u \in M(x)$;

(c) The resolvent operator J_μ^M associated with multi-valued, maximal monotone operator M and $\mu > 0$ is a single-valued mapping $J_\mu^M : \mathcal{H} \to \mathcal{H}$ defined by

$$J_\mu^M(x) := (I + \mu M)^{-1}(x), \ \ \forall x \in \mathcal{H},$$

where I is an identity operator on $\mathcal{H}$.

It should be noted that for all $\mu > 0$ the resolvent operator J_{μ}^{M} is single-valued, nonexpansive and firmly nonexpansive (reader can see [15] for more details).

Definition 2.4. ([17]) Let $\mathcal{H}$ be a real Hilbert space and C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $\mathcal{A} : C \to \mathcal{H}$ be an operator. Then, $\mathcal{A}$ is said to be demiclosed at zero if for any sequence $\{y_n\}_{n=1}^{\infty}$ in C such that y_n converges weakly to a point $\tilde{y} \in C$ and $\mathcal{A}(y_n) \to y$ then, $\mathcal{A}(\tilde{y}) = y$.

Lemma 2.5. ([28]) *Let $\mathcal{H}$ be a real Hilbert space with $u, v \in \mathcal{H}$ and $\lambda \in \mathbb{R}$. Then the following holds:*

(i) $2\langle u, v\rangle = ||u||^2 + ||v||^2 - ||u - v||^2 = ||u + v||^2 - ||u||^2 - ||v||^2$;

(ii) $||u - v||^2 \leq ||u||^2 + 2\langle v, u - v\rangle$;

(iii) $||\lambda u + (1 - \lambda)\nu||^2 = \lambda||u||^2 + (1 - \lambda)||\nu||^2 - \lambda(1 - \lambda)||u - \nu||^2$.

Lemma 2.6. ([44]) *Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. Let $S : C \to \mathcal{H}$ be a smooth pseudocontractive mapping. Then, for any $\lambda > 0$ and $u \in \mathcal{H}$ there exists $\nu \in C$ such that*

$$\langle w - \nu, S\nu\rangle - \frac{1}{\lambda}\langle w - \nu, (1 + \lambda)\nu - u\rangle \leq 0,\ \forall w \in C.$$

For $\lambda > 0$ and $u \in \mathcal{H}$, the resolvent operator of S is a mapping $S_\lambda : C \to \mathcal{H}$ defined as follows:

$$S_\lambda u := \left\{\nu \in C : \langle w - \nu, S\nu\rangle - \frac{1}{\lambda}\langle w - \nu, (1 + \lambda)\nu - u\rangle \leq 0,\ \forall w \in C\right\}.$$

Then, the following hold:

(i) *S_λ is a single-valued operator;*

(ii) *S_λ is firmly nonexpansive, that is,*

$$||u - \nu||^2 \leq \langle S_\lambda u - S_\lambda \nu, u - \nu\rangle,\quad \forall u, \nu \in \mathcal{H};$$

(iii) *$F(S_\lambda) = F(S)$;*

(iv) *$F(S)$ is a closed convex subset of C.*

Lemma 2.7. ([24]) *Let C be a closed convex subset of a real Hilbert space $\mathcal{H}$, and $S : C \to C$ be a nonexpansive mapping such that $F(S) \neq 0$. If a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup x^*$ and $x_n - Sx_n \to 0$ as $n \to \infty$, then $x^* = Sx^*$.*

Lemma 2.8. ([43]) *Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone mapping and $T : \mathcal{H} \to \mathcal{H}$ be a Lipschitz continuous mapping. Then the mapping $D := M + T : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone mapping.*

Lemma 2.9. ([27]) *Let* $\{\chi_n\}$ *be a sequence of positive real numbers. Let* $\{\delta_n\}$ *be a sequence in* $[0,1]$ *with* $\sum_{n=1}^{\infty}\delta_n=\infty$*, let* $\{\rho_n\}$ *also be a real sequence such that* $\limsup\rho_n\leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n\rho_n|<\infty$*. Suppose that* $\{\chi_n\}$ *satisfies the inequality:*

$$\chi_{n+1}\leq(1-\delta_n)\,\chi_n+\delta_n\rho_n,\ \forall\ n\in\mathbb{N}.$$

Then, $\lim_{n\to\infty}\chi_n=0$*.*

Lemma 2.10. ([27]) *Let* $\{\chi_n\}$ *be a sequence of positive real numbers such that there exists a subsequence* $\{\chi_{n_k}\}$ *of* $\{\chi_n\}$ *with* $\chi_{n_k}<\chi_{n_k+1}$ *for all* $k\in\mathbb{N}$*. Let* $\{m_j\}$ *be sequence of integers defined by* $m_j=\max\{k\leq j:\chi_k<\chi_{k+1}\}$*. Then,* $\{m_j\}$ *is a non-decreasing sequence satisfying* $\lim_{j\to\infty}m_j=\infty$ *with the properties* $\chi_{m_j}\leq\chi_{m_j+1}$ *and* $\chi_j\leq\chi_{m_j+1}$*, for all* $j\in\mathbb{N}$*.*

3. Main results

Throughout this work, we shall use $x_n\to x$ (resp. $x_n\rightharpoonup x$) to denote that the sequence $\{x_n\}$ converges strongly (resp. weakly) to a point x as $n\to\infty$. For the convergence analysis of our method, we shall make the following assumptions.

Assumption 3.1. Supposed that:

(A1) Let C be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$;

(A2) $\mathcal{A}:\mathcal{H}\to 2^{\mathcal{H}}$ is a multi-valued maximal monotone mapping;

(A3) $f:\mathcal{H}\to\mathcal{H}$ is monotone and L-Lipschitz continuous;

(A4) $S:\mathcal{H}\to\mathcal{H}$ is a pseudocontractive mapping with $F(S)\neq\emptyset$;

(A5) The solution set $\Omega:=\{z^*\in\mathcal{H}:0\in f(z^*)+\mathcal{A}(z^*)\}\neq\emptyset$ and

$$\Gamma:=\Omega\cap F(S)\neq\emptyset;$$

(A6) The operator; $S_{\lambda_n}:\mathcal{H}\to C$ is defined by

$$S_{\lambda_n}u\coloneqq\Big\{\nu\in C:\langle w-\nu,S\nu\rangle-\tfrac{1}{\lambda_n}\langle w-\nu,(1+\lambda_n)\nu-u\rangle\leq 0,\ \forall w\in C\Big\};$$

(A7) The control sequences:$\{\alpha_n\}$, $\{\lambda_n\}$ are positive real sequences in $(0,1)$ with $\{\alpha_n\}$ satisfying the property: $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\{\sigma_n\}\subset(a,1-\alpha_n)$, $a>0$ and $\{c_n\}$ a positive sequence with

$$c_n=\circ(\sigma_n),$$

that is, $\lim_{n\to\infty}\frac{c_n}{\sigma_n}=0$.

Algorithm 3.2. Initialization: Choose $\gamma \in (0,2)$, $(\mu_n) \subset [\mu_1, \mu_2] \in \left(0, \frac{1}{L}\right)$, $\theta \in [0,1]$, $x_0, x_1 \in \mathcal{H}$.

Iterative Process Steps: Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \min \left\{ \theta, \frac{c_n}{||x_n - x_{n-1}||} \right\}, \text{ if } x_n \neq x_{n-1}; \text{ otherwise, set } \bar{\theta}_n = \theta.$$

Step 1: Set $n = 1$, we calculate the iterate x_{n+1} as follows:

$$w_n = x_n + \theta_n (x_n - x_{n-1})$$

and compute

$$\nu_n = (1 - \alpha_n) w_n,$$

where α_n is as given in assumption (A7).

Step 2: Compute

$$u_n = J^{\mathcal{A}}_{\mu_n} (\nu_n - \mu_n f(\nu_n)),$$
$$d(\nu_n, u_n) = \nu_n - u_n - \mu_n (f(\nu_n) - f(u_n)).$$

Step 3: Compute

$$y_n = \nu_n - \gamma \beta_n d(\nu_n, u_n),$$

$$x_{n+1} = (1 - \sigma_n) y_n + \sigma_n S_{\lambda_n} y_n,$$

where $\beta_n := \frac{\phi(\nu_n, u_n)}{||d(\nu_n, u_n)||^2}$, for $\phi(\nu_n, u_n) := \langle \nu_n - u_n, d(\nu_n, u_n) \rangle$.

Step 4: Set $n := n + 1$ and return to **Step 1**.

Let $\tau_1, \tau_2 > 0$, then from the results in Cai et al. [12], we say that μ_n satisfies the prediction stepsize conditions in proximal and contraction methods if μ_n satisfies the inequalities:

$$\tau_1 ||\nu_n - u_n||^2 \leq \phi(\nu_n, u_n) \tag{3.1}$$

and

$$\tau_2 \leq \beta_n, \quad \forall n \geq 1. \tag{3.2}$$

Lemma 3.3. *For each $n \geq 1$, if $\nu_n = u_n$ or $d(\nu_n, u_n) = 0$, in Algorithm 3.2, then $\nu_n \in \Omega$.*

Proof. Since f is L-Lipschitzian, using the definition of $d(\nu_n, u_n)$ in Algorithm 3.2, we have

$$\begin{aligned} ||d(\nu_n, u_n)|| &= ||\nu_n - u_n - \mu_n (f(\nu_n) - f(u_n)) || \\ &\geq ||\nu_n - u_n|| - \mu_n ||f(\nu_n) - f(u_n)|| \\ &\geq ||\nu_n - u_n|| - \mu_n L ||\nu_n - u_n|| \\ &= (1 - \mu_n L) ||\nu_n - u_n||, \ \forall \, n \geq 1. \end{aligned} \tag{3.3}$$

Similarly, we have

$$\begin{aligned} ||d(\nu_n, u_n)|| &= ||\nu_n - u_n - \mu_n (f(\nu_n) - f(u_n)) || \\ &\leq ||\nu_n - u_n|| + \mu_n ||f(\nu_n) - f(u_n)|| \\ &\leq ||\nu_n - u_n|| + \mu_n L ||\nu_n - u_n|| \\ &= (1 + \mu_n L) ||\nu_n - u_n||, \ \forall \, n \geq 1. \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4) we have

$$\begin{aligned} (1 - \mu_n L) ||\nu_n - u_n|| &\leq ||d(\nu_n, u_n)|| \\ &\leq (1 + \mu_n L) ||\nu_n - u_n||, \ \forall \, n \geq 1. \end{aligned} \tag{3.5}$$

From (3.5) we observe that $\nu_n = u_n$ if and only if $d(\nu_n, u_n) = 0$. Therefore, $\nu_n = u_n$ or $d(\nu_n, u_n) = 0$. Thus, if $\nu_n = u_n$ then we have

$$\begin{aligned} \nu_n &= J^{\mathcal{A}}_{\mu_n} (\nu_n - \mu_n f(\nu_n)) \\ &= (I + \mu_n \mathcal{A})^{-1} (\nu_n - \mu_n f(\nu_n)), \\ \nu_n - \mu_n f(\nu_n) &\in (I + \mu_n \mathcal{A}) \nu_n, \\ 0 &\in \mathcal{A}(\nu_n) + f(\nu_n). \end{aligned}$$

Hence, $\nu_n \in \Omega$. □

Lemma 3.4. *Let $\{\nu_n\}$ and $\{y_n\}$ be sequences generated by Algorithm 3.2 under Assumption 3.1. Let $\mu_n \in [\mu_1, \mu_2] \subset \left(0, \frac{1}{L}\right)$, that is, $0 < \mu_1 \leq \mu_n \leq \mu_2 < +\infty$ with $\liminf_{n \to \infty} \mu_n = \mu_1 > 0$. If μ_n satisfies the conditions (3.1) and (3.2). Then, the following hold:*

(i)

$$\beta_n \geq \frac{1 - \mu_2 L}{1 + \mu_2^2 L^2}. \tag{3.6}$$

(ii)

$$||y_n - z^*||^2 \leq ||\nu_n - z^*||^2 - \frac{2 - \gamma}{\gamma} ||y_n - \nu_n||^2. \tag{3.7}$$

(iii)

$$||\nu_n - u_n||^2 \leq \frac{1+\mu_2^2 L^2}{[(1-\mu_2 L)\gamma]^2}||\nu_n - y_n||^2. \tag{3.8}$$

Proof. (i) From definition of $\phi(\nu_n, u_n)$ in Algorithm 3.2 we have

$$\begin{aligned}\phi(\nu_n, u_n) &= \langle \nu_n - u_n, d(\nu_n, u_n)\rangle \\ &= \langle \nu_n - u_n, \nu_n - u_n - \mu_n\left(f(\nu_n) - f(u_n)\right)\rangle \\ &= ||\nu_n - u_n||^2 - \mu_n \langle \nu_n - u_n f(\nu_n) - f(u_n)\rangle \\ &\geq ||\nu_n - u_n||^2 - \mu_n ||\nu_n - u_n|| ||f(\nu_n) - f(u_n)|| \\ &\geq ||\nu_n - u_n||^2 - \mu_n L||\nu_n - u_n||^2 \\ &= (1-\mu_2 L)||\nu_n - u_n||^2. \end{aligned} \tag{3.9}$$

Also, we have

$$\begin{aligned}||d(\nu_n, u_n)||^2 &= ||\nu_n - u_n - \mu_n\left(f(\nu_n) - f(u_n)\right)||^2 \\ &= ||\nu_n - u_n||^2 + \mu_n^2 ||f(\nu_n) - f(u_n)||^2 \\ &\quad - 2\mu_n \langle \nu_n - u_n, f(\nu_n) - f(u_n)\rangle \\ &\leq ||\nu_n - u_n||^2 + \mu_n^2 L^2 ||\nu_n - u_n||^2 \\ &= (1+\mu_2^2 L^2)||\nu_n - u_n||^2. \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10) we get

$$\beta_n := \frac{\phi(\nu_n, u_n)}{||d(\nu_n, u_n)||^2} \geq \frac{1-\mu_2 L}{1+\mu_2^2 L^2}. \tag{3.11}$$

(ii) Let $z^* \in \Gamma$. Then, by definition of y_n, in Algorithm 3.2, we have

$$\begin{aligned}||y_n - z^*||^2 &= ||\nu_n - \gamma\beta_n d(\nu_n, u_n) - z^*||^2 \\ &= ||\nu_n - z^*||^2 - 2\gamma\beta_n \langle \nu_n - z^*, d(\nu_n, u_n)\rangle + \gamma^2\beta_n^2 ||d(\nu_n, u_n)||^2. \end{aligned} \tag{3.12}$$

From Definition 2.3 (c), we know that the resolvent operator $J_{\mu_n}^{\mathcal{A}}$ is firmly nonexpansive and $z^* = J_{\mu_n}^{\mathcal{A}}\left(I - \mu_n f\right) z^*$. It implies that

$$\begin{aligned}&\left\langle J_{\mu_n}^{\mathcal{A}}\left(I - \mu_n f\right)\nu_n - J_{\mu_n}^{\mathcal{A}}\left(I - \mu_n f\right)z^*, (I-\mu_n f)\nu_n - (I-\mu_n f)z^*\right\rangle \\ &\quad \geq ||J_{\mu_n}^{\mathcal{A}}\left(I - \mu_n f\right)\nu_n - J_{\mu_n}^{\mathcal{A}}\left(I - \mu_n f\right)z^*||^2 \\ &\quad = ||u_n - z^*||^2. \end{aligned} \tag{3.13}$$

Observe that

$$\begin{aligned}
&\langle u_n - z^*, \nu_n - u_n - \mu_n f(\nu_n)\rangle \\
&\quad = \left\langle J^{\mathcal{A}}_{\mu_n}(I-\mu_n f)\,\nu_n - J^{\mathcal{A}}_{\mu_n}(I-\mu_n f)\,z^*, (I-\mu_n f)\nu_n - (I-\mu_n f)z^*\right\rangle \\
&\qquad + \langle u_n - z^*, z^* - u_n\rangle + \langle u_n - z^*, -\mu_n f(z^*)\rangle \\
&\quad \geq ||J^{\mathcal{A}}_{\mu_n}(I-\mu_n f)\,\nu_n - J^{\mathcal{A}}_{\mu_n}(I-\mu_n f)\,z^*||^2 \\
&\qquad - ||u_n - z^*||^2 - \mu_n \langle u_n - z^*, f(z^*)\rangle . \qquad (3.14)
\end{aligned}$$

Combining (3.13) and (3.14), we obtain

$$\begin{aligned}
&\langle u_n - z^*, \nu_n - u_n - \mu_n f(\nu_n)\rangle \\
&\qquad \geq ||u_n - z^*||^2 - ||u_n - z^*||^2 - \mu_n \langle u_n - z^*, f(z^*)\rangle . \qquad (3.15)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\langle u_n - z^*, \nu_n - u_n - \mu_n f(\nu_n)\rangle + \mu_n \langle u_n - z^*, f(z^*)\rangle \geq 0, \\
&\langle u_n - z^*, \nu_n - u_n - \mu_n (f(\nu_n) - f(z^*))\rangle \geq 0. \qquad (3.16)
\end{aligned}$$

From the monotonicity of f and $\mu_n > 0$, we have that

$$\langle u_n - z^*, \mu_n (f(u_n) - f(z^*))\rangle \geq 0. \qquad (3.17)$$

Combining (3.16) and (3.17) we get,

$$\begin{aligned}
&\langle u_n - z^*, \nu_n - u_n - \mu_n f(\nu_n) + \mu_n f(z^*) + \mu_n f(u_n) - \mu_n f(z^*)\rangle \geq 0, \\
&\langle u_n - z^*, \nu_n - u_n - \mu_n (f(\nu_n) - f(u_n))\rangle \geq 0.
\end{aligned}$$

It follows that

$$\langle u_n - z^*, d(\nu_n, u_n)\rangle = \langle u_n - z^*, \nu_n - u_n - \mu_n (f(\nu_n) - f(u_n))\rangle \geq 0. \qquad (3.18)$$

Now

$$\begin{aligned}
\langle \nu_n - z^*, d(\nu_n, u_n)\rangle &= \langle \nu_n - u_n + u_n - z^*, d(\nu_n, u_n)\rangle \\
&= \langle \nu_n - u_n, d(\nu_n, u_n)\rangle + \langle u_n - z^*, d(\nu_n, u_n)\rangle . \qquad (3.19)
\end{aligned}$$

From (3.18) and (3.19) with the definition of $\phi(\nu_n, u_n)$ in Algorithm 3.2, Step 3, we have

$$\langle \nu_n - z^*, d(\nu_n, u_n)\rangle \geq \phi(\nu_n, u_n). \qquad (3.20)$$

Combining (3.12) and (3.20) we have

$$\begin{aligned}
||y_n - z^*||^2 &\leq ||\nu_n - z^*||^2 - 2\gamma\beta_n\phi(\nu_n, u_n) + \gamma^2\beta_n\phi(\nu_n, u_n) \\
&= ||\nu_n - z^*||^2 - \gamma(2-\gamma)\beta_n\phi(\nu_n, u_n). \qquad (3.21)
\end{aligned}$$

From the definition of y_n and the relation $\beta_n := \frac{\phi(\nu_n,u_n)}{||d(\nu_n,u_n)||^2}$, we have that

$$\begin{aligned}\beta_n\phi(\nu_n,u_n) &= ||\beta_n d(\nu_n,u_n)||^2\\ &= \frac{1}{\gamma^2}||y_n-\nu_n||^2. \end{aligned} \tag{3.22}$$

Combining (3.21) and (3.22) we obtain

$$||y_n-z^*||^2 \le ||\nu_n-z^*||^2 - \frac{2-\gamma}{\gamma}||y_n-\nu_n||^2. \tag{3.23}$$

(iii) From (3.11) and (3.22) we have

$$\begin{aligned}\phi(\nu_n,u_n) &= \frac{1}{\beta_n\gamma^2}||y_n-\nu_n||^2\\ &\le \frac{1+\mu_2^2L^2}{\gamma^2\,(1-\mu_2L)}||y_n-\nu_n||^2. \end{aligned} \tag{3.24}$$

Combining (3.9) and (3.24) we obtain

$$\begin{aligned}(1-\mu_2L)||\nu_n-u_n||^2 &\le \phi(\nu_n,u_n)\\ &\le \frac{1+\mu_2^2L^2}{\gamma^2\,(1-\mu_2L)}||y_n-\nu_n||^2.\end{aligned}$$

Hence,

$$||\nu_n-u_n||^2 \le \frac{1+\mu_2^2L^2}{\left[\gamma\,(1-\mu_2L)\right]^2}||y_n-\nu_n||^2. \tag{3.25}$$

□

Lemma 3.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1. Then, $\{x_n\}$ is bounded.*

Proof. Let $z^*\in\Gamma$. Then, by definition of w_n, in Algorithm 3.2, we obtain

$$\begin{aligned}||w_n-z^*|| &= ||x_n+\theta_n\,(x_n-x_{n-1})-z^*||\\ &= ||x_n-z^*+\theta_n\,(x_n-x_{n-1})\,||\\ &\le ||x_n-z^*|| + \alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||.\end{aligned}$$

Since, $\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}|| \le \frac{c_n}{\alpha_n}$, which by (A7) implies that

$$\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}|| = 0,$$

which means $\left\{\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||\right\}$ is bounded. Then, there exists a constant $M_0 > 0$ such that

$$\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \leq M_0$$

for all $n \geq 1$. Hence,

$$||w_n - z^*|| \leq ||x_n - z^*|| + \alpha_n M_0. \tag{3.26}$$

We know from Lemma 3.4 (ii) that

$$||y_n - z^*||^2 \leq ||\nu_n - z^*||^2 - \frac{2-\gamma}{\gamma}||y_n - \nu_n||^2. \tag{3.27}$$

Since $\gamma \in (0,2)$ it follows that $2-\gamma > 0$ and so,

$$||y_n - z^*|| \leq ||\nu_n - z^*||. \tag{3.28}$$

Using the definition of sequence $\{x_n\}$ in Algorithm 3.2, Lemma 2.6 (iii), (3.26) and (3.28) we have,

$$\begin{aligned}
||x_{n+1} - z^*|| &= ||(1-\sigma_n)y_n + \sigma_n S_{\lambda_n} y_n - z^*|| \\
&\leq (1-\sigma_n)||y_n - z^*|| + \sigma_n ||y_n - z^*|| \\
&= ||y_n - z^*|| \\
&\leq ||\nu_n - z^*|| \\
&= ||\left(1-\alpha_n\right)\left(w_n - z^*\right) - \alpha_n z^*|| \\
&\leq (1-\alpha_n)||w_n - z^*|| + \alpha_n ||z^*|| \\
&\leq (1-\alpha_n)||x_n - z^*|| + \alpha_n \left[M_0 + ||z^*||\right] \\
&\leq \max\left\{||x_n - z^*||,\ \left[M_0 + ||z^*||\right]\right\}.
\end{aligned} \tag{3.29}$$

By induction we have

$$||x_n - z^*|| \leq \max\left\{||x_1 - z^*||,\ \left[M_0 + ||z^*||\right]\right\}. \tag{3.30}$$

Since the sequence $\{||x_n - z^*||\}$ is bounded, it follows that $\{x_n\}$ is bounded. Thus, ensuring the boundedness of $\{w_n\}$, $\{\nu_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{S_{\lambda_n} y_n\}$, respectively. □

Lemma 3.6. *Let $\{x_n\}$ be sequence generated by Algorithm 3.2 under Assumption 3.1. Then, $\{x_n\}$ satisfies the inequality:*

$$\Upsilon_{n+1} \leq \left(1-\alpha_n\right)\Upsilon_n + \alpha_n \rho_n, \tag{3.31}$$

where,

$$\Upsilon_n := ||x_n - z^*||^2$$

and

$$\rho_n := 2\left\langle z^*, \nu_n - z^*\right\rangle + \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||M_1,$$

with $M_1 := \sup\left(\theta_n||x_n - x_{n-1}|| + 2||x_n - z^*||\right)$.

Proof. Let $z^* \in \Gamma$. Then, using the definition of sequence $\{x_n\}$ in Step 3 of Algorithm 3.2, Assumption 3.1, Lemma 2.5, and (3.27), we obtain

$$\begin{aligned} ||x_{n+1} - z^*||^2 &= ||(1-\sigma_n)y_n + \sigma_n S_{\lambda_n} y_n - z^*||^2 \\ &= (1-\sigma_n)||y_n - z^*||^2 + \sigma_n||S_{\lambda_n}y_n - z^*||^2 \\ &\quad - \sigma_n\left(1-\sigma_n\right)||y_n - S_{\lambda_n}y_n||^2 \\ &\le (1-\sigma_n)||y_n - z^*||^2 + \sigma_n||y_n - z^*||^2 \\ &\quad - \sigma_n\left(1-\sigma_n\right)||y_n - S_{\lambda_n}y_n||^2 \\ &= ||y_n - z^*||^2 - \sigma_n\left(1-\sigma_n\right)||y_n - S_{\lambda_n}y_n||^2 \\ &\le ||\nu_n - z^*||^2 - \frac{2-\gamma}{\gamma}||\nu_n - y_n||^2 \\ &\quad - \sigma_n\left(1-\sigma_n\right)||y_n - S_{\lambda_n}y_n||^2 \\ &\le (1-\alpha_n)^2||w_n - z^*||^2 + 2\alpha_n\left\langle z^*, \nu_n - z^*\right\rangle \\ &\quad - \frac{2-\gamma}{\gamma}||\nu_n - y_n||^2 \\ &\quad - \sigma_n\left(1-\sigma_n\right)||y_n - S_{\lambda_n}y_n||^2. \end{aligned} \tag{3.32}$$

We observe that,

$$\begin{aligned} ||w_n - z^*||^2 &= ||x_n - z^* + \theta_n\left(x_n - x_{n-1}\right)||^2 \\ &= ||x_n - z^*||^2 + 2\theta_n\left\langle x_n - z^*, x_n - x_{n-1}\right\rangle \\ &\quad + \theta_n^2||x_n - x_{n-1}||^2 \\ &\le ||x_n - z^*||^2 + 2\theta_n||x_n - z^*||||x_n - x_{n-1}|| \\ &\quad + \theta n^2||x_n - x_{n-1}||^2 \\ &= ||x_n - z^*||^2 + \theta_n||x_n - x_{n-1}|| \\ &\quad \times\left(2||x_n - z^*|| + \theta_n||x_n - x_{n-1}||\right) \\ &= ||x_n - z^*||^2 + \theta_n||x_n - x_{n-1}||M_1, \end{aligned} \tag{3.33}$$

where, $M_1 := \sup\left(2||x_n - z^*|| + \theta_n||x_n - x_{n-1}||\right) < \infty$.

Combining (3.32) and (3.33) we get

$$\begin{aligned} ||x_{n+1} - z^*||^2 \le{}& (1-\alpha_n)\left[||x_n - z^*||^2 + \theta_n||x_n - x_{n-1}||M_1\right] \\ &+ 2\alpha_n\left\langle z^*, \nu_n - z^*\right\rangle - \frac{2-\gamma}{\gamma}||y_n - \nu_n||^2 \\ &- \sigma_n\left(1-\sigma_n\right)||y_n - S_{\lambda_n}y_n||^2 \end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)||x_n - z^*||^2 \\
&\quad + \alpha_n \left[2\left\langle z^*, \nu_n - z^* \right\rangle + \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||M_1 \right] \\
&\quad - \frac{2-\gamma}{\gamma}||y_n - \nu_n||^2 - \sigma_n\left(1-\sigma_n\right)||y_n - S_{\lambda_n} y_n||^2 \\
&\leq (1-\alpha_n)||x_n - z^*||^2 \\
&\quad + \alpha_n \left[2\left\langle z^*, \nu_n - z^* \right\rangle + \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||M_1 \right].
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
||x_{n+1} - z^*||^2 &\leq (1-\alpha_n)||x_n - z^*||^2 \\
&\quad + \alpha_n \left[2\left\langle z^*, \nu_n - z^* \right\rangle + \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||M_1 \right],
\end{aligned} \tag{3.34}
$$

which yields our desired result. □

Theorem 3.7. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.2 under Assumption 3.1. Then, $\{x_n\}$ converges strongly to a point $z^* \in \Gamma := \Omega \cap F(S)$.*

Proof. Let $z^* \in \Gamma$. We shall denote $\Upsilon_n := ||x_n - z^*||^2$ for all $n \geq 1,\ n \in \mathbb{N}$. Our convergence analysis will be divided into two cases.

Case I: Suppose $\{\Upsilon_n\}$ is monotonically non-increasing for all $n \in \mathbb{N}$, that is $\Upsilon_n \geq \Upsilon_{n+1}$. Then, it is obvious that $\Upsilon_n - \Upsilon_{n+1} \to 0$ as $n \to \infty$.
Using the definition of sequence $\{x_n\}$ in Algorithm 3.2, Lemma 2.5, equation (3.27) and (3.33), we obtain

$$
\begin{aligned}
||x_{n+1} - z^*||^2 &= ||(1-\sigma_n)(y_n - z^*) + \sigma_n\left(S_{\lambda_n} y_n - z^*\right)||^2 \\
&= (1-\sigma_n)||y_n - z^*||^2 + \sigma_n||S_{\lambda_n} y_n - z^*||^2 \\
&\quad - \sigma_n\left(1-\sigma_n\right)||y_n - S_{\lambda_n} y_n||^2 \\
&\leq (1-\sigma_n)||y_n - z^*||^2 + \sigma_n||y_n - z^*||^2 \\
&\quad - \sigma_n\left(1-\sigma_n\right)||y_n - S_{\lambda_n} y_n||^2 \\
&= ||y_n - z^*||^2 - \sigma_n\left(1-\sigma_n\right)||y_n - S_{\lambda_n} y_n||^2 \\
&\leq ||\nu_n - z^*||^2 - \frac{2-\gamma}{\gamma}||\nu_n - y_n||^2 \\
&\quad - \sigma_n\left(1-\sigma_n\right)||y_n - S_{\lambda_n} y_n||^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\alpha_n)^2||w_n - z^*||^2 + 2\alpha_n \langle z^*, \nu_n - z^* \rangle \\
&\quad - \frac{2-\gamma}{\gamma}||\nu_n - y_n||^2 - \sigma_n (1-\sigma_n) ||y_n - S_{\lambda_n} y_n||^2 \\
&\leq (1-\alpha_n)||x_n - z^*||^2 \\
&\quad + \alpha_n \left[2\langle z^*, \nu_n - z^* \rangle + \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||M_1 \right] \\
&\quad - \frac{2-\gamma}{\gamma}||y_n - \nu_n||^2 - \sigma_n (1-\sigma_n) ||y_n - S_{\lambda_n} y_n||^2. \qquad (3.35)
\end{aligned}
$$

It follows from (3.35) that

$$
\begin{aligned}
&\frac{2-\gamma}{\gamma}||y_n - \nu_n||^2 + \sigma_n (1-\sigma_n) ||y_n - S_{\lambda_n} y_n||^2 \\
&\quad \leq (1-\alpha_n)||x_n - z^*||^2 + \alpha_n \left[2\langle z^*, \nu_n - z^* \rangle + \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||M_1 \right] \\
&\quad - ||x_{n+1} - z^*||^2. \qquad (3.36)
\end{aligned}
$$

Hence, passing limit as $n \to \infty$ in (3.36) and using Assumption 3.1-(A7) with the fact that $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| = 0$, we obtain

$$
\lim_{n\to\infty} \left[\frac{2-\gamma}{\gamma}||y_n - \nu_n||^2 + \sigma_n (1-\sigma_n) ||y_n - S_{\lambda_n} y_n||^2 \right] = 0. \qquad (3.37)
$$

This implies that

$$
\lim_{n\to\infty} ||y_n - \nu_n|| = 0 \qquad (3.38)
$$

and

$$
\lim_{n\to\infty} ||y_n - S_{\lambda_n} y_n|| = 0. \qquad (3.39)
$$

Observe from the definition of w_n that

$$
||w_n - x_n|| = \alpha_n \times \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \to 0 \text{ as } n \to \infty. \qquad (3.40)
$$

From the definition of (y_n) and β_n we have

$$
\begin{aligned}
||y_n - \nu_n|| &= \gamma\beta_n ||d(\nu_n, u_n)|| \\
&= \gamma \frac{\langle \nu_n - u_n, d(\nu_n, u_n) \rangle}{||d(\nu_n, u_n)||}. \qquad (3.41)
\end{aligned}
$$

But, we know that

$$\begin{aligned}||d(\nu_n, u_n)|| &= ||\nu_n - u_n - \mu_n \left(f(\nu_n) - f(u)\right)|| \\ &\le ||\nu_n - u_n|| + \mu_n ||f(\nu_n) - f(u)|| \\ &\le ||\nu_n - u_n|| + \mu_n L ||\nu_n - u|| \\ &= (1 + \mu_2 L)||\nu_n - u_n||. \end{aligned} \tag{3.42}$$

It follows from (3.42) that

$$\frac{1}{||d(\nu_n, u_n)||} \ge \frac{1}{(1+\mu_2 L)||\nu_n - u_n||}. \tag{3.43}$$

Also from (3.9) we have

$$\langle \nu_n - u_n, d(\nu_n, u_n)\rangle \ge (1 - \mu_2 L)||\nu_n - u_n||^2. \tag{3.44}$$

Combining (3.41), (3.43) and (3.44) we obtain

$$||y_n - \nu_n|| \ge \gamma \frac{1 - \mu_2 L}{1 + \mu_2 L}||\nu_n - u_n||. \tag{3.45}$$

Applying (3.38) we have

$$\lim_{n\to\infty} ||\nu_n - u_n|| = 0. \tag{3.46}$$

Also, we have

$$\lim_{n\to\infty} ||\nu_n - w_n|| = \lim_{n\to\infty} (\alpha_n ||w_n||) = 0. \tag{3.47}$$

Applying (3.40) we get

$$\begin{aligned}||\nu_n - x_n|| &= ||(1-\alpha_n) w_n - x_n|| \\ &= ||(1-\alpha_n)(w_n - x_n) - \alpha_n x_n|| \\ &\le (1-\alpha_n)||w_n - x_n|| + \alpha_n ||x_n|| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.48}$$

Combining (3.46) and (3.47), we obtain

$$\begin{aligned}||y_n - u_n|| &\le ||y_n - \nu_n|| + ||\nu_n - u_n|| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.49}$$

Applying (3.38) and (3.48), we get

$$\begin{aligned}||y_n - x_n|| &\le ||y_n - \nu_n|| + ||\nu_n - x_n|| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.50}$$

Also, applying (3.39) we obtain

$$\begin{aligned}||y_n - x_{n+1}|| &= ||y_n - (1-\sigma_n)y_n - \sigma_n S_{\lambda_n} y_n|| \\ &= \sigma_n ||y_n - S_{\lambda_n} y_n|| \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence,

$$\lim_{n\to\infty} ||y_n - x_{n+1}|| = 0. \tag{3.51}$$

Combining (3.50) and (3.51) we obtain

$$\begin{aligned}||x_n - x_{n+1}|| &= ||x_n - y_n + y_n - x_{n+1}|| \\ &\leq ||x_n - y_n|| + ||y_n - x_{n+1}|| \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} ||x_n - x_{n+1}|| = 0. \tag{3.52}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z^* \in \mathcal{H}$ as $n \to \infty$. Accordingly, by Lemma 3.3, $u_{n_k} \rightharpoonup z^* \in \mathcal{H}$ as $n \to \infty$.

Next we need to show that

$$0 \in \mathcal{A}(z^*) + f(z^*).$$

Since the operator f is monotone and L-Lipschitz continuous, by Lemma 2.8, it follows that $(\mathcal{A} + f)$ is a maximal monotone mapping. Let us consider $G(\mathcal{A} + f)$ as the graph of this operator. It follows that for $(u, \omega) \in G(\mathcal{A} + f)$; $\omega - fu \in \mathcal{A}(u)$. From the definition of $u_n = J^{\mathcal{A}}_{\mu_n}(\nu_n - \mu_n f(\nu_n))$, we obtain

$$\begin{aligned}u_n &= J^{\mathcal{A}}_{\mu_n}(\nu_n - \mu_n f(\nu_n)) \\ &= (I + \mu_n \mathcal{A})^{-1}(\nu_n - \mu_n f(\nu_n)).\end{aligned}$$

It implies that

$$\nu_n - \mu_n f(\nu_n) \in (I + \mu_n \mathcal{A}) u_n.$$

That is

$$\frac{1}{\mu_n}(\nu_n - u_n - \mu_n f(\nu_n)) \in \mathcal{A}(u_n).$$

Since $(\mathcal{A}+f)$ is maximally monotone, it follows that for $(u,\omega) \in G(\mathcal{A}+f)$, we have

$$\left\langle u - u_{n_k}, \omega - fu - \frac{1}{\mu_{n_k}}(\nu_{n_k} - u_n - \mu_{n_k} f\nu_{n_k}) \right\rangle \geq 0,$$

$$\langle u - u_{n_k}, \omega \rangle - \left\langle u - u_{n_k}, fu + \frac{1}{\mu_n}(\nu_{n_k} - u_n - \mu_{n_k} f\nu_{n_k}) \right\rangle \geq 0.$$

It follows that

$$\begin{aligned}
\langle u - u_{n_k}, \omega \rangle &\geq \left\langle u - u_{n_k}, fu + \frac{1}{\mu_{n_k}}(\nu_{n_k} - u_n - \mu_{n_k} f\nu_{n_k}) \right\rangle \\
&= \left\langle u - u_{n_k}, fu - fu_{n_k} + fu_{n_k} + \frac{1}{\mu_{n_k}}(\nu_{n_k} - u_n - \mu_{n_k} f\nu_{n_k}) \right\rangle \\
&= \langle u - u_{n_k}, fu - fu_{n_k} \rangle + \langle u - u_{n_k}, fu_{n_k} - f\nu_{n_k} \rangle \\
&\quad + \frac{1}{\mu_{n_k}} \langle u - u_{n_k}, \nu_{n_k} - u_{n_k} \rangle \\
&\geq \langle u - u_{n_k}, fu_{n_k} - f\nu_{n_k} \rangle + \frac{1}{\mu_{n_k}} \langle u - u_{n_k}, \nu_{n_k} - u_{n_k} \rangle . \qquad (3.53)
\end{aligned}$$

Recall that f is L-Lipschitzian, so, combining (3.46) and (3.53) we obtain

$$\begin{aligned}
||fu_{n_k} - f\nu_{n_k}|| &\leq L||u_{n_k} - \nu_{n_k}|| \\
&\to 0 \text{ as } n \to \infty. \qquad (3.54)
\end{aligned}$$

Also, since $\{u_{n_k}\}$ converges weakly to x^*, we obtain from (3.53) that

$$\lim_{k\to\infty} \langle u - u_{n_k}, \omega \rangle = \langle u - z^*, \omega \rangle \geq 0.$$

$\mathcal{A}+f$ being maximally monotone, ensures that $0 \in (\mathcal{A}+f)\, z^*$. Thus, $z^* \in \Omega$.

Next we need to establish that $\{x_n\}$ converges strongly to $z^* \in \Omega$. To achieve this entails showing that

$$\limsup_{n\to\infty} \langle z^*, \nu_n - z^* \rangle \leq 0.$$

Since $\{\nu_n\}$ is bounded, there exists a subsequence $\{\nu_{n_k}\}$ of $\{\nu_n\}$ such that $\{\nu_n\}$ converges weakly to a point $p^* \in \mathcal{H}$ such that

$$\limsup_{n\to\infty} \langle z^*, z^* - \nu_n \rangle = \lim_{k\to\infty} \langle z^*, z^* - \nu_{n_k} \rangle . \qquad (3.55)$$

And since p^* is a unique solution of MVIP in (1.1), then $p^* \in \Omega$ and it follows that

$$\limsup_{k\to\infty} \langle z^*, z^* - \nu_{n_k} \rangle = \langle z^*, z^* - p^* \rangle \leq 0. \qquad (3.56)$$

In Lemma 3.6, if we replace z^* with p^* we obtain

$$\begin{aligned} ||x_{n+1} - p^*||^2 &\leq ||\nu_n - p^*||^2 \\ &\leq (1-\alpha_n)^2||w_n - p^*||^2 + 2\alpha_n \langle p^*, \nu_n - p^* \rangle \\ &\leq (1-\alpha_n)||w_n - p^*||^2 + 2\alpha_n \langle p^*, \nu_n - p^* \rangle . \end{aligned} \tag{3.57}$$

Combining (3.33) and (3.57) we obtain

$$\begin{aligned} ||x_{n_k+1} - p^*||^2 &\leq (1-\alpha_{n_k}) \left[||x_n - p^*||^2 + \theta_{n_k}||x_{n_k} - x_{n_k-1}||M_1\right] \\ &\quad + 2\alpha_{n_k} \langle p^*, \nu_{n_k} - p^* \rangle \\ &= (1-\alpha_{n_k}) \left[||x_{n_k} - p^*||^2 + \alpha_{n_k}\left(\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}||M_1\right)\right] \\ &\quad + 2\alpha_{n_k}(1-\alpha_{n_k}) \langle p^*, \nu_{n_k} - p^* \rangle \\ &\leq (1-\alpha_{n_k})||x_{n_k} - p^*||^2 \\ &\quad + \alpha_{n_k}\left[2\langle p^*, \nu_{n_k} - p^* \rangle + \left(\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}||M_1\right)\right] . \end{aligned} \tag{3.58}$$

We know that

$$\rho_{n_k} := 2\langle p^*, \nu_{n_k} - p^* \rangle + \frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}||M_1.$$

Hence, by $\lim_{k\to\infty} \frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}|| = 0$, (3.56) and Lemma 2.9, we have that

$$\lim_{n\to\infty} ||x_n - z^*|| = 0.$$

Therefore, we conclude that the sequence $\{x_n\}$ converges strongly to a point $z^* \in \Omega$ as $n \to \infty$.

Next we need to show that $z^* \in F(S)$ in under to conclude that $x_n \to z^* \in \Gamma := \Omega \cap F(S)$.

From (3.36) we have

$$\begin{aligned} &\frac{2-\gamma}{\gamma}||y_{n_k} - \nu_{n_k}||^2 + \sigma_{n_k}(1-\sigma_{n_k})||y_{n_k} - S_{\lambda_{n_k}} y_{n_k}||^2 \\ &\quad \leq (1-\alpha_{n_k})||x_{n_k} - z^*||^2 - ||x_{n_k+1} - z^*||^2 \\ &\quad\quad + \alpha_{n_k}\left[2\langle z^*, \nu_{n_k} - z^* \rangle + \frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}||M_1\right] . \end{aligned}$$

It follows that

$$\begin{aligned}&\sigma_{n_k}\left(1-\sigma_{n_k}\right)||y_{n_k}-S_{\lambda_{n_k}}y_{n_k}||^2\\&\quad\leq(1-\alpha_{n_k})||x_{n_k}-z^*||^2-||x_{n_k+1}-z^*||^2\\&\qquad+\alpha_{n_k}\left[2\left\langle z^*,\nu_{n_k}-z^*\right\rangle+\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k}-x_{n_k-1}||M_1\right]\\&\quad\to0\text{ as }k\to\infty.\end{aligned}\tag{3.59}$$

Thus,

$$\lim_{k\to\infty}||y_{n_k}-S_{\lambda_{n_k}}y_{n_k}||=0.\tag{3.60}$$

But

$$\begin{aligned}||\nu_{n_k}-S_{\lambda_{n_k}}\nu_{n_k}||&=||\nu_{n_k}-S_{\lambda_{n_k}}y_{n_k}+S_{\lambda_{n_k}}y_{n_k}-S_{\lambda_{n_k}}\nu_{n_k}||\\&\leq||\nu_{n_k}-S_{\lambda_{n_k}}y_{n_k}||+||S_{\lambda_{n_k}}y_{n_k}-S_{\lambda_{n_k}}\nu_{n_k}||\\&\leq||\nu_{n_k}-S_{\lambda_{n_k}}y_{n_k}||+||y_{n_k}-\nu_{n_k}||\\&=||\nu_{n_k}-y_{n_k}+y_{n_k}-S_{\lambda_{n_k}}y_{n_k}||+||y_{n_k}-\nu_{n_k}||\\&\leq||y_{n_k}-S_{\lambda_{n_k}}y_{n_k}||+2||\nu_{n_k}-y_{n_k}||.\end{aligned}\tag{3.61}$$

Combining (3.38), (3.60) and (3.61) we have

$$\lim_{k\to\infty}||\nu_{n_k}-S_{\lambda_{n_k}}\nu_{n_k}||=0.\tag{3.62}$$

Since $\nu_{n_k}\rightharpoonup z^*$, from Lemma 2.7 and (3.62), we obtain $z^*=S_{\lambda_{n_k}}z^*$. Following the same line of argument given above, and demiclosedness of S and Lemma 2.6 (iii) we conclude that $z^*\in F(S)$. Therefore, $x_n\to z^*\in\Gamma$.

Case II: Suppose $\{\Upsilon_n\}$ is not monotonically decreasing for all $n\in\mathbb{N}$. Let $\upsilon:\mathbb{N}\to\mathbb{N}$ be a sequence that is defined for all $n\geq n_0$, for some large n_0 such that

$$\upsilon(n):=\max\{m\in\mathbb{N}:m\leq n,\ \Upsilon_m\leq\Upsilon_{m+1}\}.$$

Then, it is obvious from Lemma 2.10 that $\{\upsilon(n)\}$ is non-decreasing sequence with the property $\lim_{n\to\infty}\upsilon(n)=\infty$ and $\Upsilon_{\upsilon(n)}\leq\Upsilon_{\upsilon(n)+1}$ for all $n\geq n_0$. From (3.36) and by the condition on control sequence $\{\alpha_{\upsilon(n)}\}$, we have

$$\begin{aligned}&\frac{2-\gamma}{\gamma}||y_{\upsilon(n)}-\nu_{\upsilon(n)}||^2+\sigma_{\upsilon(n)}\left(1-\sigma_{\upsilon(n)}\right)||y_{\upsilon(n)}-S_{\lambda_{\upsilon(n)}}y_{\upsilon(n)}||^2\\&\quad\leq(1-\alpha_{\upsilon(n)})||x_{\upsilon(n)}-z^*||^2-||x_{\upsilon(n)+1}-z^*||^2\\&\qquad+\alpha_{\upsilon(n)}\left[2\left\langle z^*,\nu_{\upsilon(n)}-z^*\right\rangle+\frac{\theta_{\upsilon(n)}}{\alpha_{\upsilon(n)}}||x_{\upsilon(n)}-x_{\upsilon(n)-1}||M_1\right]\\&\quad\to0,\ \text{ as }n\to\infty.\end{aligned}\tag{3.63}$$

This implies,

$$\lim_{n\to\infty} ||y_{\upsilon(n)} - S_{\lambda_{\upsilon(n)}} y_{\upsilon(n)}|| = 0 \tag{3.64}$$

and

$$\lim_{n\to\infty} ||y_{\upsilon(n)} - \nu_{\upsilon(n)}|| = 0. \tag{3.65}$$

Similarly, by adopting the same line of argument as in Case I we obtain

$$\lim_{n\to\infty} ||\nu_{\upsilon(n)} - u_{\upsilon(n)}|| = \lim_{n\to\infty} ||y_{\upsilon(n)} - u_{\upsilon(n)}|| = \lim_{n\to\infty} ||x_{\upsilon(n)} - x_{\upsilon(n)+1}|| = 0.$$

By the same line of argument in Case I, since $\{\nu_{\upsilon(n)}\}$ is bounded, there exists a subsequence $\{\nu_{\upsilon(n)_k}\}$ of $\{\nu_{\upsilon(n)}\}$ such that $\nu_{\upsilon(n)_k} \rightharpoonup p^* \in \mathcal{H}$ and

$$\limsup_{k\to\infty} \left\langle z^*, z^* - \nu_{\upsilon(n)_k} \right\rangle = \langle z^*, z^* - p^* \rangle \leq 0. \tag{3.66}$$

Also, in the view of (3.58), for all $n \geq n_0$ we have

$$\begin{aligned} &||x_{\upsilon(n)_k+1} - z^*||^2 \\ &\quad \leq \left(1 - \alpha_{\upsilon(n)_k}\right) ||x_{\upsilon(n)_k} - z^*||^2 \\ &\qquad + \alpha_{\upsilon(n)_k} \left[2 \left\langle z^*, \nu_{\upsilon(n)_k} - z^* \right\rangle + \frac{\theta_{\upsilon(n)_k}}{\alpha_{\upsilon(n)_k}} ||x_{\upsilon(n)_k} - x_{\upsilon(n)_k-1}|| M_1 \right]. \end{aligned} \tag{3.67}$$

Since $\lim_{k\to\infty} \frac{\theta_{\upsilon(n)_k}}{\alpha_{\upsilon(n)_k}} ||x_{\upsilon(n)_k} - x_{\upsilon(n)_k-1}|| = 0$, by Lemma 2.9, (3.66) and (3.67) we have that $||x_{\upsilon(n)} - z^*||^2 \to 0$, as $n \to \infty$. And since we have

$$||x_{\upsilon(n)} - z^*||^2 \geq ||x_n - z^*||^2 \geq 0,$$

we conclude that $x_n \to z^*$ as $n \to \infty$. Therefore, $\{x_n\}$ converges strongly to $z^* \in \Omega$. Adopting the same approach as in Case I, from (3.61) we have

$$\begin{aligned} ||\nu_{\upsilon(n)_k} - S_{\lambda_{\upsilon(n)_k}} \nu_{\upsilon(n)_k}|| &\leq ||y_{\upsilon(n)_k} - S_{\lambda_{\upsilon(n)_k}} y_{\upsilon(n)_k}|| \\ &\quad + 2||\nu_{\upsilon(n)_k} - y_{\upsilon(n)_k}|| \\ &\to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.68}$$

Since $\nu_{\upsilon(n)_k} \rightharpoonup z^*$, then from Lemma 2.7 and (3.62) we obtain $z^* = S_{\lambda_{\upsilon(n)_k}} z^*$. From demiclosedness of S and Lemma 2.6 (iii) we conclude that $z^* \in F(S)$. Therefore, we have that $x_n \to z^* \in \Gamma$, completing the proof. □

4. Application

In this section we are going to use our main result, Theorem 3.7 to find approximate solution of some special monotone variational inclusion and fixed point problems.

4.1. **Application to convex minimization problem.** Let $F : \mathcal{H} \to \mathbb{R}$ be a convex and differentiable function. Let $G : \mathcal{H} \to \mathbb{R}$ be a proper convex and lower semi-continuous function. Let ∇F denotes the gradient of F. Recall that ∇F is L-Lipschitz continuous. Let ∂G denotes the subdifferential of mapping G. In the work of Rockafeller [38], we see that ∂G is maximal monotone mapping and

$$F(z^*) + G(z^*) = \min_{x\in\mathcal{H}} [F(x) + G(x)] \iff 0 \in [\nabla F(z^*) + \partial G(z^*)].$$

Now let us consider a class of convex minimization problem : Find $z^* \in \mathcal{H}$ such that

$$F(z^*) + G(z^*) = \min_{x\in\mathcal{H}} [F(x) + G(x)]. \tag{4.1}$$

By application of Fermat's rule, problem (4.1) can be equivalently expressed as: Find $z^* \in \mathcal{H}$ such that

$$0 \in \nabla F(z^*) + \partial G(z^*). \tag{4.2}$$

Suppose that the solution set of (1.1) is $\Omega \neq \emptyset$ and $\Gamma := \Omega \cap F(T)$. Then, if we set $\mathcal{A} := \partial G$, and $f := \nabla F$ in Theorem 3.7, we obtain result:

Corollary 4.1. Let $\mathcal{H}$ be a real Hilbert space with nonempty, closed and convex subsets C. Let $F : \mathcal{H} \to \mathbb{R}$ be a proper convex and lower semi-continuous mapping such that the gradient ∇F is L-Lipschitzian. Let $G : \mathcal{H} \to \mathbb{R}$ be a convex and differentiable function with subdifferential ∂G. Assume $\Gamma \neq \emptyset$ and the sequence $\{x_n\}$ generated for arbitrary $x_0, x_1 \in \mathcal{H}$ is defined by

$$\begin{cases} w_n = x_n + \theta_n (x_n - x_{n-1}), \\ \nu_n = (1-\alpha_n) w_n, \\ u_n = J_{\mu_n}^{\partial G} (\nu_n - \mu_n \nabla F(\nu_n)), \\ d(\nu_n, u_n) = \nu_n - u_n - \mu_n (\nabla F(\nu_n) - \nabla F(u_n)), \\ y_n = \nu_n - \gamma \beta_n d(\nu_n, u_n), \\ x_{n+1} = (1-\sigma_n) y_n + \sigma_n S_{\lambda_n} y_n, \end{cases} \tag{4.3}$$

where, $\beta_n := \frac{\phi(\nu_n,u_n)}{||d(\nu_n,u_n)||^2}$, $\phi(\nu_n, u_n) := \langle \nu_n - u_n, d(\nu_n, u_n)\rangle$, with μ_n satisfying the prediction stepsize conditions (3.1) and (3.2). Then, the sequence $\{x_n\}$ converges strongly to a point in Γ.

Remark 4.2. Compare with the results in Haung [26], Moudafi and Oliny [33], Alvarez and Attouch [5], and Zhang and Wang [46], our results improve and generalize them in the following ways.

(i) The problem of finding the solution of monotone variational inclusion problem when the underlying operator is inverse strongly monotone in Haung [26] was extended to finding the solution of MVIP with monotone cost operator, and fixed point constraint of pseudocontractive mapping embellished with relaxation using inertial proximal and contraction techniques under prediction stepsizes conditions.

(ii) The task of finding the solution of MVIP (1.1) in Moudafi and Oliny [33], Alvarez and Attouch [5] using proximal projection algorithm, which they only obtained weak convergence results were extended to finding element(s) in $\Omega \cap F(S)$ under norm convergence, where Ω is the solution set of MVIP, and $F(S)$ is the fixed point set of pseudocontractive mapping S.

(iii) Also, the problem of finding the solution of monotone variational inclusion problem and fixed point problem of nonexpansive mapping in Zhang and Wang [46] was extended to finding element(s) in $\Omega \cap F(S)$ under self-adjustment stepsize condition without prior knowledge of Lipschitz constant of the underlying operator. The class of operators (monotone and pseudocontractive operators) considered in our work are more general than the ones in [5, 26, 33, 46]. Specifically, the class of pseudocontractive mappings include several important classes of operators like nonexpansive mappings, quasi-nonexpansive mappings, k-strictly pseudocontractive and so on.

5. Conclusion

A modified relaxed inertial proximal and contraction algorithm for solving monotone variational inclusion, and fixed point problems when the underlying operator is pseudocontractive in real Hilbert space was introduced and studied. From our convergence analysis, we establish strong convergence of Algorithm 3.2 under the prediction stepsizes conditions considered by Cai et al. [12] and, Zhang and Wang [46] and other standard mild assumptions on the algorithm parameters. The proposed iterative method does not require prior knowledge of Lipschitz constant L of the cost operator f which generally enhances its efficiency and applicability. Finally, application of our established result to convex minimization problem was presented.

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