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SOLVING QUASIMONOTONE SPLIT VARIATIONAL INEQUALITY PROBLEM AND FIXED POINT PROBLEM IN HILBERT SPACES

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Abstract. In this paper, we introduce and study an iterative technique for solving quasimonotone split variational inequality problems and fixed point problem in the framework of real Hilbert spaces. Our proposed iterative technique is self adaptive, and easy to implement. We establish that the proposed iterative technique converges strongly to a minimum-norm solution of the problem and give some numerical illustrations in comparison with other methods in the literature to support our strong convergence result.

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1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, C a nonempty, closed and convex subset of H and $A : H \to H$ be a nonlinear operator. The classical variational inequality problem (VIP) is formulated as:

$$\text{Find } x \in C \text{ such that } \langle Ax, y - x \rangle \geq 0, \ \forall \ y \in C. \tag{1.1}$$

The notion of VIP was introduced independently by Stampacchia [25] and Fichera [11, 12] for modeling problems arising from mechanics and for solving Signorini problem. It is well known that many problems in economics, mathematical sciences, mathematical physics can be formulated as VIP. We denote the solution set of a VIP by VI(A,C). Due to the fruitful applications of the VIP, many researchers in this area have developed different iterative techniques to solve VIP (1.1). In particular, Goldsten in [13] introduced an iterative technique defined as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C(x_n - \lambda A x_n), \end{cases} \tag{1.2}$$

for all $n \in \mathbb{N}$, where $\lambda \in (0, \frac{2\alpha}{L^2})$, A is α-strongly monotone and L-Lipschitz continuous and P_C is a metric projection defined from H onto C. The author established that the iterative method (1.2) converges to the solution set of VIP (1.1). However, it was observed that if A is monotone and L-Lipschitz continuous, the iterative technique (1.2) may not converge to the solution set of VIP (1.1), see [15] and the reference therein for details. In addition, computing the value of λ may be very difficult or impossible.

In the light of these drawback, Korpelevich in [17] introduced and studied the extragradient method (EM) defined as follows:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = P_C(x_n - \lambda_n A y_n), \end{cases} \tag{1.3}$$

for all $n \geq 1$, where $\lambda_n \in (0, \frac{1}{L^2})$, A is monotone and L-Lipschitz continuous and P_C is a metric projection defined from H onto C. This method was implemented with a more relaxed cost operator, however, the computation of λ_n remains a challenge. More so, another drawback of this technique is that it requires two projections onto the feasible set C per iteration, which is costly when C does not have a simple structure. Since the inception of EM, many authors have introduced, modified and studied different EM in which the cost operator A is monotone and pseudomonotone. For example, He et al. [16], Apostol et al. [2], He et al. [15], Ceng et al. [4], Censor et al. [7], Nadezhkina and Takahashi [19] and many others.

In the light of providing an affirmative answer to the set back of the EM, Censor et al. [8] introduced and studied the subgradient extragradient method (SGEM) as follows:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ T_n = \{w \in H : \langle x_n - \lambda_n A x_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_C(x_n - \lambda_n A y_n), \end{cases} \tag{1.4}$$

where $\lambda_n \in (0, \frac{1}{L})$ for all $n \geq 1$, A is monotone and L-Lipschitz continuous and P_C is a metric projection defined from H onto C. They established that the iterative method (1.4) converges to the solution of VIP (1.1). However, computing the λ_n in the above iterative method is still a setback.

An interesting generalization of VIP (1.1) was introduced and studied by Censor et al. in [9]. They introduced and studied the following split variational inequality problem (SVIP) defined as:

$$\text{Find } x^* \in C \text{ that solves } \langle Ax^*, x - x^* \rangle \geq 0, \ \forall\ x \in C \tag{1.5}$$

and

$$y^* = Tx^* \in Q \text{ that solves } \langle By^*, y - y^* \rangle \geq 0, \ \forall\ y \in Q, \tag{1.6}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_1,\ B : H_2 \to H_2$ are two operators and $T : H_1 \to H_2$ is a bounded linear operator. The SVIP has wide applications in many fields such as phase retrieval, medical image reconstruction, signal processing, and radiation therapy treatment planning see ([3, 10, 5, 6]) and the references therein. It is easy to see that, the SVIP (1.5) -(1.6) is a combination of the classical VIP (1.1) and the well-known split feasibility problem (SFP) introduced and studied by Censor and Elfving in [6]: Find $x^* \in C$

$$Tx^* = y^* \in Q. \tag{1.7}$$

In an attempt for Censor et al. in [9] to approximate the solution of SVIP (1.5)-(1.6). They needed to convert the SVIP (1.5)-(1.6) into a constrained VIP (1.1) in a product space $H_1 \times H_2$. After which they applied the SGEM to solve the equivalent SVIP (1.5)-(1.6) problem. It was observed that solving a SVIP (1.5)-(1.6) in this manner, one will be faced with the problem of converting the new product subspaces into H_1 and H_2. In addition, it was observed that this method lack the splitting structure of the SVIP (1.5)-(1.6) and in the process lacks the capacity in which the iterative method can be applied to real life problem (see [9] and the references therein).

In the light of these challenges, many authors have proposed different iterative methods to solve the SVIP (1.5)-(1.6). For example, Tian and Jiang [26], introduced and studied the following iterative method.

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \gamma_n T^*(I - P_Q(I - \nu A))Tx_n), \\ t_n = P_C(y_n - \lambda_n B y_n), \\ x_{n+1} = P_C(y_n - \lambda_n B t_n), \end{cases} \tag{1.8}$$

for $n \in \mathbb{N}$, where $\gamma_n \subset [a, b]$, for some $a, b \in (0, \frac{1}{\|T\|^2}), \lambda_n \subset [c, d]$ for some $c, d \in (0, \frac{1}{L}), \nu \in (0, 2\alpha), T : H_1 \to H_2$ is a bounded linear operator, A is α-inversely strongly monotone and Lipschitz continuous, B is monotone and and Lipschitz continuous. They established that the proposed iterative method converges weakly to the solution set of SVIP (1.5)-(1.6). In addition, Pham et al. [20] introduced a Halpern type iterative technique for solving the SVIP (1.5)-(1.6) in real Hilbert spaces. They established that the iterative technique converges strongly to the solution set of the SVIP (1.5)-(1.6).

In this area of research approximating a solution of split variational inequality problems (SVIP) has been an interesting problem to consider. However, the iterative techniques that have been considered for this problem in the literature require that the underlying operators to be α-inversely strongly, or monotone, or pseudomonotone. It is well known that the underlying cost operators have crucial roles to play in real applications of these iterative methods. In the light of this introducing an iterative technique with weaker monotonicity condition on cost operators and better rate of convergence is highly sorted after.

Remark 1.1. We observe the following drawback in the iterative processes introduced and studied by different authors.

(1) In [21, 26, 27], this method requires three projections onto the feasible set C per iteration, which will be expensive if C is not simple.
(2) In [9, 20, 26], the implementation of their iterative technique depends on the knowledge of the bounded linear operator norm. This property is crucial because any iterative technique that depends on the operator norm require the value during the process of computation, which is a very difficult or sometimes impossible to get. Hence, this make it difficult to apply the iterative technique to real life problems.
(3) In [1, 9, 15, 16, 20, 26], the cost operators A and B are α-inversely strongly or monotone, or pseudomonotone.

The purpose of this paper is to introduce and study a modified split variational inequality problem and fixed point problem (SVIPFPP), which is a generalization of SVIP (1.5)-(1.6) in infinite dimensional real Hilbert spaces, in which the underlying cost operators are quaismonotone and Lipschitz continuous. The problem is defined as follows:

$$\text{Find } x^* \in C \text{ that solves } F(S) \cap \langle Ax^*, x - x^* \rangle \geq 0,\ \forall\, x \in C \tag{1.9}$$

and

$$y^* = Tx^* \in Q \text{ that solves } \langle By^*, y - y^* \rangle \geq 0,\ \forall\, y \in Q, \tag{1.10}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, $S : H_1 \to H_1$ is a quasinonexpansive mapping, $A : H_1 \to H_1,\ B : H_2 \to H_2$ are two quaismonotone operators and $T : H_1 \to H_2$ is a bounded linear operator. As such, we propose a two SGEM for solving the SVIPFPP with the following properties:

(1) It is easy to see that if $F(S) = I$(identity mappy), problem(1.9)-(1.10) becomes SVIP (1.5)-(1.6).
(2) In comparison with different iterative techniques for solving SVIP (1.5)-(1.6), iterative method is designed in such a way that the underlying cost operators are quasimonotone, Lipschitz continuous, and sequentially weakly continuous.
(3) Our methods do not require any product space reformulation of the classical SVIP (1.5)-(1.6), thus, overcoming the challenges faced by the authors in [9].
(4) Our proposed iterative method does not depend on the knowledge of the bounded linear operator $\|T\|$ unlike the following iterative methods in which knowledge of the bounded linear operator is relevant for their implementation (see [9, 20, 26]).
(5) The sequence generated by the proposed methods converges strongly to a minimum-norm solution of the SVIPFPP in real Hilbert spaces unlike [9, 20, 26].
(6) Our proposed iterative technique include inertial extrapolation steps. We emphasize that the inertial extrapolation step helps to improve the rate of convergence of an iterative method. The inertial steps remarkably increase the convergence speed of these algorithm when compared with others without extrapolation step of Algorithm 31 of [24] and Algorithm 1 of [20].

The rest of this paper is organized as follows: In Section 2, we recall some useful definitions and results that are relevant for our study. In Section 3, we present our proposed method. In Section 4, we establish strong convergence of our method and in Section 5, we present some numerical experiments to show the efficiency and applicability of our method in the framework of infinite dimensional Hilbert spaces. Lastly in Section 6, we give the conclusion of the paper.

2. Preliminaries

In this section, we begin by recalling some known and useful results which are needed in the sequel. Let H be a real Hilbert space. The set of fixed points of a nonlinear mapping $T : H \to H$ will be denoted by $F(T)$, that is

$$F(T) = \{x \in H : Tx = x\}.$$

We denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. For any $x, y \in H$ and $\alpha \in [0, 1]$, it is well known that

$$\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2, \tag{2.1}$$

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \tag{2.2}$$

$$\|x - y\|^2 \le \|x\|^2 + 2\langle y, x - y\rangle \tag{2.3}$$

and

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2. \tag{2.4}$$

Definition 2.1. Let $T : H \to H$ be an operator. Then T is called

(a) L-Lipschitz continuous if there exists $L > 0$ such that
$$\|Tx - Ty\| \le L\|x - y\|, \ \forall x, y \in H;$$

(b) nonexpansive if
$$\|Tx - Ty\| \le \|x - y\|, \ \forall x, y \in H;$$

(c) quasinonexpansive, if
$$\|Tx - y\| \le \|x - y\|, \ \forall x \in H, y \in F(T);$$

(d) monotone if
$$\langle Tx - Ty, x - y\rangle \ge 0, \ \forall x, y \in H;$$

(e) pseudomonotone if
$$\langle Tx, y - x\rangle \ge 0 \Rightarrow \langle Ty, y - x\rangle \ge 0, \ \forall x, y \in H;$$

(f) α-strongly monotone if there exists $\alpha > 0$, such that
$$\langle Tx - Ty, x - y\rangle \ge \alpha\|x - y\|^2, \ \forall\ x, y \in H;$$

(g) quasimonotone
$$\langle Tx, x - y\rangle > 0 \Rightarrow \langle Ty, x - y\rangle \ge 0, \ \forall\ x, y \in H;$$

(h) sequentially weakly continuous if for each sequence $\{x_n\}$, we obtain that $\{x_n\}$ converges weakly to x implies that Tx_n converges weakly to Tx.

Remark 2.2. It is well known that α-strongly monotone is monotone, motone is pseudomonotone, pseudomonotone is quasimonotone. However, the converses are not generally true.

Let C be a nonempty, closed and convex subset of H. For any $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\|, \ \forall y \in C.$$

The operator P_C is called the metric projection of H onto C. It is well-known that P_C is a nonexpansive mapping and that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.5}$$

for all $x, y \in H$. Furthermore, P_C is characterized by the property

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$$

and

$$\langle x - P_C x, y - P_C x\rangle \leq 0 \tag{2.6}$$

for all $x \in H$ and $y \in C$.

Lemma 2.3. ([14, 28]) *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : H \to H$ b a L-Lipschitz and quasimonotone operator. Suppose that $y \in C$ and for some $p \in C$, we have $\langle Ay, p - y\rangle \geq 0$. Then at least one of the following hold*

$$\langle Ap, p - y\rangle \geq 0 \ \text{or} \ \langle Ay, q - y\rangle \leq 0$$

for all $q \in C$.

Lemma 2.4. ([22]) *Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n, \ \ n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty}\{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then $\lim\limits_{k\to\infty} a_n = 0$.

3. Proposed algorithm

In this section, we present our proposed method for solving a quasimonotone variational inequality problem and a fixed point problem.

Assumption 3.1. Suppose that the following conditions A and B are hold:

Condition A:

(1) The feasible sets C and Q are nonempty, closed and convex subsets of the real Hilbert spaces H_1 and H_2, respectively.
(2) $\{S_n\}$ is a sequence of nonexpansive mapping on H_1.
(3) $A : H_2 \to H_2$ and $B : H_1 \to H_1$ are quasimonotone, sequentially weakly continuous and Lipschitz continuous with Lipschitz constant L_2 and L_1 respectively.
(4) $S : H_1 \to H_1$ is a quasinonexpansive operator and $f : H_1 \to H_1$ is a contraction mapping with coefficient $\tau \in (0, 1)$.
(5) $T : H_1 \to H_2$ is a bounded linear operator.
(6) The solution set

$$\Omega := \{x \in VI(B, C) \cap F(S) : Tx \in VI(A, Q)\} \neq \emptyset.$$

Condition B:

(1) $\alpha_n \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.
(2) $\{\eta_n\} \subset (0, \eta_0) \in (0, 1), \eta \in (1, \frac{13}{10}), \alpha \in (1, \frac{13}{10}), \nu, \delta \in (0, \frac{1}{2})$ such that $2 - \eta - \nu\eta > 0, 2 - \alpha - \delta\alpha > 0$, $\{\omega_n\} \subset (0, 1)$ witht $\alpha_n + \eta_n + \omega_n = 1$, $\lambda_0 > 0, \mu_0 > 0$, and choose the nonnegative real sequence $\{\Gamma_n\}$ and $\{\zeta_n\}$ such that $\sum_{n=1}^{\infty} \Gamma_n < \infty$ and $\sum_{n=1}^{\infty} \zeta_n < \infty$.

We present the following iterative algorithm.

Algorithm 3.2. Initialization Step:

Step 1: Choose $x_0, x_1 \in H_1$, given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\beta-1}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{n-1}{n+\beta-1}, & \text{otherwise}, \end{cases} \tag{3.1}$$

with $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n)$.

Step 2: Set

$$w_n = x_n + \theta_n(S_n x_n - S_n x_{n-1}).$$

Then, compute

$$y_n = P_Q(Tw_n - \lambda_n ATw_n), \tag{3.2}$$

$$z_n = P_{\Phi_n}(Tw_n - \eta\lambda_n Ay_n), \tag{3.3}$$

where

$$\Phi_n = \{x \in H_2 : \langle Tw_n - \lambda_n ATw_n - y_n, x - y_n\rangle \leq 0\}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\nu(\|Tw_n - y_n\|^2 + \|y_n - z_n\|^2)}{2\langle ATw_n - Ay_n, y_n - z_n\rangle}, \lambda_n + \zeta_n\right\}, & \text{if } \langle ATw_n - Ay_n, y_n - z_n\rangle > 0, \\ \lambda_n + \zeta_n, & \text{otherwise.} \end{cases} \tag{3.4}$$

Step 3: Compute

$$v_n = w_n + \gamma_n T^*(z_n - Tw_n), \tag{3.5}$$

$$u_n = P_C(v_n - \nu_n Bv_n), \tag{3.6}$$

$$t_n = P_{\psi_n}(v_n - \alpha\nu_n Bu_n), \tag{3.7}$$

where γ_n is chosen such that for small enough $\epsilon > 0$, $\gamma_n \in \left[\epsilon, \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right]$ if $Tw_n \neq z_n$, otherwise $\gamma_n = \gamma$, $\psi_n = \{x \in H_1 : \langle v_n - \nu_n Bv_n - u_n, x - u_n\rangle \leq 0\}$ and

$$\mu_{n+1} = \begin{cases} \min\left\{\frac{\delta(\|v_n - u_n\|^2 + \|u_n - t_n\|^2)}{2\langle Bv_n - Bu_n, u_n - t_n\rangle}\ \mu_n + \Gamma_n\right\}, & \text{if } \langle Bv_n - Bu_n, u_n - t_n\rangle > 0, \\ \mu_n + \Gamma_n, & \text{otherwise.} \end{cases} \tag{3.8}$$

Step 4: Compute

$$x_{n+1} = \alpha_n f(x_n) + \omega_n x_n + \eta_n St_n. \tag{3.9}$$

4. Convergence analysis

Lemma 4.1. *The step-sizes γ_n, μ_{n+1} and λ_{n+1} in Algorithm 3.2 are well defined.*

Proof. The proof that λ_{n+1}, μ_{n+1} and γ_n are well define follows similar approach as in Lemma 3.1 of [18] and Lemma 3.6 of [19], thus we omit it. □

Lemma 4.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1. Then, $\{x_n\}$ is bounded.*

Proof. Let $p \in \Omega$. Then $Tp \in VI(A, Q) \subset Q$. Since $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, there exists $N_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq N_1$, for all $n \in \mathbb{N}$. Then using

Algorithm 3.2, we have

$$\begin{aligned}\|w_n - p\| &= \|x_n + \theta_n(S_n x_n - S_n x_{n-1}) - p\| \\ &\le \|x_n - p\| + \theta_n\|S_n x_n - S_n x_{n-1}\| \\ &\le \|x_n - p\| + \alpha_n \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &\le \|x_n - p\| + \alpha_n N_1. \end{aligned} \tag{4.1}$$

Also, using Algorithm 3.2, we have

$$\begin{aligned}\|z_n - Tp\|^2 &= \|P_{Q_n}(Tw_n - \eta ATw_n) - Tp\|^2 \\ &\le \|Tw_n - \eta\lambda_n Ay_n - Tp\|^2 - \|Tw_n - \eta\lambda_n Ay_n - z_n\|^2 \\ &= \|Tw_n - Tp\|^2 + (\eta\lambda_n)^2\|Ay_n\|^2 - 2\langle Tw_n - Tp, \eta\lambda_n Ay_n\rangle \\ &\quad - \|Tw_n - z_n\|^2 - (\eta\lambda_n)^2\|Ay_n\|^2 + 2\langle Tw_n - z_n, \eta\lambda_n Ay_n\rangle \\ &= \|Tw_n - Tp\|^2 - \|Tw_n - z_n\|^2 - 2\langle \eta\lambda_n Ay_n, z_n - Tp\rangle \\ &= \|Tw_n - Tp\|^2 - \|Tw_n - z_n\|^2 - 2\langle \eta\lambda_n Ay_n, z_n - y_n\rangle \\ &\quad - 2\langle \eta\lambda_n Ay_n, y_n - Tp\rangle. \end{aligned} \tag{4.2}$$

Since $Tp \in VI(Q, A)$ and $y_n \in Q$, we have $\langle ATp, y_n - Tp\rangle \ge 0$ and using Lemma 2.3, we obtain $\langle Ay_n, y_n - Tp\rangle \ge 0$. Thus, (4.2) becomes

$$\|z_n - Tp\|^2 \le \|Tw_n - Tp\|^2 - \|Tw_n - z_n\|^2 - 2\langle \eta\lambda_n Ay_n, z_n - y_n\rangle. \tag{4.3}$$

Now, observe that

$$\begin{aligned}-\|Tw_n - z_n\|^2 &= -\|Tw_n - y_n + y_n - z_n\|^2 \\ &= -\|Tw_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\langle Tw_n - y_n, z_n - y_n\rangle \\ &= -\|Tw_n - y_n\|^2 - \|y_n - z_n\|^2 \\ &\quad + 2\langle Tw_n - y_n - \lambda_n ATw_n + \lambda_n ATw_n - \lambda_n Ay_n + \lambda_n Ay_n, z_n - y_n\rangle \\ &= -\|Tw_n - y_n\|^2 - \|y_n - z_n\|^2 \\ &\quad + \langle Tw_n - \lambda_n ATw_n - y_n, z_n - y_n\rangle \\ &\quad + \langle \lambda_n ATw_n - \lambda_n Ay_n, z_n - y_n\rangle + \langle \lambda_n Ay_n, z_n - y_n\rangle. \end{aligned} \tag{4.4}$$

Since $z_n \in Q \subset H_2$, we have $\langle Tw_n - \lambda_n ATw_n - y_n, z_n - y_n\rangle \le 0$ and using the step-size, we have (4.4) becomes

$$\begin{aligned}-\|Tw_n - z_n\|^2 &\le -(1 - \frac{\lambda_n \nu}{\lambda_{n+1}})\|Tw_n - y_n\|^2 - (1 - \frac{\lambda_n \nu}{\lambda_{n+1}})\|y_n - z_n\|^2 \\ &\quad + 2\langle \lambda_n Ay_n, z_n - y_n\rangle, \end{aligned} \tag{4.5}$$

this implies that

$$-2\langle \lambda_n A y_n, z_n - y_n\rangle \le -(1-\frac{\lambda_n \nu}{\lambda_{n+1}})\|Tw_n - y_n\|^2 - (1-\frac{\lambda_n \nu}{\lambda_{n+1}})\|y_n - z_n\|^2 + \|Tw_n - z_n\|^2. \tag{4.6}$$

Hence

$$-2\langle \eta\lambda_n A y_n, z_n - y_n\rangle \le -\eta(1-\frac{\lambda_n \nu}{\lambda_{n+1}})\|Tw_n - y_n\|^2 - \eta(1-\frac{\lambda_n \nu}{\lambda_{n+1}})\|y_n - z_n\|^2 + \eta\|Tw_n - z_n\|^2. \tag{4.7}$$

Substituting (4.7) into (4.3), we have

$$\|z_n - Tp\|^2 \le \|Tw_n - Tp\|^2 - \eta(1-\frac{\lambda_n \nu}{\lambda_{n+1}})\|Tw_n - y_n\|^2 - \eta(1-\frac{\lambda_n \nu}{\lambda_{n+1}})\|y_n - z_n\|^2 - (1-\eta)\|Tw_n - z_n\|^2. \tag{4.8}$$

Since

$$\|Tw_n - z_n\|^2 \le 2\|Tw_n - y_n\|^2 + 2\|z_n - y_n\|^2 \text{ and } \quad -(1-\eta) > 0,$$

we have

$$-(1-\eta)\|Tw_n - z_n\|^2 \le -2(1-\eta)\|Tw_n - y_n\|^2 - 2(1-\eta)\|z_n - y_n\|^2,$$

thus, we have

$$\begin{aligned}\|z_n - Tp\|^2 &\le \|Tw_n - Tp\|^2 - \eta(1-\frac{\lambda_n \nu}{\lambda_{n+1}})\|Tw_n - y_n\|^2 \\ &\quad - \eta(1-\frac{\lambda_n \nu}{\lambda_{n+1}})\|y_n - z_n\|^2 - 2(1-\eta)\|Tw_n - y_n\|^2 \\ &\quad - 2(1-\eta)\|z_n - y_n\|^2 \\ &= \|Tw_n - Tp\|^2 - (2-\eta-\frac{\nu\lambda_n\eta}{\lambda_{n+1}})\|Tw_n - y_n\|^2 \\ &\quad - (2-\eta-\frac{\nu\lambda_n\eta}{\lambda_{n+1}})\|z_n - y_n\|^2.\end{aligned} \tag{4.9}$$

Considering the limit $(2-\eta-\frac{\nu\lambda_n\eta}{\lambda_{n+1}}) = 2-\eta-\nu\eta > 0$. Hence, there exists n_0 such that for all $n \ge n_0$, we have $2-\eta-\frac{\nu\lambda_n\eta}{\lambda_{n+1}} \ge 0$. Thus, it follows that, for all $n \geq n_0$, we obtain

$$\|z_n - Tp\|^2 \leq \|Tw_n - Tp\|^2 \tag{4.10}$$

and this implies that

$$\|z_n - Tp\| \leq \|Tw_n - Tp\|. \tag{4.11}$$

Furthermore, using Algorithm 3.2 with step-size γ_n and (4.11), we have

$$\begin{aligned}
\|v_n - p\|^2 &= \|w_n + \gamma_n T^*(z_n - Tw_n) - p\| \\
&= \|w_n - p\|^2 + \gamma_n^2 \|T^*(z_n - Tw_n)\|^2 \\
&\quad + 2\gamma_n \langle w_n - p, T^*(z_n - Tw_n)\rangle \\
&= \|w_n - p\|^2 + \gamma_n^2 \|T^*(z_n - Tw_n)\|^2 \\
&\quad + 2\gamma_n \langle Tw_n - Tp, z_n - Tw_n\rangle \\
&= \|w_n - p\|^2 + \gamma_n^2 \|T^*(z_n - Tw_n)\|^2 \\
&\quad + \gamma_n \|z_n - Tp\|^2 - \gamma_n \|Tw_n - Tp\|^2 - \gamma_n \|z_n - Tw_n\|^2 \\
&\leq \|w_n - p\|^2 + \gamma_n^2 \|T^*(z_n - Tw_n)\|^2 \\
&\quad + \gamma_n \|Tw_n - Tp\|^2 - \gamma_n \|Tw_n - Tp\|^2 - \gamma_n \|z_n - Tw_n\|^2 \\
&\leq \|w_n - p\|^2 + \gamma_n^2 \|T^*(z_n - Tw_n)\|^2 \\
&\quad - \gamma_n(\gamma_n + \epsilon) \|T^*(z_n - Tw_n)\|^2 \\
&= \|w_n - p\|^2 - \gamma_n \epsilon \|T^*(z_n - Tw_n)\|^2 \\
&\leq \|w_n - p\|^2,
\end{aligned} \tag{4.12}$$

which implies that

$$\|v_n - p\| \leq \|w_n - p\|. \tag{4.13}$$

Using a similar approach as in (4.9), we obtain

$$\begin{aligned}
\|t_n - p\|^2 \leq \|v_n - p\|^2 &- (2 - \alpha - \frac{\delta \mu_n \alpha}{\mu_{n+1}}) \|v_n - u_n\|^2 \\
&- (2 - \alpha - \frac{\delta \mu_n \alpha}{\mu_{n+1}}) \|t_n - u_n\|^2,
\end{aligned} \tag{4.14}$$

which implies that

$$\|t_n - p\| \leq \|v_n - p\|. \tag{4.15}$$

Finally, using Algorithm 3.2, (4.15), (4.13) and (4.1) we have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n f(x_n)+\omega_n x_n+\eta_n St_n - p\| \\
&= \left\|\alpha_n(f(x_n)-p)+\omega_n(x_n-p)+\eta_n\Big(St_n-p\Big)\right\| \\
&\le \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\| \\
&\quad +(1-\alpha_n-\omega_n)\|St_n-p\| \\
&\le \alpha_n\tau\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n-\omega_n)\|t_n-p\| \\
&\le \alpha_n\tau\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n-\omega_n)\|v_n-p\| \\
&\le \alpha_n\tau\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n-\omega_n)\|w_n-p\| \\
&\le \alpha_n\tau\|x_n-p\|+\alpha_n\|f(p)-p\| \\
&\quad +(1-\alpha_n-\omega_n)[\|x_n-p\|+\alpha_n N_1] \\
&\le \alpha_n\tau\|x_n-p\|+\alpha_n\|f(p)-p\| \\
&\quad +(1-\alpha_n)\|x_n-p\|+\alpha_n N_1 \\
&= (1-\alpha_n(1-\tau))\|x_n-p\| \\
&\quad +\alpha_n(1-k)\left[\frac{N_1+\|f(p)-p\|}{(1-\tau)}\right] \\
&\le \max\left\{\|x_n-p\|,\frac{N_1+\|f(p)-p\|}{(1-\tau)}\right\}. \qquad (4.16)
\end{aligned}$$

It follows by induction

$$\|x_n-p\|\le \max\left\{\|x_0-p\|,\frac{N_1+\|f(p)-p\|}{(1-\tau)}\right\}. \qquad (4.17)$$

Hence $\{x_n\}$ is bounded. □

Lemma 4.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1 and suppose that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to $x^*\in H_1$ and*

$$\lim_{k\to\infty}\|w_{n_k}-v_{n_k}\|=0=\lim_{k\to\infty}\|t_{n_k}-v_{n_k}\|.$$

Then $x^*\in\Omega$.

Proof. Let $p\in\Omega$. We suppose that $z_{n_k}\neq Tw_{n_k}$. It is easy to see from (4.12) that

$$\begin{aligned}
\|v_{n_k}-p\|^2 &\le \|w_{n_k}-p\|^2-\gamma_{n_k}\epsilon\|T^*(z_{n_k}-Tw_{n_k})\|^2 \\
&\le \|w_{n_k}-p\|^2-\epsilon^2\|T^*(z_{n_k}-Tw_{n_k})\|, \qquad (4.18)
\end{aligned}$$

which implies that

$$\begin{aligned}\epsilon^2\|T^*(z_{n_k}-Tw_{n_k})\|^2 &\le \|w_{n_k}-p\|^2-\|v_{n_k}-p\|^2\\ &\le (\|w_{n_k}-v_{n_k}\|+\|v_{n_k}-p\|)^2-\|v_{n_k}-p\|^2\\ &\le \|w_{n_k}-v_{n_k}\|^2+2\|w_{n_k}-v_{n_k}\|\|v_{n_k}-p\|\\ &\quad+\|v_{n_k}-p\|^2-\|v_{n_k}-p\|^2\\ &= \|w_{n_k}-v_{n_k}\|^2+2\|w_{n_k}-v_{n_k}\|\|v_{n_k}-p\|. \end{aligned} \tag{4.19}$$

By using the hypothesis, we have

$$\lim_{k\to\infty}\|T^*(z_{n_k}-Tw_{n_k})\|=0. \tag{4.20}$$

Thus

$$\|v_{n_k}-p\|^2\le\|w_{n_k}-p\|^2+\gamma_n^2\|T^*(z_{n_k}-Tw_{n_k})\|^2-\gamma_n\|z_{n_k}-Tw_{n_k}\|^2, \tag{4.21}$$

and this implies that

$$\begin{aligned}\gamma_{n_k}\|z_{n_k}-Tw_{n_k}\|^2 &\le \|w_{n_k}-p\|^2-\|v_{n_k}-p\|^2+\gamma_{n_k}^2\|T^*(z_{n_k}-Tw_{n_k})\|^2\\ &\le \|w_{n_k}-v_{n_k}\|^2+2\|w_{n_k}-v_{n_k}\|\|v_{n_k}-p\|\\ &\quad+\gamma_{n_k}^2\|T^*(z_{n_k}-Tw_{n_k})\|^2. \end{aligned} \tag{4.22}$$

From our hypothesis, we have

$$\lim_{k\to\infty}\|z_{n_k}-Tw_{n_k}\|=0. \tag{4.23}$$

From (4.9), we have

$$\begin{aligned}\|z_{n_k}-Tp\|^2 \le \|Tw_{n_k}-Tp\|^2 &-(2-\eta-\frac{\nu\lambda_{n_k}\eta}{\lambda_{n_k+1}})\|Tw_{n_k}-y_{n_k}\|^2\\ &-(2-\eta-\frac{\nu\lambda_{n_k}\eta}{\lambda_{n_k+1}})\|z_{n_k}-y_{n_k}\|^2. \end{aligned} \tag{4.24}$$

Now, observe that

$$\begin{aligned}\|z_{n_k}-Tp\|^2 &= \|z_{n_k}-Tw_{n_k}+Tw_{n_k}-Tp\|^2\\ &= \|Tw_{n_k}-Tp-(Tw_{n_k}-z_{n_k})\|^2\\ &= \|Tw_{n_k}-Tp\|^2-2\langle Tw_{n_k}-Tp, Tw_{n_k}-z_n\rangle+\|Tw_{n_k}-z_{n_k}\|^2\\ &\ge \|Tw_{n_k}-Tp\|^2-2\|T(w_{n_k}-p)\|\|Tw_{n_k}-z_{n_k}\|+\|Tw_n-z_n\|^2\\ &\ge \|Tw_{n_k}-Tp\|^2-2\|T\|\|w_{n_k}-p\|\|Tw_{n_k}-z_{n_k}\|\\ &\quad+\|Tw_{n_k}-z_{n_k}\|^2 \end{aligned} \tag{4.25}$$

and this implies that

$$-\|z_{n_k} - Tp\|^2 \leq -\|Tw_{n_k} - Tp\|^2 + 2\|T\|\|w_{n_k} - p\|\|Tw_{n_k} - z_{n_k}\| - \|Tw_{n_k} - z_{n_k}\|^2. \tag{4.26}$$

Adding (4.24) and (4.26), we have

$$(2 - \eta - \frac{\nu\lambda_{n_k}\eta}{\lambda_{n_k+1}})\|Tw_{n_k} - y_{n_k}\|^2 + (2 - \eta - \frac{\nu\lambda_{n_k}\eta}{\lambda_{n_k+1}})\|z_{n_k} - y_{n_k}\|^2 \leq 2\|T\|\|w_{n_k} - p\|\|Tw_{n_k} - z_{n_k}\| - \|Tw_{n_k} - z_{n_k}\|^2. \tag{4.27}$$

By using (4.23), we have

$$\lim_{k\to\infty} \|Tw_{n_k} - y_{n_k}\| = 0 = \lim_{k\to\infty} \|z_{n_k} - y_{n_k}\|. \tag{4.28}$$

Since $y_{n_k} = P_Q(Tw_{n_k} - \lambda_{n_k} ATw_{n_k})$, from the characteristic of the metric projection, we have

$$\langle Tw_{n_k} - \lambda_{n_k} ATw_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0, \ \forall\ x \in Q \tag{4.29}$$

and this implies that

$$\langle Tw_{n_k} - y_{n_k}, x - y_{n_k}\rangle - \lambda_{n_k}\langle ATw_{n_k}, x - y_{n_k}\rangle \leq 0. \tag{4.30}$$

Hence we obtain that

$$\begin{aligned}\langle Tw_{n_k} - y_{n_k}, x - y_{n_k}\rangle &\leq \lambda_{n_k}\langle ATw_{n_k}, x - y_{n_k}\rangle \\ &= \lambda_{n_k}\langle ATw_{n_k}, Tw_{n_k} - y_{n_k}\rangle \\ &\quad + \lambda_{n_k}\langle ATw_{n_k}, x - Tw_{n_k}\rangle.\end{aligned} \tag{4.31}$$

Since $\lambda_{n_k} > 0$, we have

$$\frac{1}{\lambda_{n_k}}\langle Tw_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle ATw_{n_k}, y_{n_k} - Tw_{n_k}\rangle \leq \langle ATw_{n_k}, x - Tw_{n_k}\rangle. \tag{4.32}$$

Using (4.28), we have

$$0 \leq \liminf_{k\to\infty}\langle ATw_{n_k}, x - Tw_{n_k}\rangle \leq \limsup_{k\to\infty}\langle ATw_{n_k}, x - Tw_{n_k}\rangle. \tag{4.33}$$

Now, observe that

$$\begin{aligned}\langle Ay_{n_k}, x - y_{n_k}\rangle &= \langle Ay_{n_k}, x - Tw_{n_k}\rangle + \langle Ay_{n_k}, Tw_{n_k} - y_{n_k}\rangle \\ &= \langle Ay_{n_k} - ATw_{n_k}, x - Tw_{n_k}\rangle + \langle ATw_{n_k}, x - Tw_{n_k}\rangle \\ &\quad + \langle Ay_{n_k}, Tw_{n_k} - y_{n_k}\rangle.\end{aligned} \tag{4.34}$$

Since A is Lipschitz continuous on H_2,

$$\lim_{k\to\infty} \|ATw_{n_k} - Ay_{n_k}\| \leq L_2 \lim_{k\to\infty} \|Tw_{n_k} - y_{n_k}\| = 0. \tag{4.35}$$

Combining (4.33), (4.34) and (4.35), we have

$$0 \leq \liminf_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \leq \limsup_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle. \tag{4.36}$$

In what follows, we now establish that $Tx^* \in VI(A, Q)$. To start with, we consider the case in which $\limsup_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle > 0$ for all $x \in Q$. Then there exists a subsequence $\{y_{n_{k_m}}\}$ of sequence $\{y_{n_k}\}$ such that

$$\limsup_{m\to\infty}\langle Ay_{n_{k_m}}, x - y_{n_{k_m}}\rangle > 0$$

for all $x \in Q$. It follows that we can find N_0 such that

$$\langle Ay_{n_{k_m}}, x - y_{n_{k_m}}\rangle > 0,\ \forall m > N_0. \tag{4.37}$$

Since A is quasimonotone, it follows that

$$\langle Ax, x - y_{n_{k_m}}\rangle > 0,\ \forall m > N_0. \tag{4.38}$$

Now observe that

$$\begin{aligned}\|w_{n_{k_m}} - x_{n_{k_m}}\| &= \alpha_{n_{k_m}}\frac{\theta_{n_{k_m}}}{\alpha_{n_{k_m}}}\|S_{n_{k_m}}x_{n_{k_m}} - S_{n_{k_m}}x_{n_{k_m}-1}\| \\ &\to 0,\ \text{as } m \to \infty.\end{aligned} \tag{4.39}$$

Since, the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is weakly convergent to a point $x^* \in H_1$. Again, since T is a bounded linear operator, we obtain that $\{Tw_{n_k}\}$ converges weakly to Tx^*. Hence, using the fact that $\lim_{n\to\infty}\|Tw_{n_{k_m}} - y_{n_{k_m}}\| = 0$, we have that $\{y_{n_{k_m}}\}$ also converges to Tx^*.

Now passing the limit as $m \to \infty$ in (4.38), we have

$$\lim_{m\to\infty}\langle Ax, x - y_{n_{k_m}}\rangle = \langle Ax, x - Tx^*\rangle > 0. \tag{4.40}$$

Hence, $Tx^* \in VI(A, Q)$.

Secondly, we consider the case in which $\limsup_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle = 0$ for $x \in Q$. Let $\{\delta_k\}$ be a non-increasing positive sequence defined by

$$\delta_k = |\langle Ay_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}. \tag{4.41}$$

Then, we obtain

$$\lim_{k\to\infty}\delta_k = \lim_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle + \lim_{k\to\infty}\frac{1}{k+1} = 0. \tag{4.42}$$

This implies by (4.41), that

$$\langle Ay_{n_k}, x - y_{n_k}\rangle + \delta_k > 0 \tag{4.43}$$

for each $k \geq 1$, since $\{y_{n_k}\} \subset Q$, it implies that $\{Ay_{n_k}\}$ is strictly non-zero and $\liminf_{k\to\infty} \|Ay_{n_k}\| = N_0 > 0$. We therefore deduce that

$$\|Ay_{n_k}\| > \frac{N_0}{2}. \tag{4.44}$$

In addition, let $\{\epsilon_{n_k}\}$ be a sequence defined by $\epsilon_{n_k} = \frac{Ay_{n_k}}{\|Ay_{n_k}\|^2}$. It implies that

$$\langle Ay_{n_k}, \epsilon_{n_k}\rangle = 1. \tag{4.45}$$

Combining (4.43) and (4.45), we have

$$\langle Ay_{n_k}, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle > 0. \tag{4.46}$$

By quasimonotonicity of the operator A on H_2, we get that

$$\langle A(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle \geq 0. \tag{4.47}$$

Now, observe that

$$\begin{aligned}\langle Ax, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle &= \langle Ax - A(x + \delta_k\epsilon_{n_k}) \\ &\quad + A(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle \\ &= \langle Ax - A(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle \\ &\quad + \langle A(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle.\end{aligned} \tag{4.48}$$

Combining (4.47), (4.48) and applying the well-known Cauchy-Schwarz inequality, we have

$$\begin{aligned}\langle Ax, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle &\geq \langle Ax - A(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_n\rangle \\ &\geq -\|Ax - A(x + \delta_k\epsilon_{n_k})\|\|x + \delta_k\epsilon_{n_k} - y_{n_k}\|.\end{aligned} \tag{4.49}$$

Since A is Lipschitz continuous, we have

$$\langle Ax, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle + L_2\|\delta_k\epsilon_{n_k}\|\|x + \delta_k\epsilon_{n_k} - y_{n_k}\| \geq 0. \tag{4.50}$$

Combining (4.44) and (4.50) and using the definition of ϵ_{n_k}, we have

$$\langle Ax, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle + \frac{2L_2}{N_0}\delta_k\|x + \delta_k\epsilon_{n_k} - y_{n_k}\| \geq 0. \tag{4.51}$$

Since, the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is weakly convergent to a point $x^* \in H_1$, and T is a bounded linear operator, we obtain that $\{Tw_{n_k}\}$ converges to Tx^*. Hence, using the fact that $\lim_{n\to\infty} \|Tw_{n_k} - y_{n_k}\| = 0$, we have that $\{y_{n_k}\}$ also converges to Tx^*. Taking limit as $k \to \infty$, since $\delta_k \to 0$, we have

$$\lim_{k\to\infty}\left[\langle Ax, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle + \frac{2L_2}{N_0}\delta_k\|x + \delta_k\epsilon_{n_k} - y_{n_k}\|\right] = \langle Ax, x - Tx^*\rangle > 0. \tag{4.52}$$

Hence $Tx^* \in VI(A, Q)$.

Using a similar approach, we have $x^* \in VI(B, C)$. Hence, we conclude that $x^* \in \Omega$. □

Theorem 4.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1. Then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = P_\Omega f(p)$.*

Proof. Let $p \in \Omega$. Using Algorithm 3.2, we have

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(S_n x_n - S_n x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + 2\theta_n\langle S_n x_n - p, S_n x_n - S_n x_{n-1}\rangle \\
&\quad + \theta_n^2\|S_n x_n - S_n x_{n-1}\|^2 \\
&\leq \|x_n - p\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - p\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \theta_n\|x_n - x_{n-1}\|] \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| \\
&\quad + \beta_n\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\|] \\
&\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \alpha_n N_1] \\
&\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2. \qquad (4.53)
\end{aligned}$$

In addition, using Algorithm 3.2 and (4.53), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + \omega_n x_n + \eta_n St_n - p\|^2 \\
&= \|\alpha_n f(x_n) + \omega_n x_n + \eta_n St_n - p\|^2 \\
&\leq \|\omega_n(x_n - p) + \eta_n(St_n - p)\|^2 + 2\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle \\
&\leq \omega_n^2\|x_n - p\|^2 + \eta_n^2\|St_n - p\|^2 + 2\eta_n\omega_n\|x_n - p\|\|St_n - p\| \\
&\quad + 2\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle \\
&\leq \omega_n^2\|x_n - p\|^2 + \eta_n^2\|t_n - p\|^2 + \omega_n\eta_n(\|x_n - p\|^2 + \|t_n - p\|^2) \\
&\quad + 2\alpha_n\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\leq \omega_n(\omega_n + \eta_n)\|x_n - p\|^2 + \eta_n(\omega_n + \eta_n)\|t_n - p\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\leq \omega_n(\omega_n + \eta_n)\|x_n - p\|^2 + \eta_n(\omega_n + \eta_n)\|v_n - p\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\leq \omega_n(\omega_n + \eta_n)\|x_n - p\|^2 + \eta_n(\omega_n + \eta_n)\|w_n - p\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \omega_n(\omega_n+\eta_n)\|x_n-p\|^2 \\
&\quad + \eta_n(\omega_n+\eta_n)\|x_n-p\|^2 + \eta_n(\omega_n+\eta_n)\theta_n\|x_n-x_{n-1}\|N_2 \\
&\quad + 2\alpha_n\tau\|x_n-p\|\|x_{n+1}-p\| + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\leq (\omega_n+\eta_n)^2\|x_n-p\|^2 + \eta_n(\omega_n+\eta_n)\theta_n\|x_n-x_{n-1}\|N_2 \\
&\quad + \alpha_n\tau\|x_n-p\| + \alpha_n\tau\|x_{n+1}-p\| + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\leq (1-2\alpha_n+\alpha_n\tau)\|x_n-p\|^2 + \alpha_n^2\|x_n-p\|^2 \\
&\quad + \eta_n(\omega_n+\eta_n)\theta_n\|x_n-x_{n-1}\|N_2 \\
&\quad + \alpha_n\tau\|x_{n+1}-p\| + 2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle, \qquad (4.54)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \Big(1-\frac{2\alpha_n(1-\tau)}{1-\alpha_n\tau}\Big)\|x_n-p\|^2 \\
&\qquad + \frac{2\alpha_n(1-\tau)}{1-\alpha_n\tau}\Bigg[\frac{\eta_n(1-\alpha_n)\theta_n}{2\alpha_n(1-\tau)}\|x_n-x_{n-1}\|N_2 \\
&\qquad + \frac{\alpha_n N_3}{2(1-\tau)} + \frac{1}{(1-\tau)}\langle f(p)-p, x_{n+1}-p\rangle\Bigg] \\
&= \Big(1-\frac{2\alpha_n(1-\tau)}{1-\alpha_n\tau}\Big)\|x_n-p\|^2 + \frac{2\alpha_n(1-\tau)}{1-\alpha_n\tau}\Psi_n, \qquad (4.55)
\end{aligned}
$$

where

$$
N_3 = \sup_{n\in\mathbb{N}}\{\|x_n-p\|^2 : n \geq \mathbb{N}\}
$$

and

$$
\begin{aligned}
\Psi_n &= \frac{\eta_n(1-\alpha_n)}{2(1-\tau)}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|N_2 \\
&\quad + \frac{\alpha_n N_3}{2(1-\tau)} + \frac{1}{(1-\tau)}\langle f(p)-p, x_{n+1}-p\rangle.
\end{aligned}
$$

According to Lemma 2.4, to conclude our proof, it is sufficient to establish that $\limsup_{k\to\infty}\Psi_{n_k} \leq 0$ for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ satisfying the condition:

$$
\liminf_{k\to\infty}\{\|x_{n_k+1}-p\| - \|x_{n_k}-p\|\} \geq 0. \qquad (4.56)
$$

To establish that $\limsup_{k\to\infty}\Psi_n \leq 0$, we suppose that for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ such that (4.56) holds. Then,

$$
\begin{aligned}
&\liminf_{k\to\infty}\{\|x_{n_k+1}-p\|^2 - \|x_{n_k}-p\|^2\} \\
&\quad = \liminf_{k\to\infty}\{(\|x_{n_k+1}-p\| - \|x_{n_k}-p\|)(\|x_{n_k+1}-p\| + \|x_{n_k}-p\|)\} \\
&\quad \geq 0. \qquad (4.57)
\end{aligned}
$$

It is easy to see from (4.54) and (4.14), that

$$\begin{aligned}
\|x_{n_k+1} - p\|^2 &\leq \omega_n(\omega_n + \eta_n)\|x_n - p\|^2 + \eta_n(\omega_n + \eta_n)\|t_n - p\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - f(p), x_{n+1} - p\rangle \\
&\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\leq \omega_n(\omega_n + \eta_n)\|x_n - p\|^2 + \eta_n(\omega_n + \eta_n)\|v_n - p\|^2 \\
&\quad - \eta_n(\omega_n + \eta_n)(2 - \alpha - \frac{\delta\mu_n\alpha}{\mu_{n+1}})\|v_n - u_n\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - f(p), x_{n+1} - p\rangle \\
&\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\leq \omega_n(\omega_n + \eta_n)\|x_n - p\|^2 + \eta_n(\omega_n + \eta_n)\|w_n - p\|^2 \\
&\quad - \eta_n(\omega_n + \eta_n)(2 - \alpha - \frac{\delta\mu_n\alpha}{\mu_{n+1}})\|v_n - u_n\|^2 \\
&\quad - \eta_n(\omega_n + \eta_n)(2 - \alpha - \frac{\delta\mu_n\alpha}{\mu_{n+1}})\|t_n - u_n\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - f(p), x_{n+1} - p\rangle \\
&\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\leq (1 - \frac{2\alpha_n(1-\tau)}{1-\alpha_n\tau})\|x_n - p\|^2 \\
&\quad + \frac{2\alpha_n(1-\tau)}{1-\alpha_n\tau}\Big[\frac{\eta_n(1-\alpha_{n_k})\theta_n}{2\alpha_n(1-\tau)}\|x_n - x_{n-1}\|N_2 \\
&\quad + \frac{\alpha_n N_3}{2\alpha_n(1-\tau)} - \frac{\eta_n(1-\alpha_n)}{2\alpha_n(1-\tau)}(2 - \alpha - \frac{\delta\mu_n\alpha}{\mu_{n+1}})\|t_n - u_n\|^2 \\
&\quad - \frac{\eta_n(1-\alpha_n)}{2\alpha_n(1-\tau)}(2 - \alpha - \frac{\delta\mu_n\alpha}{\mu_{n_k+1}})\|v_{n_k} - u_{n_k}\|^2 \\
&\quad + \frac{1}{(1-\tau)}\langle f(p) - p, x_{n_k+1} - p\rangle\Big] \\
&\leq \|x_n - p\|^2 + \frac{\alpha_n\eta_n(1-\alpha_n)}{1-\alpha_n\tau}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|N_2 + \alpha_{n_k}N_3 \\
&\quad - \eta_n(1-\alpha_n)(2 - \alpha - \frac{\delta\mu_n\alpha}{\mu_{n+1}})\|t_n - u_n\|^2 \\
&\quad - \eta_n(1-\alpha_n)(2 - \alpha - \frac{\delta\mu_n\alpha}{\mu_{n+1}})\|v_n - u_n\|^2 \\
&\quad + \frac{2\alpha_n}{(1-\alpha_n\tau)}\langle f(p) - p, x_{n+1} - p\rangle\Big],
\end{aligned} \tag{4.58}$$

which implies that

$$
\begin{aligned}
&\limsup_{k\to\infty}\Bigg(\eta_{n_k}(1-\alpha_{n_k})(2-\alpha-\frac{\delta\mu_{n_k}\alpha}{\mu_{n_k+1}})\|t_{n_k}-u_{n_k}\|^2\\
&\quad+\eta_{n_k}(1-\alpha_{n_k})(2-\alpha-\frac{\delta\mu_{n_k}\alpha}{\mu_{n_k+1}})\|v_{n_k}-u_{n_k}\|^2\Bigg)\\
&\le\limsup_{k\to\infty}\Bigg[\|x_{n_k}-p\|^2+\frac{\alpha_{n_k}\eta_{n_k}(1-\alpha_{n_k})}{1-\alpha_{n_k}\tau}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2\\
&\quad+\alpha_{n_k}N_3+\frac{2\alpha_{n_k}}{(1-\alpha_{n_k}\tau)}\langle f(p)-p,x_{n_k+1}-p\rangle-\|x_{n_k+1}-p\|^2\Bigg]\\
&\le-\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\le 0.
\end{aligned}
$$

Thus, we have

$$
\lim_{k\to\infty}\|t_{n_k}-u_{n_k}\|=0=\lim_{k\to\infty}\|v_{n_k}-u_{n_k}\|. \tag{4.59}
$$

Using the triangular inequality and (4.59), we have

$$
\lim_{k\to\infty}\|t_{n_k}-v_{n_k}\|\le\lim_{k\to\infty}\|t_{n_k}-u_{n_k}\|+\lim_{k\to\infty}\|u_{n_k}-v_{n_k}\|=0. \tag{4.60}
$$

Now using similar approach as in (4.58), we have

$$
\begin{aligned}
\|x_{n_k+1}-p\|^2&\le\omega_{n_k}(\omega_{n_k}+\eta_{n_k})\|x_{n_k}-p\|^2+\eta_{n_k}(\omega_{n_k}+\eta_{n_k})\|t_{n_k}-p\|^2\\
&\quad+2\alpha_{n_k}\langle f(x_{n_k})-f(p),x_{n_k+1}-p\rangle+2\alpha_{n_k}\langle f(p)-p,x_{n_k+1}-p\rangle\\
&\le\omega_{n_k}(\omega_{n_k}+\eta_{n_k})\|x_{n_k}-p\|^2+\eta_{n_k}(\omega_{n_k}+\eta_{n_k})\|v_{n_k}-p\|^2\\
&\quad+2\alpha_{n_k}\langle f(x_{n_k})-f(p),x_{n_k+1}-p\rangle+2\alpha_{n_k}\langle f(p)-p,x_{n_k+1}-p\rangle\\
&\le\omega_{n_k}(\omega_{n_k}+\eta_{n_k})\|x_{n_k}-p\|^2+\eta_{n_k}(\omega_{n_k}+\eta_{n_k})[\|w_{n_k}-p\|^2\\
&\quad-\gamma_{n_k}\epsilon\|T^*(z_{n_k}-Tw_{n_k})\|^2]\\
&\quad+2\alpha_{n_k}\langle f(x_{n_k})-f(p),x_{n_k+1}-p\rangle+2\alpha_{n_k}\langle f(p)-p,x_{n_k+1}-p\rangle\\
&\le(1-\alpha_{n_k})^2\|x_{n_k}-p\|^2+\theta_{n_k}\|x_{n_k}-x_{n_k-1}\|N_2\\
&\quad-\eta_{n_k}(1-\alpha_{n_k})\epsilon^2\|T^*(z_{n_k}-Tw_{n_k})\|^2\\
&\quad+2\alpha_{n_k}\langle f(x_{n_k})-f(p),x_{n_k+1}-p\rangle+2\alpha_{n_k}\langle f(p)-p,x_{n_k+1}-p\rangle\\
&\le\|x_{n_k}-p\|^2+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2\\
&\quad-\eta_{n_k}(1-\alpha_{n_k})\epsilon^2\|T^*(z_{n_k}-Tw_{n_k})\|^2\\
&\quad+2\alpha_{n_k}\langle f(x_{n_k})-f(p),x_{n_k+1}-p\rangle+2\alpha_{n_k}\langle f(p)-p,x_{n_k+1}-p\rangle,
\end{aligned} \tag{4.61}
$$

which implies that

$$\begin{aligned}
&\limsup_{k\to\infty}\left(\eta_{n_k}(1-\alpha_{n_k})\epsilon^2\|T^*(z_{n_k}-Tw_{n_k})\|^2\right)\\
&\quad\le \limsup_{k\to\infty}\Big[\|x_{n_k}-p\|^2+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2\\
&\qquad+2\alpha_{n_k}\langle f(x_{n_k})-f(p),x_{n_k+1}-p\rangle\\
&\qquad+2\alpha_{n_k}\langle f(p)-p,x_{n_k+1}-p\rangle-\|x_{n_k+1}-p\|^2\Big]\\
&\quad\le -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\le 0.
\end{aligned}$$

Hence, we obtain

$$\lim_{k\to\infty}\|T^*(z_{n_k}-Tw_{n_k})\|=0. \tag{4.62}$$

In the proof of Lemma 4.1 in [19] (establishing that γ_n is well defined), the authors obtained that

$$\|Tw_n-z_n\|^2\le 2\|T^*(z_{n_k}-Tw_{n_k})\|\|w_n-z_n\|, \tag{4.63}$$

see Equation (3.14) of [19]. Using (4.62) and with the above inequality, we have

$$\lim_{k\to\infty}\|z_{n_k}-Tw_{n_k}\|=0. \tag{4.64}$$

From Algorithm 3.2 and (4.62), we have

$$\begin{aligned}
\lim_{k\to\infty}\|v_{n_k}-w_{n_k}\|&=\lim_{k\to\infty}\|w_{n_k}+\gamma_{n_k}T^*(z_{n_k}-Tw_{n_k})-w_{n_k}\|\\
&=\gamma_{n_k}\lim_{k\to\infty}\|T^*(z_{n_k}-Tw_{n_k})\|=0.
\end{aligned} \tag{4.65}$$

In addition, we have

$$\begin{aligned}
\|z_{n_k}-Tp\|^2&=\|Tw_{n_k}-Tp-Tw_{n_k}+z_{n_k}\|^2\\
&=\|Tw_{n_k}-Tp\|^2-2\langle T(w_{n_k}-p),Tw_{n_k}-z_{n_k}\rangle+\|Tw_{n_k}-z_{n_k}\|^2\\
&\ge\|Tw_{n_k}-Tp\|^2-2\|T\|\|w_{n_k}-p\|\|Tw_{n_k}-z_{n_k}\|\\
&\quad+\|Tw_{n_k}-z_{n_k}\|^2,
\end{aligned} \tag{4.66}$$

which implies that

$$\begin{aligned}
-\|z_{n_k}-Tp\|^2\le -\|Tw_{n_k}-Tp\|^2+2\|T\|\|w_{n_k}-p\|\|Tw_{n_k}-z_{n_k}\|\\
-\|Tw_{n_k}-z_{n_k}\|^2.
\end{aligned} \tag{4.67}$$

Adding (4.67) and (4.9), we have

$$\begin{aligned}
&(2-\eta-\frac{\nu\lambda_{n_k}\eta}{\lambda_{n_k+1}})\|Tw_{n_k}-y_{n_k}\|^2+(2-\eta-\frac{\nu\lambda_{n_k}\eta}{\lambda_{n_k+1}})\|z_{n_k}-y_{n_k}\|^2\\
&\quad\le 2\|T\|\|w_{n_k}-p\|\|Tw_{n_k}-z_{n_k}\|-\|Tw_{n_k}-z_{n_k}\|^2.
\end{aligned} \tag{4.68}$$

Taking limit as $k \to \infty$, we have

$$\lim_{k\to\infty} \|Tw_{n_k} - y_{n_k}\| = \lim_{k\to\infty} \|z_{n_k} - y_{n_k}\| = 0. \tag{4.69}$$

In addition, we have

$$\begin{aligned}\lim_{k\to\infty} \|Tw_{n_k} - z_{n_k}\| &\leq \lim_{k\to\infty} \|Tw_{n_k} - y_{n_k}\| + \lim_{k\to\infty} \|y_{n_k} - z_{n_k}\| \\ &= 0. \end{aligned}\tag{4.70}$$

And also, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \alpha_n\|f(x_n) - p\|^2 + \omega_n\|x_n - p\|^2 \\ &\quad + \eta_n\|St_n - p\|^2 - \eta_n\delta_n\|x_n - St_n\|^2 \\ &\leq \alpha_n\|f(x_n) - p\|^2 + \omega_n\|x_n - p\|^2 \\ &\quad + \eta_n\|t_n - p\|^2 - \omega_n\eta_n\|x_n - St_n\|^2 \\ &\leq \alpha_n\|f(x_n) - p\|^2 + \omega_n\|x_n - p\|^2 \\ &\quad + \eta_n\|v_n - p\|^2 - \omega_n\eta_n\|x_n - St_n\|^2 \\ &\leq \alpha_n\|f(x_n) - p\|^2 + \omega_n\|x_n - p\|^2 \\ &\quad + \eta_n\|w_n - p\|^2 - \omega_n\eta_n\|x_n - St_n\|^2 \\ &\leq \alpha_n\|f(x_n) - p\|^2 + \omega_n\|x_n - p\|^2 + \eta_n\|x_n - p\|^2 \\ &\quad + \eta_n\theta_n\|x_n - x_{n-1}\|N_2 - \omega_n\eta_n\|x_n - St_n\|^2 \\ &= (\omega_n + \eta_n)\|x_n - p\|^2 + \alpha_n\|f(x_n) - p\|^2 \\ &\quad + \eta_n\theta_n\|x_n - x_{n-1}\|N_2 - \omega_n\eta_n\|x_n - St_n\|^2 \\ &\leq \|x_n - p\|^2 + \alpha_n\|f(x_n) - p\|^2 \\ &\quad + \eta_n\theta_n\|x_n - x_{n-1}\|N_2 - \omega_n\eta_n\|x_n - St_n\|^2, \end{aligned}\tag{4.71}$$

which implies that

$$\begin{aligned}&\limsup_{k\to\infty}\left(\omega_{n_k}\eta_{n_k}\|x_{n_k} - St_{n_k}\|^2\right) \\ &\quad\leq \limsup_{k\to\infty}\Bigg[\|x_{n_k} - p\|^2 + \eta_{n_k}\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\|N_2 \\ &\qquad + \alpha_{n_k}\|f(x_{n_k}) - p\|^2 - \|x_{n_k+1} - p\|^2\Bigg] \\ &\quad\leq -\liminf_{k\to\infty}[\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2] \\ &\quad\leq 0. \end{aligned}\tag{4.72}$$

Thus, we have

$$\lim_{k\to\infty} \|x_{n_k} - St_{n_k}\| = 0. \tag{4.73}$$

It is easy to see that, as $k \to \infty$, we have

$$\begin{aligned}\|w_{n_k} - x_{n_k}\| &= \theta_{n_k}||S_{n_k}x_{n_k} - S_{n_k}x_{n_k-1}|| \\ &= \alpha_{n_k} \cdot \frac{\theta_{n_k}}{\alpha_{n_k}}||S_{n_k}x_{n_k} - S_{n_k}x_{n_k-1}|| \to 0.\end{aligned} \tag{4.74}$$

In addition, we have that

$$\|v_{n_k} - x_{n_k}\| \le \|w_{n_k} - x_{n_k}\| + \gamma_n\|T^*(z_{n_k} - Tw_{n_k})\| \to 0, \text{ as } k \to \infty, \tag{4.75}$$

$$\|w_{n_k} - v_{n_k}\| \le \|w_{n_k} - x_{n_k}\| + \|x_{n_k} - v_{n_k}\| \to 0, \text{ as } k \to \infty, \tag{4.76}$$

$$\|t_{n_k} - x_{n_k}\| \le \|t_{n_k} - v_{n_k}\| + \|v_{n_k} - x_{n_k}\| \to 0, \text{ as } k \to \infty, \tag{4.77}$$

$$\|t_{n_k} - w_{n_k}\| \le \|tn_k - x_{n_k}\| + \|x_{n_k} - w_{n_k}\| \to 0, \text{ as } k \to \infty, \tag{4.78}$$

$$\|u_{n_k} - x_{n_k}\| \le \|u_{n_k} - v_{n_k}\| + \|v_{n_k} - x_{n_k}\| \to 0, \text{ as } k \to \infty \tag{4.79}$$

and

$$\begin{aligned}\|t_{n_k} - St_{n_k}\| \le{}& \|t_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \\ &+ \|x_{n_k} - St_{n_k}\| \to 0, \text{ as } k \to \infty.\end{aligned} \tag{4.80}$$

Thus, we have

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| \le{}& \alpha_n\|f(x_{n_k}) - x_{n_k})\| + \omega_n\|x_{n_k} - x_{n_k}\| \\ &+ \eta_{n_k}\|St_{n_k} - x_{n_k}\| \to 0, \text{ as } k \to \infty.\end{aligned} \tag{4.81}$$

Now, since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to $x^* \in H$. In addition, using (4.77) and the boundedness of $\{t_{n_k}\}$, there exists a subsequence $\{t_{n_{k_j}}\}$ of $\{t_{n_k}\}$ such that $\{t_{n_{k_j}}\}$ converges weakly to $x^* \in H_1$ and since S is demiclosed with (4.80), we have that $x^* \in F(S)$. Hence, by (4.60), (4.65) and Lemma 4.3, we obtain that $x^* \in \Omega$. Furthermore, since $\{x_{n_{k_j}}\}$ converges weakly to x^*, we obtain that

$$\begin{aligned}\limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle &= \lim_{j\to\infty}\langle f(p) - p, x_{n_{k_j}} - p\rangle \\ &= \langle f(p) - p, x^* - p\rangle.\end{aligned} \tag{4.82}$$

Hence, since p is a unique solution of Ω, it follows that

$$\limsup_{k\to\infty}\langle f(p)-p, x_{n_k}-p\rangle = \langle f(p)-p, x^*-p\rangle \le 0, \tag{4.83}$$

we have obtain from (4.83) and (4.81)

$$\limsup_{k\to\infty}\langle f(p)-p, x_{n_k+1}-p\rangle \le 0. \tag{4.84}$$

Using our assumption and (4.84), we have that

$$\begin{aligned}\lim_{k\to\infty}\Psi_{n_k} &= \lim_{k\to\infty}\Big(\frac{\eta_{n_k}(1-\alpha_{n_k})}{2(1-\tau)}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2+\frac{\alpha_{n_k}N_3}{2(1-\tau)}\\ &\quad+\frac{1}{(1-\tau)}\langle f(p)-p, x_{n_k+1}-p\rangle\Big)\\ &\le 0.\end{aligned}$$

Thus, From Lemma 2.4, we have that $\lim\limits_{n\to\infty}\|x_n-p\|=0$. □

5. Numerical example

In this section, we will give some numerical examples which will show the applicability and the efficiency of our proposed iterative method in comparison to Algorithm 31 in [24] and Algorithm 1 in [20], respectively.

Example 5.1. Let $H_1 = H_2 = L_2([0,1])$ be equipped with the inner product

$$\langle x,y\rangle = \int_0^1 x(t)y(t)dt,\ \forall\ x,y\in L_2([0,1])$$

and norm

$$\|x\|^2 = \int_0^1 |x(t)|^2dt,\ \forall x,y\in L_2([0,1]).$$

Let $B; A; f; T : L_2([0,1]) \to L_2([0,1])$ be defined by

$$\begin{aligned}Ax(t) &= \max\{0, x(t)\},\ t\in[0,1], x\in L_2([0,1]);\\ Bx(t) &= \frac{x(t)}{2},\ t\in[0,1], x\in L_2([0,1]);\\ fx(t) &= \int_0^t \frac{t}{2}x(s)\ dt\ t\in[0,1], x\in L_2([0,1]);\end{aligned}$$

and

$$Tx(s) = \int_0^1 K(s,t)x(t)dt\ x\in L_2([0,1]),$$

where K is a continuous real valued function on $[0,1]\times[0,1]$. It is easy to see that A is 1-Lipschitz continuous and monotone, B is γ-strongly monotone,

f is a contraction on $L_2([0,1])$ and T is a bounded linear operator with the adjoint operator

$$T^*x(s) = \int_0^1 K(t,s)x(t)dt, \ x \in L_2([0,1])$$

(we use this example due to Remark 2.2).

Let $S_n; S : L_2([0,1]) \to L_2([0,1])$ be defined by

$$Sx(s) = \int_0^1 tx(s)ds, \ \forall \ t \in [0,1]$$

and

$$S_n x(t) = \sin x(t).$$

Let C be defined by $C = Q = \{x \in L_2 : \langle a, x\rangle = b\}$ where $a \neq 0$ and $b = 2$. Then, we have

$$P_C(\bar{x}) = P_Q(\bar{x}) = \max\left\{0, \frac{b - \langle a, \bar{x}\rangle}{\|a\|^2}\right\} a + \bar{x}.$$

We choose $\alpha_n = \frac{2}{200n+5}, \omega_n = \frac{2n}{100n^2+8}, \eta_n = 1 - \omega_n - \alpha_n, \theta_n = \overline{\theta}, \eta = 1.2, \alpha = 1.1, \nu = 0.3, \delta = 0.1, \lambda_0 = \frac{1}{3}, \Gamma_n = \frac{100}{(n+1)^{1.3}}, \epsilon_n = \frac{\alpha_n}{n^{0.01}}, \mu = \frac{1}{2}, \zeta_n = \frac{100}{(n+1)^{1.2}}$ for all $n \in \mathbb{N}$. Also if we consider $\epsilon = \|x_n - x_{n_1}\| \leq 10^{-5}$ as the stopping criterion and choose the following as starting points:

Case (1): $x_0(t) = 2t^2 + t + 2, \ x_1(t) = t;$

Case (2): $x_0(t) = 2t^2 + e^{2t} + 1, \ x_1(t) = 3t^3 + 3;$

Case (3): $x_0(t) = t^3 + e^{3t} + 2, \ x_1(t) = \cos(t).$

		Alg. 3.2	Alg. 31 in [24]	Alg. 1 in [20]
Case(1)	No of Iter.	10	28	26
	CPU time(s)	0.1704	0.20101	0.1745
Case(2)	No of Iter.	10	29	21
	CPU time(s)	0.1713	0.2130	0.1810
Case(3)	No of Iter.	15	30	27
	CPU time(s)	0.1710	0.2201	0.1821

Table 1. Computation result for Example 5.1.

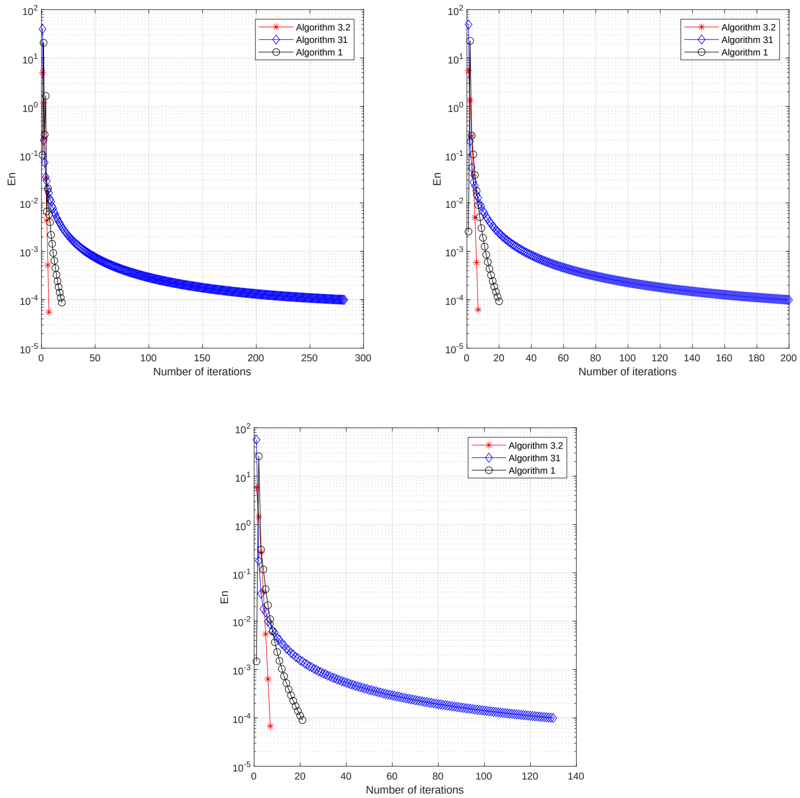

FIGURE 1. Example 5.1, Top Left: Case(1); Top Right: Case(2); Case (3); Bottom.

Example 5.2. ([18, 23]) Let $H_1 = H_2 = l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \cdots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $\|x\| = (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}$ for all $x \in l_2(\mathbb{R})$. Suppose the operators $T, A, B; f : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ are defined by

$$\begin{aligned} Tx &= (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots),\ x \in l_2(\mathbb{R}); \\ Ax &= (7 - \|x\|)x,\ \forall\ x \in l_2(\mathbb{R}); \\ Bx &= (5 - \|x\|)x,\ \forall\ x \in l_2(\mathbb{R}) \end{aligned}$$

and

$$f(x) = \frac{x}{3},\ \forall\ x \in l_2(\mathbb{R}).$$

Then, it is easy to see that T is a bounded linear operator with the adjoint operator $T^*y = (0, y_1, \frac{y_2}{2}, \frac{y_3}{3}, \cdots)$ $y \in l_2(\mathbb{R})$ and A, B are quasimonotone, Lipschitz continuous and weakly sequentially continuous on $l_2(\mathbb{R})$, see [23]. Let $C = Q = \{x \in l_2(\mathbb{R}) : \|x\| \leq 3\}$. Clearly, C and Q are nonempty, closed and convex subsets of $l_2(\mathbb{R})$. Hence, we have

$$P_C(x) = P_Q(x) = \begin{cases} x, & \text{if } \|x\| \leq 3, \\ \frac{3x}{\|x\|}, & \text{if otherwise.} \end{cases} \tag{5.1}$$

In addition, we define $S, S_n : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ are defined by $Sx = (0, \frac{x_1}{2}, \frac{x_2}{2}, \cdots)$ and $S_n x = (0, x_1, x_2, x_3, \cdots)$. We choose $\alpha_n = \frac{2}{200n+5}, \omega_n = \frac{2n}{100n^2+8}, \eta_n = 1 - \omega_n - \alpha_n, \theta_n = \overline{\theta}, \eta = 1.2, \alpha = 1.1, \nu = 0.3, \delta = 0.1, \lambda_0 = \frac{1}{3}, \Gamma_n = \frac{100}{(n+1)^{1.3}}, \epsilon_n = \frac{\alpha_n}{n^{0.01}}, \mu = \frac{1}{2}, \zeta_n = \frac{100}{(n+1)^{1.2}}$ for all $n \in \mathbb{N}$. Also if we consider $\epsilon = \|x_n - x_{n_1}\| \leq 10^{-5}$ as the stopping criterion and choose the following as starting points:

Case (1): $x_0 = (2, 2, 2, \cdots),\ x_1 = (0.5, 0.5, 0.5, \cdots)$;

Case (2): $x_0 = (1, 2, 3, 4, \cdots),\ x_1 = (1, 1, 1, \cdots)$;

Case (3): $x_0 = (0.1, 0.2, 0.3, \cdots),\ x_1 = (2, 4, 6, \cdots)$;

		Alg. 3.2	Alg. 31 in [24]	Alg.1 in [20]
Case(1)	No of Iter.	7	22	14
	CPU time	0.0812	0.1345	0.0823
Case(2)	No of Iter.	3	20	8
	CPU time	0.0821	0.1430	0.0913
Case(3)	No of Iter.	5	50	12
	CPU time	0.0810	0.0833	0.0819

Table 2. Computation result for Example 5.2.

6. Conclusion

A SEGM with an inertial extrapolation step is introduced and studied for solving the SVIPFPP (1.9)-(1.10) in infinite dimensional real Hilbert spaces when the cost operators are quasimonotone, sequentially weakly continuous and Lipschitz continuous. In addition, we established that the proposed iterative method converges strongly to the solution set of SVIPFPP (1.9)-(1.10). Our method uses stepsizes that are generated at each iteration by some simple computations, which allows it to be easily implemented without the prior knowledge of the operator norm or the coefficient of an underlying operator.

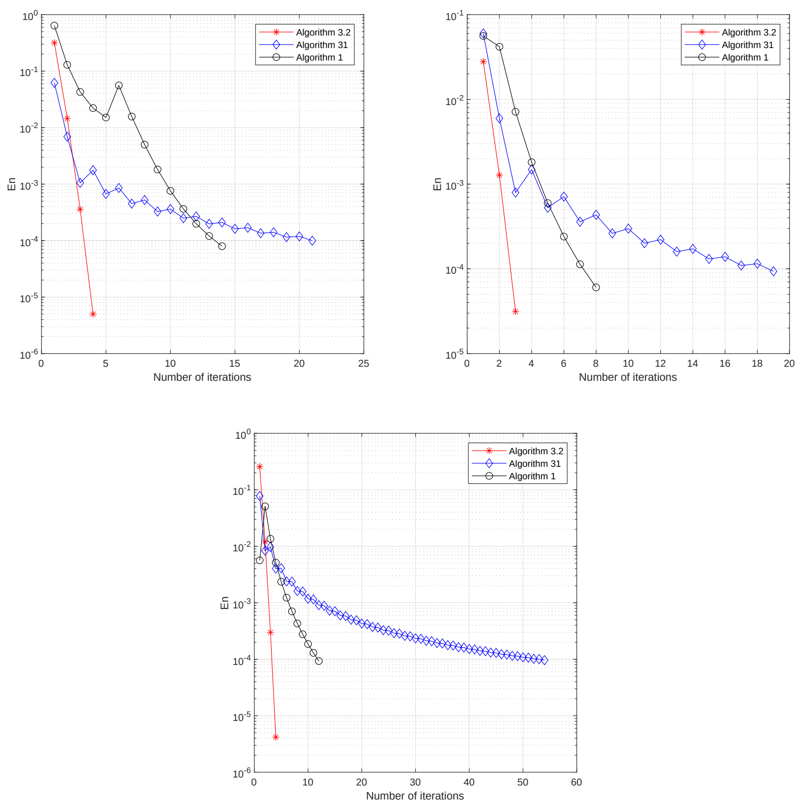

FIGURE 2. Example 5.2, Top Left: Case (1) ; Top Right: Case (2); Bottom Case (3).

In addition, we present some examples and numerical experiment to show the efficiency and implementation of our method in the framework of infinite and finite dimensional Hilbert spaces.

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