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AN INVESTIGATION ON THE EXISTENCE AND UNIQUENESS ANALYSIS OF THE FRACTIONAL NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, by means of the Schauder fixed point theorem and Arzela-Ascoli theorem, the existence and uniqueness of solutions for a class of not instantaneous impulsive problems of nonlinear fractional functional Volterra-Fredholm integro-differential equations are investigated. An example is given to illustrate the main results.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary noninteger order, so fractional differential equations have wider application. Fractional integro-differential equations have gained considerable importance; it can describe many phenomena in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, and electromagnetic [10, 12, 13, 14, 15, 16, 17, 23, 26, 29].

In the recent years, there has been a significant development in fractional calculus and fractional differential equations; see Kilbas et al. [24], Miller and Ross [29], Podlubny [30], Baleanu et al. [3], and so forth. Research on the solutions of fractional differential equations is very extensive, such as numerical solutions, see El-Mesiry et al. [9] and Hashim et al. [18], mild solutions, see Chang et al. [10] and Chen et al. [6], the existence and uniqueness of solutions

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for initial and boundary value problem, see [1, 2, 8, 11, 20, 21, 22, 25, 26, 27, 28, 31, 32, 34], and so on.

Recently, not instantaneous impulsive condition first time used by author's Hernandez and O'Regan [19] for the following problem of the form:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), t \in (s_i, t_{i+1}], i = 0, 1, \cdots, N, \\ u(t) = g_i(t, u(t)), t \in (t_i, s_i], i = 1, 2, \cdots, N, \\ u(0) = x_0. \end{cases}$$

Motivated by the previous results, we discuss in this paper the existence and uniqueness of solutions for the following impulsive nonlinear fractional Volterra-Fredholm integro-differential equation:

$$D_t^\alpha y(t) = J_t^{2-\alpha} f\left(t, y_{\rho(t,y_t)}, A(y_{\rho(t,y_t)}), B(y_{\rho(t,y_t)})\right), t \in (s_i, t_{i+1}], \quad (1.1)$$

$$y(t) = g_i(t, y(t)), y'(t) = q_i(t, y(t)), t \in (t_i, s_i], i = 1, 2, \cdots, N, \quad (1.2)$$

$$y(t) + u(y) = \phi(t), y'(t) + v(y) = \varphi(t), t \in (-\infty, 0], \quad (1.3)$$

where D_t^α is Caputo fractional derivative of order $\alpha \in (1, 2]$ and $J^{2-\alpha}$ is Riemann-Liouville fractional integral. y' denotes the derivative of y with respect to t and operational interval $J = [0, T], 0 < T < \infty$. $f : J \times \mathfrak{B}_h \times \mathfrak{B}_h \to X, u, v : X \to X$ are given functions. $\mathfrak{B}_h$ is an abstract phase space and y_t the element of $\mathfrak{B}_h$ defined by $y_t(\theta) = y(t + \theta), \theta \in (-\infty, 0]$. The terms $A\left(y_{\rho(t,y_t)}\right), B\left(y_{\rho(t,y_t)}\right)$ are given by

$$A\left(y_{\rho(t,y_t)}\right) = \int_0^t K_1(t, s)\left(y_{\rho(s,y_s)}\right) ds$$

and

$$B\left(y_{\rho(t,y_t)}\right) = \int_0^T K_2(t, s)\left(y_{\rho(s,y_s)}\right) ds,$$

where $K \in C\left(D, \mathbb{R}^+\right)$, is the set of all positive functions which are continuous on $D = \left\{(t, s) \in \mathbb{R}^2 : 0 \le s \le t < T\right\}$ and $A^* = \sup_{t\in[0,T]} \int_0^t K_1(t, s)ds < \infty$, $B^* = \sup_{t\in[0,T]} \int_0^T K_2(t, s)ds < \infty$. Here $0 = t_0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < t_N \le s_N \le t_{N+1} = T$, are pre-fixed numbers, $g_i, q_i \in C\left((t_i, s_i] \times X; X\right)$ for all $i = 1, 2, \cdots, N$.

The rest of the paper is organized as follows. In Section 2, we give some definitions and lemmas that will be useful to our main results. In Section 3, we give two main results: the first result based on the Schauder's fixed point theorem and the second result based on the Banach contraction principle. In Section 4, an example is presented to illustrate the main results.

2. Preliminary

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $(X, \|\cdot\|_X)$ be a complex Banach space of functions with the norm $\|y\|_X = \sup_{t\in J}\{|y(t)| : y \in X\}$. For infinite delay we use abstract phase space $\mathfrak{B}_h$ details are as follow:

Assume that $h : (-\infty, 0] \to (0, \infty)$ is a continuous functions with

$$l = \int_{-\infty}^{0} h(s)ds < \infty, t \in (-\infty, 0].$$

For any $a > 0$, we define $\mathfrak{B} = \{\psi : [-a, 0] \to X$ such that $\psi(t)$ is bounded and measurable$\}$, and equipped the space $\mathfrak{B}$ with the norm

$$\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} \|\psi(s)\|_X, \ \ \forall\psi \in \mathfrak{B}.$$

Let us define

$$\mathfrak{B}_h = \left\{\psi : (-\infty, 0] \to X, \text{s.t. for } c > 0,\, \psi|_{[-c,0]} \in \mathfrak{B}, \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds < \infty\right\}.$$

If $\mathfrak{B}_h$ is endowed with the norm $\|\psi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds$, for all $\psi \in \mathfrak{B}_h$, then it is clear that $(\mathfrak{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a complete Banach space.

To treat the impulsive conditions, we consider the following setting

$$\mathfrak{B}'_h := PC((-\infty, T]; X), \ \ T < \infty$$

is a Banach space of all such functions $y : (-\infty, T] \to X$, which are continuous every where except for a finite number of points $t_i \in (0, T), i = 1, 2, \dots, N$, at which $y\left(t_i^+\right)$and $y\left(t_i^-\right)$exists and endowed with the norm

$$\|y\|_{\mathfrak{B}'_h} = \sup\left\{\|y(s)\|_X : s \in J\right\} + \|\phi\|_{\mathfrak{B}_h}, \ \ y \in \mathfrak{B}'_h,$$

where $\|\cdot\|_{\mathfrak{B}'_h}$ to be a semi-norm in $\mathfrak{B}'_h$.

For a function $y \in \mathfrak{B}'_h$ and $i \in \{0, 1, \dots, N\}$, we introduce the function $\bar{y}_i \in C\left([t_i, t_{i+1}]\,; X\right)$ given by

$$\bar{y}_i(t) = \begin{cases} y(t), \text{ for } t \in (t_i, t_{i+1}], \\ y\left(t_i^+\right), \text{for } t = t_i, \end{cases}$$

and setting

$$\mathfrak{B}''_h := PC^1((-\infty, T]; X), \ \ T < \infty$$

is a Banach space of all such functions $y : (-\infty, T] \to X$, which are continuously differentiable every where except for a finite number of points $t_i \in$

$(0,T), i=1,2,\ldots,N$, at which $y'\left(t_i^+\right)$and $y'\left(t_i^-\right)$exists and endowed with the semi-norm

$$\|y\|_{\mathfrak{B}_h''} = \sup_{t\in[0,T]} \left\{\|y(s)\|_X, \left\|y'(s)\right\|_X\right\} + \|\phi\|_{\mathfrak{B}_h}, \ \ y \in \mathfrak{B}_h''.$$

For a function $y \in \mathfrak{B}_h''$ and $i \in \{0,1,\ldots,N\}$, we introduce the function $\bar{y}_i \in C^1\left([t_i, t_{i+1}];X\right)$ given by

$$\bar{y}_i(t) = \begin{cases} y'(t), \text{ for } t \in (t_i, t_{i+1}], \\ y'\left(t_i^+\right), \text{for } t = t_i. \end{cases}$$

If function $y : (-\infty, T] \to X$ such that $y \in \mathfrak{B}_h''$ then for all $t \in [0,T]$, the following conditions hold:

(C_1) $y_t \in \mathfrak{B}_h$.

(C_2) $\|y(t)\|_X \leq H \|y_t\|_{\mathfrak{B}_h}$.

(C_3) $\|y_t\|_{\mathfrak{B}_h} \leq K(t) \sup\{\|y(s)\|_X : 0 \leq s \leq t\} + M(t)\|\phi\|_{\mathfrak{B}_h}$,

where $H > 0$ is constant; $K, M : [0,\infty) \to [0,\infty), K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded and K, M are independent of $y(t)$.

$\left(C_{4_\phi}\right)$ The function t $\to \phi_t$ is well defined and continuous from the set

$$\Re\left(\rho^-\right) = \{\rho(s,\psi) : (s,\psi) \in [0,T] \times \mathfrak{B}_h\}$$

into $\mathfrak{B}_h$ and there exists a continuous and bounded function $J^\phi : \Re\left(\rho^-\right) \to (0,\infty)$ such that $\|\phi_t\|_{\mathcal{B}_h} \leq J^\phi(t)\|\phi\|_{\mathfrak{B}_h}$ for every $t \in \Re\left(\rho^-\right)$.

Lemma 2.1. ([4, Lemma 3.6]) *Let $y : (-\infty, T] \to X$ be a function such that $y \in \mathfrak{B}_h''$ with $y_0 = \phi$, $y|_{J_k} \in C^1(J_k, X)$ and if $(C_{4\phi})$ hold. Then*

$$\|y_s\|_{\mathfrak{B}_h} \leq \left(M_b + J^\phi\right) \|\phi\|_{\mathfrak{B}_h} + K_b \sup\{\|y(\theta)\|_X ; \theta \in [0, \max\{0,s\}]\}, s \in \Re\left(\rho^-\right) \cup J,$$

where

$$J^\phi = \sup_{t\in\Omega(\rho-)} J^\phi(t), \ \ M_b = \sup_{s\in[0,T]} M(s) \ \ \text{and} \ \ K_b = \sup_{s\in[0,T]} K(s).$$

Definition 2.2. Caputo's derivative of order $\alpha > 0$ for a function $f : [a,\infty) \to \mathbb{R}$ is defined as

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = {}_aJ_t^{n-\alpha} f^{(n)}(t),$$

where $a \geq 0, n \in N$. It is clear that derivative of constant function is zero.

Definition 2.3. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a function $f \in L^1\left(\mathbb{R}^+, X\right)$ is defined by

$${}_aJ_t^0 f(t) = f(t), {}_aJ_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0, t > 0,$$

where $a \geq 0, n \in N$ and $\Gamma(\cdot)$ is the Euler gamma function.

Lemma 2.4. ([4]) *For $\alpha > 0$, solution of fractional differential equations with lower limit not zero ${}_aJ_t^{\alpha} D_t^{\alpha} y(t) = y(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + c_3(t-a)^3 + \cdots + c_{n-1}(t-a)^{n-1}$ where $c_i \in \mathbb{R}, i = 0, 1, \cdots, n-1, n = [\alpha]+1$ and $[\alpha]$ represent the integral part of the real number α.*

Our following result is based Definition 2.1 in [19].

Definition 2.5. A function $y : (-\infty, T] \to X$ such that $y \in \mathfrak{B}_h''$ is called a solution of the problem (1.1)-(1.3) if $y(0) = \phi(0), y'(0) = \varphi(0), y(t) = g_j(t, y(t)), y'(t) = q_j(t, y(t))$ for $t \in (t_j, s_j], j = 1, 2, \cdots, N$, and satisfying the following integral equation

$$y(t) = \begin{cases} \phi(0) - u(y) + (\varphi(0) - v(y))t \\ \quad + \int_0^t (t-s) f\left(s, y_{\rho(s,y_s}, Ay_{\rho(s,y_s)}, By_{\rho(s,y_s)}\right) ds, t \in [0, t_1], \\ \\ g_i(s_i, y(s_i)) + q_i(s_i, y(s_i))\, t \\ \quad + \int_{s_i}^t (t-s) f\left(s, y_{\rho(s,y_s)}, Ay_{\rho(s,y_s)}, By_{\rho(s,y_s)}\right) ds, t \in [s_i, t_{i+1}] \end{cases}$$

for every $i = 1, 2, \cdots, N$.

3. Main results

In this section, we state and prove our main results. To prove our results we shall assume the function $\rho : [0, T] \times \mathfrak{B}_h \to (-\infty, T]$ is continuous and $\phi, \varphi \in \mathfrak{B}_h$. If $y \in \mathfrak{B}_h$ we defined $\bar{y} : (-\infty, T) \to X$ as the extension of y to $(-\infty, T]$ such that $\bar{y}(t) = \phi$. We defined $\tilde{y} : (-\infty, T) \to X$ such that $\tilde{y} = y + x$ where $x : (-\infty, T) \to X$ is the extension of $\phi \in \mathfrak{B}_h$ such that $x(t) = \phi(0)$ for $t \in [0, T]$. In additional if $y' \in \mathfrak{B}_h$ we defined $\bar{y}' : (-\infty, T) \to X$ as the extension of y' to $(-\infty, T]$ such that $y'(t) = \varphi$. We defined $\tilde{y}' : (-\infty, T) \to X$ such that $\tilde{y}' = y' + x'$ where $x' : (-\infty, T) \to X$ is the extension of $\varphi \in \mathfrak{B}_h$ such that $x'(t) = \varphi(0)$ for $t \in [0, T]$.

Now we introduce the following assumptions.

(H_1) $f : J \times \mathfrak{B}_h \times \mathfrak{B}_h \times \mathfrak{B}_h \to X$ is jointly continuous function and there exist positive constants L_{f1}, L_{f1} and L_{f3} such that

$$\begin{aligned} &\|f(t, \psi_1, \varphi_1, \chi_1) - f(t, \psi_2, \varphi_2, \chi_2)\|_X \\ &\leq L_{f1}\|\psi_1 - \psi_2\|_{\mathfrak{B}_h} + L_{f2}\|\varphi_1 - \varphi_2\|_{\mathfrak{B}_h} \\ &\quad + L_{f3}\|\chi_1 - \chi_2\|_{\mathfrak{B}_h}, \ \forall\ \psi_i, \varphi_i, \chi_i \in \mathfrak{B}_h. \end{aligned}$$

(H_2) f is continuous and there exist positive constants M_1, M_2 and M_3 such that

$$\|f(t,\psi,\varphi,\chi)\|_X \leq M_1\|\psi\|_{\mathfrak{B}_h} + M_2\|\varphi\|_{\mathfrak{B}_h} + M_3\|\chi\|_{\mathfrak{B}_h}, \ \ \forall\psi,\varphi,\chi \in \mathfrak{B}_h.$$

(H_3) The functions u, v are continuous and there are positive constants L_u, L_v such that

$$\|u(x) - u(y)\|_X \leq L_u\|x - y\|_X$$

and

$$\|v(x) - v(y)\|_X \leq L_v\|x - y\|_X$$

for all $x, y \in X$.

(H_4) The functions u, v are continuous and there are positive constants M_u, M_v such that

$$\|u(y)\|_X \leq M_u\|y\|_X; \ \|v(y)\|_X \leq M_v\|y\|_X, \ \ \forall x, y \in X.$$

(H_5) The functions g_i, q_i are continuous and there are positive constants L_{g_i}, L_{q_i} such that

$$\|g_i(t,x) - g_i(t,y)\|_X \leq L_{g_i}\|x - y\|_X$$

and

$$;\|q_i(t,x) - q_i(t,y)\|_X \leq L_{q_i}\|x - y\|_X$$

for all $x, y \in X, t \in (t_i, s_i]$ and each $i = 1, 2, \cdots, N$.

(H_6) The functions g_i, q_i are continuous and there are positive constants M_5, M_6 such that

$$\|g_i(t,y)\|_X \leq M_5\|y\|_X \text{ and } \|q_i(t,y)\|_X \leq M_6\|y\|_X$$

for all $x, y \in X, t \in (t_i, s_i]$ and each $i = 1, 2, \cdots, N$.

Theorem 3.1. *Assume the condition* $(H_1), (H_3)$ *and* (H_5) *are satisfied and constant*

$$\begin{aligned}\Delta = \max\Big\{&(L_u + TL_v + K_b\frac{T^2}{2}(L_{f_1} + L_{f_2}A^* + L_{f_3}B^*)),\\ &(L_{g_i} + TL_{q_i} + K_b\frac{T^2}{2}(L_{f1} + L_{f_2}A^* + L_{f_3}B^*))\Big\}\\ &< 1, \ \textit{for } i = 1, \cdots, N.\end{aligned}$$

Then there exists a unique solution $y(t)$ *of the problem* (1.1)-(1.3) *on* J.

Proof. Let $\bar{\phi}$ and $\bar{\varphi} : (-\infty, T) \to X$ be the extensions of ϕ and φ to $(-\infty, T]$, respectively, such that $\bar{\phi(t)} = \phi(0), \varphi(0) = \varphi(0)$ on J.

Consider the space $\mathfrak{B}_h''' = \{y \in \mathfrak{B}_h'' : y(0) = \phi(0), y'(0) = \varphi(0)\}$ and $y(t) = \phi(t), y'(t) = \varphi(t)$ for $t \in (-\infty, 0\ ]$ endowed with the uniform convergence topology. Let us consider an operator $P : \mathfrak{B}_h''' \to \mathfrak{B}_h'''$ defined as $Py(t) = g_i(t, \bar{y}(t))$ for $t \in (t_i, s_i]$ and

$$Py(t) = \begin{cases} \phi(0) - u(\bar{y}) + (\varphi(0) - v(\bar{y}))t \\ \quad + \int_0^t (t-s) f\left(s, \bar{y}_{\rho(s,\bar{y}_s)}, A\bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)}\right) ds, \quad t \in [0, t_1], \\ \\ g_i\left(s_i, \bar{y}\left(s_i\right)\right) + q_i\left(s_i, \bar{y}\left(s_i\right)\right) t \\ \quad + \int_{s_i}^t (t-s) f\left(s, \bar{y}_{\rho(s,\bar{y}_s)}, A\bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)}\right) ds, \quad t \in [s_i, t_{i+1}], \end{cases} \tag{3.1}$$

where $\bar{y} : (-\infty, T] \to X$ is such that $\bar{y(0)} = \phi, y'(0) = \varphi$ and $\bar{y} = y$ on J. Then it is obvious that P is well defined. Now, we show that the operator P has a fixed point. Let $y(t), y^*(t) \in \mathfrak{B}_h'''$ and $t \in [0, t_1]$, we have

$$\begin{aligned} \|Py - Py^*\|_X \leq& \|u(\bar{y}) - u\left(\bar{y}^*\right)\|_X + T\left\|v(\bar{y}) - v\left(\bar{y}^*\right)\right\|_X \\ &+ \int_0^t (t-s) \Big\| f\Big(s, \bar{y}_{\rho(s,\bar{y}_s)}, A\bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)}\Big) \\ &- f\Big(s, \bar{y}^*_{\rho(s,\bar{y}^*_s)}, A\bar{y}^*_{\rho(s,\bar{y}^*_s)}, B\bar{y}^*_{\rho(s,\bar{y}^*_s)}\Big)\Big\|_X ds \\ \leq& \left(L_u + TL_v + K_b \frac{T^2}{2}\left(L_{f1} + L_{f2}A^* + L_{f3}B^*\right)\right)\|y - y^*\|_{\mathfrak{B}_h'''}. \end{aligned}$$

For $t \in [s_i, t_{i+1}]$, we have

$$\begin{aligned} \|Py - Py^*\|_X \leq& \left\|g_i\left(s_i, \bar{y}\left(s_i\right)\right) - g_i\left(s_i, \bar{y}^*\left(s_i\right)\right)\right\|_X \\ &+ \left\|q_i\left(s_i, \bar{y}\left(s_i\right)\right) - q_i\left(s_i, \bar{y}^*\left(s_i\right)\right)\right\|_X T \\ &+ \int_{s_i}^t (t-s) \Big\| f\Big(s, \bar{y}_{\rho(s,\bar{y}_s}, A\bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)}\Big) \\ &- f\Big(s, \bar{y}^*_{\rho(s,\bar{y}^*_s)}, A\bar{y}^*_{\rho(s,\bar{y}^*_s)}, B\bar{y}^*_{\rho(s,\bar{y}^*_s)}\Big)\Big\|_X ds \\ \leq& \left(L_{g_i} + TL_{q_i} + K_b \frac{T^2}{2}\left(L_{f1} + L_{f2}A^* + L_{f3}B^*\right)\right)\|y - y^*\|_{\mathfrak{B}_h'''}. \end{aligned}$$

For $t \in (t_j, s_j]$, we get

$$\|Py - Py^*\|_X \leq L_{g_j}\|y - y^*\|_{\mathfrak{B}_h'''}, \quad j = 1, 2, \cdots, N.$$

Gathering above results, we obtain

$$\|Py - Py^*\|_X \leq \Delta\|y - y^*\|_{\mathfrak{B}_h'''}.$$

Since $\Delta < 1$, which implies that P is a contraction map and there exists a unique fixed point which is the solution of problem (1.1)-(1.3) on J. This completes the proof. □

Theorem 3.2. *Let the assumptions* $(H_2),(H_4)$ *and* (H_6) *are satisfied. Then the system* (1.1)-(1.3) *has at least one solution* $y(t)$ *on* J.

Proof. Consider the operator $P:\mathfrak{B}_h''' \to \mathfrak{B}_h'''$, defined by (3.1) in Theorem 3.1. We shall show P has a fixed point in $\mathfrak{B}_h'''$. First, we shall show that P is continuous, so we consider a sequence $y^n \to y$ in $\mathfrak{B}_h'''$, then for $[0,t_1]$

$$\begin{aligned}\|P(y^n)-P(y)\|_X \leq{}& \|u(\bar{y}^n)-u(\bar{y})\|_X + T\|v(\bar{y}^n)-v(\bar{y})\|_X \\ &+ \int_0^t (t-s)\Big\|f\Big(s,\bar{y}^n_{\rho(s,y^n{}_s)},A\bar{y}^n_{\rho(s,y^n{}_s)},B\bar{y}^n_{\rho(s,y^n{}_s)}\Big) \\ &- f\Big(s,\bar{y}_{\rho(s,\bar{y}_s)},A\bar{y}_{\rho(s,\bar{y}_s)},B\bar{y}_{\rho(s,\bar{y}_s)}\Big)\Big\|_X ds.\end{aligned}$$

For $t\in[s_i,t_{i+1}]$, we have

$$\begin{aligned}\|P(y^n)-P(y)\|_X \leq{}& \|g_i(s_i,\overline{y}^n(s_i))-g_i(s_i,\bar{y}(s_i))\|_X \\ &+ T\|q_i(s_i,\overline{y}^n(s_i))-q_i(s_i,\bar{y}(s_i))\|_X \\ &+ \int_{s_i}^t (t-s)\Big\|f\Big(s,\bar{y}^n_{\rho(s,\bar{y}^n_s)},A\bar{y}^n_{\rho(s,\bar{y}^n_s)},B\bar{y}^n_{\rho(s,\bar{y}^n_s)}\Big) \\ &- f\Big(s,\bar{y}_{\rho(s,\bar{y}_s)},A\bar{y}_{\rho(s,\bar{y}_s)},B\bar{y}_{\rho(s,\bar{y}_s)}\Big)\Big\|_X ds.\end{aligned}$$

Since f,u,v,g_i and q_i are continuous functions, we have

$$\|P(y^n)-P(y)\|_X \to 0, \text{ as } n\to\infty$$

which show that P is continuous. Let $B_r=\{y\in\mathfrak{B}_h''' : \|y\|_X \leq r\}$ be a closed bounded and convex subset of $\mathfrak{B}_h'''$. Now, it is easy to prove that P maps bounded set into bounded set in B_r. To do this we have for $[0,t_1]$

$$\begin{aligned}\|P(y)(t)\|_X \leq{}& \|\phi(0)\|+\|u(\bar{y})\|+T(\|\varphi(0)\|+\|v(\bar{y})\|) \\ &+ \int_0^t (t-s)f\left(s,\bar{y}_{\rho(s,\bar{y}_s)},A\bar{y}_{\rho(s,\bar{y}_s)},B\bar{y}_{\rho(s,\bar{y}_s)}\right)ds \\ \leq{}& \|\phi(0)\|+L_u r+T(\|\varphi(0)\|+L_\nu r) \\ &+ \frac{T^2}{2}(M_1+M_2A^*+M_3B^*)r^*.\end{aligned}$$

For $t\in[s_i,t_{i+1}]$, we have

$$\begin{aligned}\|P(y)(t)\|_X \leq{}& \|g_i(s_i,\bar{y}(s_i))\|_X + T\|q_i(s_i,\bar{y}(s_i))\|_X \\ &+ \int_{s_i}^t (t-s)\left\|f\left(s,\bar{y}_{\rho(s,\bar{y}_s},A\bar{y}_{\rho(s,\bar{y}_s)},B\bar{y}_{\rho(s,\bar{y}_s)}\right)\right\|_X ds \\ \leq{}& M_5 r+TM_6 r+\frac{T^2}{2}(M_1+M_2A^*+M_3B^*)r^*,\end{aligned}$$

where $r^* = \left(M_b + J^{\phi}\right) \|\phi\|_{\mathfrak{B}_h} + K_b r$. Which implies that P maps bounded set into bounded set in B_r.

Next, we shall show that P maps bounded sets into equi-continuous sets in B_r. Let $l_1, l_2 \in [0, t_1]$ with $l_1 < l_2$, we have

$$\begin{aligned}
\|(Py)(l_2) - (Py)(l_1)\|_X &\le (l_2 - l_1)\left(\|\varphi(0)\| + \|v(\bar{y})\|\right) \\
&\quad + \int_0^{l_1} (l_2 - l_1) \left\| f\left(s, \bar{y}_{\rho(s,\bar{y}_s}, A\bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)}\right)\right\|_X ds \\
&\quad + \int_{l_1}^{l_2} (l_2 - s) \left\| f\left(s, \bar{y}_{\rho(s,\bar{y}_s)}, A\bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)}\right)\right\|_X ds \\
&\le (l_2 - l_1)\left(\|\varphi(0)\| + M_v r\right) \\
&\quad + (l_2 - l_1) T \left(M_1 + M_2 A^* + M_3 B^*\right) r^* \\
&\quad + \frac{(l_2 - l_1)^2}{2} \left(M_1 + M_2 A^* + M_3 B^*\right) r^*.
\end{aligned}$$

Let $l_1, l_2 \in (s_i, t_{k+1}]$ with $l_1 < l_2, k = 1, 2, \cdots, m$. Then we have

$$\begin{aligned}
\|(Py)(l_2) - (Py)(l_1)\|_X &\le (l_2 - l_1) \left\|q_i\left(s_i, \bar{y}(s_i)\right)\right\|_X \\
&\quad \times \int_{s_i}^{l_1} (l_2 - l_1) \left\| f\left(s, \bar{y}_{\rho(s,\bar{y}_s}, A\bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)}\right)\right\|_X ds \\
&\quad + \int_{l_1}^{l_2} (l_2 - s) \left\| f\left(s, \bar{y}_{\rho(s,\bar{y}_s)}, A\bar{y}_{\rho(s,\bar{y}_s)}, B\bar{y}_{\rho(s,\bar{y}_s)}\right)\right\|_X ds \\
&\le (l_2 - l_1) M_6 r + (l_2 - l_1) T \left(M_1 + M_2 A^* + M_3 B^*\right) r^* \\
&\quad + \frac{(l_2 - l_1)^2}{2} \left(M_1 + M_2 A^* + M_3 B^*\right) r^*.
\end{aligned}$$

Letting $l_2 \to l_1$. Then

$$\|P(y)(l_2) - P(y)(l_1)\|_X \to 0.$$

This implies that P is equi-continuous on all $t \in J$ in B_r. Thus, by Arzela-Ascoli Theorem, it follows that P is completely continuous. Therefore, by Schauder fixed point theorem, the operator P has a fixed point, which in turn implies that problem (1.1)-(1.3) has at least one solution on J. This is complete the proof of theorem. □

4. Example

Consider the following nonlinear impulsive fractional functional initial value problem:

$$D_t^{\frac{3}{2}} y(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \left[\int_{-\infty}^{s} e^{2(\nu-s)} \frac{y(v-\sigma(\|y\|))}{24} dv \right.$$
$$+ \int_0^{\xi} \cos(\gamma-\xi) \frac{y(\gamma-\sigma(\|y\|))}{25} d\gamma$$
$$\left. + \int_0^{\pi} \sin(\gamma-\xi) \frac{y(\gamma-\sigma(\|y\|))}{26} d\gamma \right] ds, \tag{4.1}$$

$$y(t) + \sum_{k=1}^{r} c_k y\left(s_k\right) = \phi(t), t \in (-\infty, 0], y \in [0, \pi], \tag{4.2}$$

$$y'(t) + \sum_{k=1}^{r} d_k y\left(s_k\right) = \psi(t), t \in (-\infty, 0], y \in [0, \pi], \tag{4.3}$$

$$y(t) = G_i(t, y); y'(t) = H_i(t, y),$$
$$t \in \left(t_i, s_i\right], (t, y) \in \cup_{i=1}^{N} \left[s_i, t_{i+1}\right] \times [0, \pi]. \tag{4.4}$$

For the phase space $\mathfrak{B}_h$, let $h(s) = e^{2s}, s < 0$. Then $l = \int_{-\infty}^{0} h(s) ds = \frac{1}{2} < \infty$, for $t \in (-\infty, 0]$ and define

$$\|\phi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta \in [s, 0]} \|\phi(\theta)\|_{L^2} ds.$$

Hence for $(t, \phi) \in [0, 1] \times \mathfrak{B}_h$, let $y : (-\infty, T] \to L^2[0, \pi]$ such that $y \in \mathfrak{B}_h$. Setting

$$\rho(t, \phi) = t - \sigma(\|\phi(0)\|), (t, \phi) \in J \times \mathfrak{B}_h,$$

then, we have

$$f(t, \phi, A\phi, B\phi)$$
$$= \int_{-\infty}^{0} e^{2(v)} \left[\frac{\phi}{24} + \int_0^{\xi} \cos(\xi-\gamma) \frac{\phi}{25} d\gamma + \int_0^{\pi} \sin(\xi-\gamma) \frac{\phi}{26} d\gamma \right] dv,$$
$$u(y) = \sum_{k=1}^{r} c_k y\left(s_k\right); \; v(y) = \sum_{k=1}^{r} d_k y\left(s_k\right),$$
$$g_i(t, y) = G_i(t, y); \; q_i(t, y) = H_i(t, y),$$

hence the above equations (4.1)-(4.4) can be written in the abstract form as (1.1)-(1.3). Further more, we can see that for $(t, \phi, A\phi, B\phi)$, $(t, \psi, A\psi, B\psi) \in J \times \mathfrak{B}_h \times \mathfrak{B}_h \times \mathfrak{B}_h$, we get

$$\begin{aligned}
&\|f(t,\phi,A\phi,B\phi)-f(t,\psi,A\psi,B\psi)\|_{L^2}\\
&\quad=\Big[\int_0^\pi\Big\{\int_{-\infty}^0 e^{2(s)}\Big\|\frac{\phi}{24}-\frac{\psi}{24}\Big\|ds\\
&\qquad+\int_{-\infty}^0 e^{2(s)}\int_0^\xi\|\cos(\gamma-\xi)\Big\|\frac{\phi}{25}-\frac{\psi}{25}\Big\|d\gamma ds\\
&\qquad+\int_{-\infty}^0 e^{2(s)}\int_0^\xi\|\sin(\gamma-\xi)\Big\|\frac{\phi}{26}-\frac{\psi}{26}\Big\|d\gamma ds\Big\}^2dy\Big]^{1/2}\Big]\\
&\quad\le\Big[\int_0^\pi\Big\{\int_{-\infty}^0 e^{2(s)}\Big\|\frac{\phi}{24}-\frac{\psi}{24}\Big\|ds+\int_{-\infty}^0 e^{2(s)}\Big\|\frac{\phi}{25}-\frac{\psi}{25}\Big\|ds\\
&\qquad+\int_{-\infty}^0 e^{2(s)}\Big\|\frac{\phi}{26}-\frac{\psi}{26}\Big\|ds\Big\}^2dy\Big]^{1/2}\\
&\quad\le\Big[\int_0^\pi\Big\{\frac{1}{24}\int_{-\infty}^0 e^{2(s)}\sup\|\phi-\psi\|ds\\
&\qquad+\frac{1}{25}\int_{-\infty}^0 e^{2(s)}\sup\|\phi-\psi\|ds\\
&\qquad+\frac{1}{26}\int_{-\infty}^0 e^{2(s)}\sup\|\phi-\psi\|ds\Big\}^2dy\Big]^{1/2}\\
&\quad\le\frac{\sqrt{\pi}}{24}\|\phi-\psi\|+\frac{\sqrt{\pi}}{25}\|\phi-\psi\|+\frac{\sqrt{\pi}}{26}\|\phi-\psi\|.
\end{aligned}$$

Hence the function f satisfies (H_1). Similarly we can show that the functions g_i, q_i, u, v satisfy $(H_3)\,,(H_5)$. All the condition of theorem 3.1 have fulfilled so we deduced that the system (4.1)-(4.4) has a unique solution on $[0,1]$.

5. Conclusion

In this work, we have examined the existence and uniqueness of solutions for a class of not instantaneous impulsive problems of nonlinear fractional functional Volterra-Fredholm integro-differential equations by means of the Schauder fixed point theorem and Arzela-Ascoli theorem.

The problem considered in this paper can be generalized to a higher dimension involving a general formulation of fractional derivative with respect to another function. Also, study nonlinear fractional systems of Volterra-Fredholm integro-differential equations with nonlocal conditions is a direction which we are working on.

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