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ON α-GERAGHTY CONTRACTIVE MAPPINGS IN BIPOLAR METRIC SPACES

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Abstract. In this paper, we introduce the notion of α-Geraghty contractive type covariant and contravariant mappings in the bipolar metric spaces. In addition, we prove some fixed point theorems, which give existence and uniqueness of fixed point, for α-Geraghty contractive type covariant and contravariant mappings in complete bipolar metric spaces. Finally, we show some examples to support our main results.

1. Introduction

The Banach's fixed point theory is very important in nonlinear analysis. Many authors proved various fixed point theorems for such mappings in complete metric spaces. Geraghty [6] studied a generalized of Banach contraction

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principle. Samet et al. [21] introduced and studied new concepts, called contractive and α-admissible mapping. Many researchers have focused to study such type mappings which extended the mentioned mappings to various metric spaces as cone, generalized, modular, e.g. [2, 3, 4, 5, 8, 9, 14, 15, 16, 17, 22].

Mutlu and Gürdal [12] added a new one to them and introduced bipolar metric spaces. Moreover, they proved some fixed point theorems as Banach and Kannan in such spaces. Afterward, Mutlu, Özkan and Gürdal extended the certain coupled fixed point theorems, previously introduced in metric spaces as standard, cone and modular, to bipolar metric spaces [1, 13, 20].

2. Preliminaries

In this paper, $\mathbb{R}^+$ and $\mathbb{N}$ symbolize the set of all non-negative real numbers and the set of positive integers, respectively.

Definition 2.1. ([12]) Let $\Gamma, \Upsilon \neq \emptyset$ and $\Lambda : \Gamma \times \Upsilon \to \mathbb{R}^+$ be a function. Λ is called a bipolar metric on pair (Γ, Υ), if the following properties are satisfied

(B0) if $\Lambda(\eta, \mu) = 0$, then $\eta = \mu$;
(B1) if $\eta = \mu$, then $\Lambda(\eta, \mu) = 0$;
(B2) if $\eta, \mu \in \Gamma \cap \Upsilon$, then $\Lambda(\eta, \mu) = \Lambda(\mu, \eta)$;
(B3) $\Lambda(\eta_1, \mu_2) \leq \Lambda(\eta_1, \mu_1) + \Lambda(\eta_2, \mu_1) + \Lambda(\eta_2, \mu_2)$
for all (η, μ), (η_1, μ_1), $(\eta_2, \mu_2) \in \Gamma \times \Upsilon$.

Then the triple $(\Gamma, \Upsilon, \Lambda)$ is called a bipolar metric space.

Definition 2.2. ([12]) Let (Γ_1, Υ_1) and (Γ_2, Υ_2) be pairs of sets and given a function $f : \Gamma_1 \cup \Upsilon_1 \to \Gamma_2 \cup \Upsilon_2$.

(i) If $f(\Gamma_1) \subseteq \Gamma_2$ and $f(\Upsilon_1) \subseteq \Upsilon_2$, then f is called a covariant map from (Γ_1, Υ_1) to (Γ_2, Υ_2) and denoted this with $f : (\Gamma_1, \Upsilon_1) \rightrightarrows (\Gamma_2, \Upsilon_2)$.
(ii) If $f(\Gamma_1) \subseteq \Upsilon_2$ and $f(\Upsilon_1) \subseteq \Gamma_2$, then f is called a contravariant map from (Γ_1, Υ_1) to (Γ_2, Υ_2) and denoted this $f : (\Gamma_1, \Upsilon_1) \Rrightarrow (\Gamma_2, \Upsilon_2)$.

If Λ_1 and Λ_2 are bipolar metrics on (Γ_1, Υ_1) and (Γ_2, Υ_2), respectively, we also use the notations.

$$f : (\Gamma_1, \Upsilon_1, \Lambda_1) \rightrightarrows (\Gamma_2, \Upsilon_2, \Lambda_2) \quad \text{and} \quad f : (\Gamma_1, \Upsilon_1, \Lambda_1) \Rrightarrow (\Gamma_2, \Upsilon_2, \Lambda_2).$$

Definition 2.3. ([12]) Let $(\Gamma, \Upsilon, \Lambda)$ be a bipolar metric space.

(i) A point $z \in \Gamma \cup \Upsilon$ is called a left point if $z \in \Gamma$, a right point if $z \in \Upsilon$ and a central point if it is both left and right point.
(ii) A sequence $\{\eta_n\}$ on the set Γ is called a left sequence and a sequence $\{\mu_n\}$ on Υ is called a right sequence. In a bipolar metric space, a left or a right sequence is called simply a sequence.

(iii) A sequence $\{u_n\}$ is called convergent to a point z, if $\{u_n\}$ is a left sequence, z is a right point and $\lim_{n\to\infty} \Lambda(u_n, z) = 0$ or $\{u_n\}$ is a right sequence, z is a left point and $\lim_{n\to\infty} \Lambda(z, u_n) = 0$. A bisequence $\{\eta_n, \mu_n\}$ on $(\Gamma, \Upsilon, \Lambda)$ is a sequence on the set $\Gamma \times \Upsilon$. If the sequences $\{\eta_n\}$ and $\{\mu_n\}$ are convergent, then the bisequence $\{\eta_n, \mu_n\}$ is called convergent, and if $\{\eta_n\}$ and $\{\mu_n\}$ converge to a common point, then $\{\eta_n, \mu_n\}$ is called biconvergent.

(iv) $\{\eta_n, \mu_n\}$ is a Cauchy bisequence, if $\lim_{n,m\to\infty} \Lambda(\eta_n, \mu_m) = 0$. In a bipolar metric space, every convergent Cauchy bisequence is biconvergent.

(v) A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Definition 2.4. ([12]) Let $(\Gamma_1, \Upsilon_1, \Lambda_1)$ and $(\Gamma_2, \Upsilon_2, \Lambda_2)$ be bipolar metric spaces.

(i) A map $f : (\Gamma_1, \Upsilon_1, \Lambda_1) \rightrightarrows (\Gamma_2, \Upsilon_2, \Lambda_2)$ is called left-continuous at a point $\eta_0 \in \Gamma_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\Lambda_1(\eta_0, \mu) < \delta, \Lambda_2(f(\eta_0), f(\mu)) < \varepsilon$ as $\mu \in \Upsilon_1$.

(ii) A map $f : (\Gamma_1, \Upsilon_1, \Lambda_1) \rightrightarrows (\Gamma_2, \Upsilon_2, \Lambda_2)$ is called right-continuous at a point $\mu_0 \in \Upsilon_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\Lambda_1(\eta, \mu_0) < \delta, \Lambda_2(f(\eta), f(\mu_0)) < \varepsilon$ as $\eta \in \Gamma_1$.

(iii) A map f is called continuous, if it is left-continuous at each point $\eta \in \Gamma_1$ and right-continuous at each point $\mu \in \Upsilon_1$.

(iv) A contravariant map $f : (\Gamma_1, \Upsilon_1, \Lambda_1) \Rrightarrow (\Gamma_2, \Upsilon_2, \Lambda_2)$ is called continuous if it is continuous as a covariant map $f : (\Gamma_1, \Upsilon_1, \Lambda_1) \rightrightarrows (\Upsilon_2, \Gamma_2, \Lambda_2)$.

From the definition we know that a covariant or a contravariant map f from $(\Gamma_1, \Upsilon_1, \Lambda_1)$ to $(\Gamma_2, \Upsilon_2, \Lambda_2)$ is continuous if and only if

$$u_n \to v \text{ on } (\Gamma_1, \Upsilon_1, \Lambda_1) \Rightarrow \ f(u_n) \ \to f(v) \text{ on } (\Gamma_2, \Upsilon_2, \Lambda_2).$$

Definition 2.5. ([13]) Let $(\Gamma, \Upsilon, \Lambda)$ be a bipolar metric space, $u \in \Gamma$, $r \in \Upsilon$ and $F : (\Gamma \times \Upsilon, \Upsilon \times \Gamma) \rightrightarrows (\Gamma, \Upsilon)$ be a covariant mapping. (u, r) is said to be a coupled fixed point of F if

$$F(u, r) = u \quad \text{and} \quad F(r, u) = r.$$

Lemma 2.6. ([7]) *Let $F : (\Gamma \times \Upsilon, \Upsilon \times \Gamma) \rightrightarrows (\Gamma, \Upsilon)$ be a covariant mapping. If we define the covariant mapping $T : (\Gamma \times \Upsilon, \Upsilon \times \Gamma) \rightrightarrows (\Gamma \times \Upsilon, \Upsilon \times \Gamma)$ with*

$$T(\eta, \mu) = (F(\eta, \mu), F(\mu, \eta)) \text{ for all } (\eta, \mu) \in \Gamma \times \Upsilon, \tag{2.1}$$

then (η, μ) is a coupled fixed point of F if and only if (η, μ) is a fixed point of T.

Definition 2.7. ([21]) For a nonempty set Γ, let $T : \Gamma \to \Gamma$ and $\alpha : \Gamma \times \Gamma \to [0, \infty)$ be given mappings. We say that T is α-admissible, if for all $\eta, \mu \in \Gamma$ we have $\alpha(\eta, \mu) \geq 1$ implies $\alpha(T\eta, T\mu) \geq 1$.

Definition 2.8. ([7]) Let $T : (\Gamma, \Upsilon) \rightrightarrows (\Gamma, \Upsilon)$ and $\alpha : \Gamma \times \Upsilon \to [0, +\infty)$. Then T is called α-admissible (covariant) if for $\alpha(\eta, \mu) \geq 1$,

$$\alpha(T\eta, T\mu) \geq 1 \text{ for all } \eta \in \Gamma \text{ and } \mu \in \Upsilon.$$

Definition 2.9. ([7]) Let $T : (\Gamma, \Upsilon) \Rrightarrow (\Gamma, \Upsilon)$ and $\alpha : \Gamma \times \Upsilon \to [0, +\infty)$. Then T is called α-admissible (contravariant) if for $\alpha(\eta, \mu) \geq 1$,

$$\alpha(T\mu, T\eta) \geq 1 \text{ for all } \eta \in \Gamma \text{ and } \mu \in \Upsilon.$$

We denote by Ψ the family of all functions $\gamma : [0, +\infty) \to [0, 1)$ such that, for any bounded sequence $\{s_n\}$ of positive reals, $\gamma(s_n) \to 1$ implies $s_n \to 0$.

Let Ω be the family of functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(i) φ is nondecreasing.
(ii) $\sum_{n=1}^{\infty} \varphi^n(s) < \infty$ for all $s > 0$, where φ^n is the n-th iterate of φ.

From [18] and [19], for every function $\varphi : [0, +\infty) \to [0, +\infty)$ the following holds:

(i) If φ is nondecreasing, then for each $s > 0$,

$$\lim_{n\to\infty} \varphi^n(s) = 0 \Rightarrow \varphi(s) < s \Rightarrow \varphi(0) = 0.$$

(ii) If $\varphi \in \Omega$, then for each $s > 0$, $\varphi(s) < s$ and $\varphi(0) = 0$.

3. Main Results

Definition 3.1. Let $(\Gamma, \Upsilon, \Lambda)$ be a bipolar metric space and $T : (\Gamma, \Upsilon) \rightrightarrows (\Gamma, \Upsilon)$ be a covariant mapping. If there exist $\alpha : \Gamma \times \Upsilon \to [0, +\infty)$, $\gamma \in \Psi$ and $\varphi \in \Omega$ with $\alpha(\eta, \mu) \geq 1$ such that

$$\varphi(\Lambda(T\eta, T\mu)) \leq \gamma(\varphi(\Lambda(\eta, \mu)))\, \varphi(\Lambda(\eta, \mu)) \text{ for all } \eta \in \Gamma \text{ and } \mu \in \Upsilon, \quad (3.1)$$

then T is called an α-Geraghty contractive covariant mapping.

Definition 3.2. Let $(\Gamma, \Upsilon, \Lambda)$ be a bipolar metric space and $T : (\Gamma, \Upsilon) \Rrightarrow (\Gamma, \Upsilon)$ be a contravariant mapping. If there exist $\alpha : \Gamma \times \Upsilon \to [0, +\infty)$, $\gamma \in \Psi$ and $\varphi \in \Omega$ with $\alpha(\eta, \mu) \geq 1$ such that

$$\varphi(\Lambda(T\eta, T\mu)) \leq \gamma(\varphi(\Lambda(\eta, \mu)))\, \varphi(\Lambda(\eta, \mu)) \text{ for all } \eta \in \Gamma \text{ and } \mu \in \Upsilon, \quad (3.2)$$

then T is called an α-Geraghty contractive contravariant mapping.

Theorem 3.3. *Let* $(\Gamma, \Upsilon, \Lambda)$ *be a complete bipolar metric space and* $T : (\Gamma, \Upsilon) \rightrightarrows (\Gamma, \Upsilon)$ *be an* α*-Geraghty contractive covariant mapping. Suppose that the following conditions are satisfied:*

(i) T *is* α*-admissible*;
(ii) *There exist* $\eta_0 \in \Gamma$, $\mu_0 \in \Upsilon$ *such that* $\alpha(\eta_0, \mu_0) \geq 1$ *and* $\alpha(\eta_0, T\mu_0) \geq 1$;
(iii) T *is continuous.*

Then T *has a fixed point.*

Proof. Let $\eta_0 \in \Gamma$, $\mu_0 \in \Upsilon$ such that $\alpha(\eta_0, T\mu_0) \geq 1$. We define the bisequence $\{\eta_n, \mu_n\}$ by $\eta_{n+1} = T\eta_n$, $\mu_{n+1} = T\mu_n$ for all $n \in \mathbb{N}$. Because T is α-admissible, using condition (ii), we get

$$\begin{aligned}
&\alpha(\eta_0, \mu_0) \geq 1 \Rightarrow \alpha(T\eta_0, T\mu_0) \geq 1, \\
&\alpha(\eta_0, \mu_1) = \alpha(\eta_0, T\mu_0) \geq 1 \Rightarrow \alpha(T\eta_0, T\mu_1) = \alpha(\eta_1, \mu_2) \geq 1, \\
&\alpha(\eta_1, \mu_1) = \alpha(T\eta_0, T\mu_0) \geq 1 \Rightarrow \alpha(T\eta_1, T\mu_1) = \alpha(\eta_2, \mu_2) \geq 1, \\
&\alpha(\eta_1, \mu_2) = \alpha(\eta_1, T\mu_1) \geq 1 \Rightarrow \alpha(T\eta_1, T\mu_2) = \alpha(\eta_2, \mu_3) \geq 1, \\
&\alpha(\eta_2, \mu_2) = \alpha(T\eta_1, T\mu_1) \geq 1 \Rightarrow \alpha(T\eta_2, T\mu_2) = \alpha(\eta_3, \mu_3) \geq 1.
\end{aligned}$$

Similarly, we obtain

$$\alpha(\eta_n, \mu_{n+1}) \geq 1 \text{ and } \alpha(\eta_{n+1}, \mu_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

From equations (3.1) and (3.3), we can find that for $\eta = \eta_{n-1}$, $\mu = \mu_n$,

$$\begin{aligned}
\varphi(\Lambda(\eta_n, \mu_{n+1})) &= \varphi(\Lambda(T\eta_{n-1}, T\mu_n)) \\
&\leq \gamma(\varphi(\Lambda(\eta_{n-1}, \mu_n)))\varphi(\Lambda(\eta_{n-1}, \mu_n)) \\
&< \varphi(\Lambda(\eta_{n-1}, \mu_n))
\end{aligned} \tag{3.4}$$

and for $\eta = \eta_n$, $\mu = \mu_n$,

$$\begin{aligned}
\varphi(\Lambda(\eta_{n+1}, \mu_{n+1})) &= \varphi(\Lambda(T\eta_n, T\mu_n)) \\
&\leq \gamma(\varphi(\Lambda(\eta_n, \mu_n)))\varphi(\Lambda(\eta_n, \mu_n)) \\
&< \varphi(\Lambda(\eta_n, \mu_n))
\end{aligned} \tag{3.5}$$

for $n \geq 1$. So, by the properties of φ, we conclude that

$$\Lambda(\eta_n, \mu_{n+1}) < \Lambda(\eta_{n-1}, \mu_n) \quad \text{for all } n \in \mathbb{N} \tag{3.6}$$

and

$$\Lambda(\eta_{n+1}, \mu_{n+1}) < \Lambda(\eta_n, \mu_n) \quad \text{for all } n \in \mathbb{N}. \tag{3.7}$$

Therefore, the sequence $\{\Lambda(\eta_n, \mu_{n+1})\}$ and $\{\Lambda(\eta_n, \mu_n)\}$ are nonnegative and nonincreasing sequence. Consequently, there exists $\tau_1 \geq 0$ and $\tau_2 \geq 0$ such that $\lim_{n\to\infty} \Lambda(\eta_n, \mu_{n+1}) = \tau_1$ and $\lim_{n\to\infty} \Lambda(\eta_n, \mu_n) = \tau_2$. We claim that

$\tau_1 = \tau_2 = 0$. Suppose, on the contrary, that $\tau_1 > 0$ and $\tau_2 > 0$. Then, due to (3.4) and (3.5), we have

$$\frac{\varphi(\Lambda(\eta_n, \mu_{n+1}))}{\varphi(\Lambda(\eta_{n-1}, \mu_n))} \le \gamma(\varphi(\Lambda(\eta_{n-1}, \mu_n))) < 1$$

and

$$\frac{\varphi(\Lambda(\eta_{n+1}, \mu_{n+1}))}{\varphi(\Lambda(\eta_n, \mu_n))} \le \gamma(\varphi(\Lambda(\eta_n, \mu_n))) < 1.$$

In what follows that

$$\lim_{n\to\infty} \gamma(\varphi(\Lambda(\eta_{n-1}, \mu_n))) = 1 = \lim_{n\to\infty} \gamma(\varphi(\Lambda(\eta_n, \mu_n))).$$

Owing to the fact that $\varphi \in \Omega$, we obtain

$$\lim_{n\to\infty} \varphi(\Lambda(\eta_{n-1}, \mu_n)) = 0 = \lim_{n\to\infty} \varphi(\Lambda(\eta_n, \mu_n)), \tag{3.8}$$

which yields that

$$\tau_1 = \lim_{n\to\infty} \Lambda(\eta_{n-1}, \mu_n) = 0 = \lim_{n\to\infty} \Lambda(\eta_n, \mu_n) = \tau_2. \tag{3.9}$$

We assert that $\{\eta_n, \mu_n\}$ is a Cauchy sequence. Suppose, on the contrary, that we have

$$\varepsilon = \lim_{n,m\to\infty} \Lambda(\eta_n, \mu_m) > 0. \tag{3.10}$$

Now, for $n, m \in \mathbb{N}$ with $m > n$, by applying the property (B3), we get

$$\Lambda(\eta_n, \mu_m) \le \Lambda(\eta_n, \mu_n) + \Lambda(\eta_m, \mu_n) + \Lambda(\eta_m, \mu_m). \tag{3.11}$$

Combining (3.1) and (3.11) with the properties of φ, we obtain

$$\begin{aligned} \varphi(\Lambda(\eta_n, \mu_m)) &\le \varphi(\Lambda(\eta_n, \mu_n) + \Lambda(\eta_m, \mu_n) + \Lambda(\eta_m, \mu_m)) \\ &\le \varphi(\Lambda(\eta_n, \mu_n)) + \varphi(\Lambda(T\eta_{m-1}, T\mu_{n-1})) + \varphi(\Lambda(\eta_m, \mu_m)) \\ &\le \varphi(\Lambda(\eta_n, \mu_n)) + \gamma(\varphi(\Lambda(\eta_{m-1}, \mu_{n-1})))\varphi(\Lambda(\eta_{m-1}, \mu_{n-1})) \\ &\quad + \varphi(\Lambda(\eta_m, \mu_m)). \end{aligned} \tag{3.12}$$

Together with (3.8) and (3.12), we deduce that

$$\begin{aligned} \lim_{n,m\to\infty} \varphi(\Lambda(\eta_n, \mu_m)) \le &\lim_{n,m\to\infty} \gamma(\varphi(\Lambda(\eta_{m-1}, \mu_{n-1}))) \\ &\times \lim_{n,m\to\infty} \varphi(\Lambda(\eta_{n-1}, \mu_{m-1})). \end{aligned}$$

Therefore, using (3.10), we obtain

$$1 \le \lim_{n,m\to\infty} \gamma(\varphi(\Lambda(\eta_{m-1}, \mu_{n-1}))),$$

which implies $\lim_{n,m\to\infty} \gamma(\varphi(\Lambda(\eta_{m-1}, \mu_{n-1}))) = 1$. Consequently, we have $\lim_{n,m\to\infty} \Lambda(\eta_{m-1}, \mu_{n-1}) = 0$. It is a contradiction. Since $(\Gamma, \Upsilon, \Lambda)$ is complete bipolar metric space, $\{\eta_n, \mu_n\}$ biconverges. That is, there exists $z \in \Gamma \cap \Upsilon$ such that $\eta_n \to z$ and $\mu_n \to z$ as $n \to \infty$. Since the covariant map T is continuous, $\mu_n \to z$ implies that $\mu_{n+1} = T\mu_n \to Tz$ and $\eta_n \to z$ implies that $\eta_{n+1} = T\eta_n \to Tz$. By uniqueness of the limit, we obtain immediately that $Tz = z$. Therefore, z is a fixed point of T. □

Theorem 3.4. *Let $(\Gamma, \Upsilon, \Lambda)$ be a complete bipolar metric space and $T : (\Gamma, \Upsilon) \rightrightarrows (\Gamma, \Upsilon)$ be an α-Geraghty contractive contravariant mapping. Suppose that the following conditions are satisfied*

(i) *T is α-admissible*;
(ii) *There exist $\eta_0 \in \Gamma$ such that $\alpha(\eta_0, T\eta_0) \geq 1$;*
(iii) *T is continuous.*

Then T has a fixed point.

Proof. Let $\eta_0 \in \Gamma$ such that $\alpha(\eta_0, T\eta_0) \geq 1$. We define the bisequence $\{\eta_n, \mu_n\}$ by $\eta_{n+1} = T\mu_n$, $\mu_n = T\eta_n$ for all $n \in \mathbb{N}$. Because T is α-admissible, using condition (ii), we get

$$\begin{aligned}
&\alpha(\eta_0, \mu_0) = \alpha(\eta_0, T\eta_0) \geq 1 \Rightarrow \alpha(T\mu_0, T\eta_0) = \alpha(\eta_1, \mu_0) \geq 1,\\
&\alpha(\eta_1, \mu_0) \geq 1 \Rightarrow \alpha(T\mu_0, T\eta_1) = \alpha(\eta_1, \mu_1) \geq 1,\\
&\alpha(\eta_1, \mu_1) \geq 1 \Rightarrow \alpha(T\mu_1, T\eta_1) = \alpha(\eta_2, \mu_1) \geq 1,\\
&\alpha(\eta_2, \mu_1) \geq 1 \Rightarrow \alpha(T\mu_1, T\eta_2) = \alpha(\eta_2, \mu_2) \geq 1.
\end{aligned}$$

Similarly, we obtain

$$\alpha(\eta_n, \mu_n) \geq 1 \text{ and } \alpha(\eta_{n+1}, \mu_n) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{3.13}$$

From equations (3.1) and (3.3), we can find that for $\eta = \eta_n$, $\mu = \mu_{n-1}$,

$$\begin{aligned}
\varphi(\Lambda(\eta_n, \mu_n)) &= \varphi(\Lambda(T\mu_{n-1}, T\eta_n))\\
&\leq \gamma(\varphi(\Lambda(\eta_n, \mu_{n-1})))\varphi(\Lambda(\eta_n, \mu_{n-1}))\\
&< \varphi(\Lambda(\eta_n, \mu_{n-1}))
\end{aligned} \tag{3.14}$$

and for $\eta = \eta_{n+1}$, $\mu = \mu_n$,

$$\begin{aligned}
\varphi(\Lambda(\eta_{n+1}, \mu_n)) &= \varphi(\Lambda(T\mu_n, T\eta_n))\\
&\leq \gamma(\varphi(\Lambda(\eta_n, \mu_n)))\varphi(\Lambda(\eta_n, \mu_n))\\
&< \varphi(\Lambda(\eta_n, \mu_n))
\end{aligned} \tag{3.15}$$

for $n \geq 1$. So, by the properties of φ, we conclude that

$$\Lambda(\eta_n, \mu_n) < \Lambda(\eta_n, \mu_{n-1}) \quad \text{for all } n \in \mathbb{N} \tag{3.16}$$

and

$$\Lambda(\eta_{n+1}, \mu_n) < \Lambda(\eta_n, \mu_n) \quad \text{for all } n \in \mathbb{N}. \tag{3.17}$$

Therefore, the sequence $\{\Lambda(\eta_n, \mu_n)\}$ and $\{\Lambda(\eta_{n+1}, \mu_n)\}$ are nonnegative and nonincreasing sequences. Consequently, there exists $\tau_1 \geq 0$ and $\tau_2 \geq 0$ such that

$$\lim_{n\to\infty} \Lambda(\eta_n, \mu_n) = \tau_1$$

and

$$\lim_{n\to\infty} \Lambda(\eta_{n+1}, \mu_n) = \tau_2.$$

We claim that $\tau_1 = \tau_2 = 0$. Suppose, on the contrary, that $\tau_1 > 0$ and $\tau_2 > 0$. Then, due to (3.14) and (3.15), we have

$$\frac{\varphi(\Lambda(\eta_n, \mu_n))}{\varphi(\Lambda(\eta_n, \mu_{n-1}))} \leq \gamma(\varphi(\Lambda(\eta_n, \mu_{n-1}))) < 1$$

and

$$\frac{\varphi(\Lambda(\eta_{n+1}, \mu_n))}{\varphi(\Lambda(\eta_n, \mu_n))} \leq \gamma(\varphi(\Lambda(\eta_n, \mu_n))) < 1.$$

In what follows that

$$\lim_{n\to\infty} \gamma(\varphi(\Lambda(\eta_n, \mu_{n-1}))) = 1 = \lim_{n\to\infty} \gamma(\varphi(\Lambda(\eta_n, \mu_n))).$$

Owing to the fact that $\varphi \in \Omega$, we obtain

$$\lim_{n\to\infty} \varphi(\Lambda(\eta_n, \mu_{n-1})) = 0 = \lim_{n\to\infty} \varphi(\Lambda(\eta_n, \mu_n)), \tag{3.18}$$

which yields that

$$\tau_1 = \lim_{n\to\infty} \Lambda(\eta_n, \mu_{n-1}) = 0 = \lim_{n\to\infty} \Lambda(\eta_n, \mu_n) = \tau_2. \tag{3.19}$$

We assert that $\{\eta_n, \mu_n\}$ is a Cauchy sequence. Suppose, on the contrary, that we have

$$\varepsilon = \lim_{n,m\to\infty} \Lambda(\eta_n, \mu_m) > 0. \tag{3.20}$$

Now, for $n, m \in \mathbb{N}$ with $m > n$, by applying the property (B3), we get

$$\Lambda(\eta_n, \mu_m) \leq \Lambda(\eta_n, \mu_n) + \Lambda(\eta_m, \mu_n) + \Lambda(\eta_m, \mu_m). \tag{3.21}$$

Combining (3.2) and (3.21) with the properties of φ, we obtain

$$\begin{aligned} \varphi(\Lambda(\eta_n, \mu_m)) &\leq \varphi(\Lambda(\eta_n, \mu_n) + \Lambda(\eta_m, \mu_n) + \Lambda(\eta_m, \mu_m)) \\ &\leq \varphi(\Lambda(\eta_n, \mu_n)) + \varphi(\Lambda(T\eta_{m-1}, T\mu_{n-1})) + \varphi(\Lambda(\eta_m, \mu_m)) \\ &\leq \varphi(\Lambda(\eta_n, \mu_n)) + \gamma(\varphi(\Lambda(\eta_{m-1}, \mu_{n-1})))\varphi(\Lambda(\eta_{m-1}, \mu_{n-1})) \\ &\quad + \varphi(\Lambda(\eta_m, \mu_m)). \end{aligned} \tag{3.22}$$

Together with (3.18) and (3.22), we deduce that

$$\begin{aligned} \lim_{n,m\to\infty} \varphi(\Lambda(\eta_n, \mu_m)) &\leq \lim_{n,m\to\infty} \gamma(\varphi(\Lambda(\eta_{m-1}, \mu_{n-1}))) \\ &\quad \times \lim_{n,m\to\infty} \varphi(\Lambda(\eta_{m-1}, \mu_{n-1})). \end{aligned}$$

Therefore, using (3.20), we obtain

$$1 \leq \lim_{n,m\to\infty} \gamma(\varphi(\Lambda(\eta_{m-1}, \mu_{n-1}))),$$

which implies

$$\lim_{n,m\to\infty} \gamma(\varphi(\Lambda(\eta_{m-1}, \mu_{n-1}))) = 1.$$

Consequently, we have

$$\lim_{n,m\to\infty} \Lambda(\eta_{m-1}, \mu_{n-1}) = 0,$$

which is a contradiction. Since $(\Gamma, \Upsilon, \Lambda)$ is a complete bipolar metric space, $\{\eta_n, \mu_n\}$ is biconvergent. That is, there exists $z \in \Gamma \cap \Upsilon$ such that $\eta_n \to z$ and $\mu_n \to z$ as $n \to \infty$. Since the contravariant map T is continuous, $\eta_n \to z$ implies that $\mu_n = T\eta_n \to Tz$ and combining this with $\mu_n \to z$ gives $Tz = z$. Therefore, z is a fixed point of T. □

Theorem 3.5. *Let $(\Gamma, \Upsilon, \Lambda)$ be a complete bipolar metric space and $T : (\Gamma, \Upsilon) \rightrightarrows (\Gamma, \Upsilon)$ be an α-Geraghty contractive covariant mapping. Suppose that the following conditions are satisfied:*

(i) *T is α-admissible;*
(ii) *There exist $\eta_0 \in \Gamma$, $\mu_0 \in \Upsilon$ such that $\alpha(\eta_0, \mu_0) \geq 1$ and $\alpha(\eta_0, T\mu_0) \geq 1$;*
(iii) *If $\{\eta_n, \mu_n\}$ is a bisequence such that $\alpha(\eta_n, \mu_n) \geq 1$ for all n and $\eta_n \to u$, $\mu_n \to u$, $u \in X \cap Y$ as $n \to \infty$, then $\alpha(u, \mu_n) \geq 1$ for all n.*

Then T has a fixed point.

Proof. Following the proof of Theorem 3.3, we get the sequence $\{\eta_n, \mu_n\}$ defined by $\eta_{n+1} = T\eta_n$, $\mu_{n+1} = T\mu_n$ for all $n \geq 0$, which is Cauchy bisequence in the complete metric space $(\Gamma, \Upsilon, \Lambda)$ and converges to some $z \in \Gamma \cap \Upsilon$ such that $\eta_n \to z$, $\mu_n \to z$ as $n \to \infty$. Using (3.3) and condition (iii), we obtain $\alpha(z, \mu_n) \geq 1$ for all $n \in \mathbb{N}$. By applying (B3), (3.1) and the above inequality, using condition (ii), we obtain

$$\begin{aligned}
\varphi(\Lambda(Tz, z)) &= \varphi(\Lambda(Tz, T\mu_n) + \Lambda(T\eta_n, T\mu_n) + \Lambda(T\eta_n, z)) \\
&\leq \varphi(\Lambda(Tz, T\mu_n)) + \varphi(\Lambda(T\eta_n, T\mu_n)) + \varphi(\Lambda(T\eta_n, z)) \\
&\leq \gamma(\varphi(\Lambda(z, \mu_n)))\varphi(\Lambda(z, \mu_n)) + \varphi(\Lambda(\eta_{n+1}, z)) \\
&\quad + \gamma(\varphi(\Lambda(\eta_n, \mu_n)))\varphi(\Lambda(\eta_n, \mu_n)) \\
&< \varphi(\Lambda(z, \mu_n)) + \varphi(\Lambda(\eta_n, \mu_n)) + \varphi(\Lambda(\eta_{n+1}, z)) \qquad (3.23) \\
&\leq \varphi(\Lambda(z, \mu_n)) + \varphi(\Lambda(\eta_n, z)) + \varphi(\Lambda(z, z) \\
&\quad + \Lambda(z, \mu_n) + \Lambda(\eta_{n+1}, z)) \\
&\leq \varphi(\Lambda(z, \mu_n)) + \varphi(\Lambda(\eta_n, z)) \\
&\quad + \varphi(\Lambda(z, \mu_n)) + \varphi(\Lambda(\eta_{n+1}, z)).
\end{aligned}$$

Taking $n \to \infty$ in above inequality and using the continuity of φ at $s = 0$, we obtain $\Lambda(Tz, z) = 0$, that is, $Tz = z$. Therefore, z is a fixed point of T. □

Theorem 3.6. *Let $(\Gamma, \Upsilon, \Lambda)$ be a complete bipolar metric space and $T : (\Gamma, \Upsilon) \rightrightarrows (\Gamma, \Upsilon)$ be an α-Geraghty contractive contravariant mapping. Suppose that the following conditions are satisfied:*

(i) *T is α-admissible*;
(ii) *There exist $\eta_0 \in \Gamma$ such that $\alpha(\eta_0, T\eta_0) \geq 1$;*
(iii) *If $\{\eta_n, \mu_n\}$ is a bisequence such that $\alpha(\eta_n, \mu_n) \geq 1$ for all n and $\eta_n \to u$, $\mu_n \to u$, $u \in X \cap Y$ as $n \to \infty$, then $\alpha(u, \mu_n) \geq 1$ for all n.*

Then T has a fixed point.

Proof. Following the proof of Theorem 3.4, we get the sequence $\{\eta_n, \mu_n\}$ defined by $\eta_{n+1} = T\mu_n$, $\mu_n = T\eta_n$, for all $n \geq 0$, which is a Cauchy bisequence in the complete metric space $(\Gamma, \Upsilon, \Lambda)$ and converges to some $z \in \Gamma \cap \Upsilon$ such that $\eta_n \to z$, $\mu_n \to z$ as $n \to \infty$. Using (3.13) and condition (iii), we obtain $\alpha(\eta_n, z) \geq 1$ for all $n \in \mathbb{N}$. By applying (B3), (3.2) and above inequality, we obtain

$$\begin{aligned}
\varphi(\Lambda(Tz, z)) &= \varphi(\Lambda(Tz, T\mu_n) + \Lambda(T\eta_n, T\mu_n) + \Lambda(T\eta_n, z)) \\
&\leq \varphi(\Lambda(Tz, T\mu_n)) + \varphi(\Lambda(T\eta_n, T\mu_n)) + \varphi(\Lambda(T\eta_n, z)) \\
&\leq \gamma(\varphi(\Lambda(z, \mu_n)))\varphi(\Lambda(z, \mu_n)) + \varphi(\Lambda(\eta_{n+1}, z)) \\
&\quad + \gamma(\varphi(\Lambda(\eta_n, \mu_n)))\varphi(\Lambda(\eta_n, \mu_n)) \\
&< \varphi(\Lambda(z, \mu_n)) + \varphi(\Lambda(\eta_n, \mu_n)) + \varphi(\Lambda(\eta_{n+1}, z)) \qquad (3.24) \\
&\leq \varphi(\Lambda(z, \mu_n)) + \varphi(\Lambda(\eta_n, z)) + \varphi(\Lambda(z, z) \\
&\quad + \Lambda(z, \mu_n) + \Lambda(\eta_{n+1}, z)) \\
&\leq \varphi(\Lambda(z, \mu_n)) + \varphi(\Lambda(\eta_n, z)) \\
&\quad + \varphi(\Lambda(z, \mu_n)) + \varphi(\Lambda(\eta_{n+1}, z)).
\end{aligned}$$

Taking $n \to \infty$ in above inequality and using the continuity of φ at $s = 0$, we obtain $\Lambda(Tz, z) = 0$, it means that $Tz = z$. Therefore, T has a fixed point. □

In the following, we give a hypothesis to obtain the uniqueness of the fixed point.

($\mathcal{H}$) There exists $a \in \Gamma \cap \Upsilon$ such that $\alpha(\eta, a) \geq 1$ and $\alpha(a, \mu) \geq 1$ for all $\eta \in \Gamma$ and $\mu \in \Upsilon$.

Theorem 3.7. *If the condition ($\mathcal{H}$) is added to the hypotheses of Theorem 3.3 or Theorem 3.5 (resp. Theorem 3.4 or Theorem 3.6), we obtain that z is a unique fixed point of the covariant mapping (resp. the contravariant mapping) T.*

Proof. We will show the uniqueness of a fixed point of the covariant mapping (resp., the contravariant mapping) T. We suppose the contrary, that is, u is an another fixed point of T with $z \neq u$. Then, from the condition $(\mathcal{H})$, there exists $a \in \Gamma \cap \Upsilon$ such that

$$\alpha(z,a) \geq 1 \quad \text{and} \quad \alpha(a,u) \geq 1. \tag{3.25}$$

Because T is α-admissible, using (3.25), we get

$$\begin{aligned} \varphi(\Lambda(z,a)) &= \varphi(\Lambda(Tz,Ta))) \\ &\leq \gamma(\varphi(\Lambda(z,a)))\varphi(\Lambda(z,a)) \\ &< \varphi(\Lambda(z,a)) \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \varphi(\Lambda(a,u)) &= \varphi(\Lambda(Ta,Tu))) \\ &\leq \gamma(\varphi(\Lambda(a,u)))\varphi(\Lambda(a,u)) \\ &< \varphi(\Lambda(a,u)), \end{aligned} \tag{3.27}$$

which is a contradiction to the uniqueness of the limit. Hence, $z = a \in \Gamma \cap \Upsilon$. Therefore, T has a unique fixed point. □

Corollary 3.8. *Let $(\Gamma, \Upsilon, \Lambda)$ be a complete bipolar metric space and $T : (\Gamma, \Upsilon) \rightrightarrows (\Gamma, \Upsilon)$ be a covariant mapping. Suppose that there exists a function $\gamma \in \Psi$ and $\varphi \in \Omega$ such that*

$$\Lambda(T\eta, T\mu) \leq \gamma(\varphi\Lambda(\eta,\mu))\varphi(\Lambda(\eta,\mu))$$

for all $\eta \in \Gamma$ and $\mu \in \Upsilon$. Then T has a unique fixed point.

Proof. We take the mapping $\alpha : \Gamma \times \Upsilon \to [0, +\infty)$ as $\alpha(\eta,\mu) = 1$ for all $\eta \in \Gamma$, $\mu \in \Upsilon$ in Theorem 3.3 and Theorem 3.7. It is obvious that all the conditions of Theorem 3.3 and Theorem 3.7 are satisfied. Then the proof is completed. □

Corollary 3.9. *Let $(\Gamma, \Upsilon, \Lambda)$ be a complete bipolar metric space and $T : (\Gamma, \Upsilon) \Rrightarrow (\Gamma, \Upsilon)$ be a contravariant mapping. Suppose that there exists a function $\gamma \in \Psi$ and $\varphi \in \Omega$ such that*

$$\Lambda(T\eta, T\mu) \leq \gamma(\varphi\Lambda(\eta,\mu))\varphi(\Lambda(\eta,\mu))$$

for all $\eta \in \Gamma$ and $\mu \in \Upsilon$. Then T has a unique fixed point.

Proof. Using similar method with proof of Corollary 3.8, we say that it suffices to take $\alpha(\eta,\mu) = 1$ for all $\eta \in \Gamma$ and $\mu \in \Upsilon$ in Theorem 3.4 and Theorem 3.7 to prove the corollary. □

Theorem 3.10. *Let* $(\Gamma, \Upsilon, \Lambda)$ *be a complete bipolar metric space and* $F : (\Gamma \times \Upsilon, \Upsilon \times \Gamma) \rightrightarrows (\Gamma, \Upsilon)$ *be a covariant mapping. We suppose that there exist functions* $\gamma \in \Psi$, $\varphi \in \Omega$ *and* $\alpha : (\Gamma \times \Upsilon) \times (\Upsilon \times \Gamma) \to [0, +\infty)$ *with* $\alpha((\eta, \mu), (u, z)) \geq 1$ *such that*

$$\varphi(\Lambda(F(\eta, \mu), F(u, z))) \leq \gamma\left(\varphi\left(\frac{\Lambda(\eta, u) + d(z, \mu)}{2}\right)\right)\varphi\left(\frac{\Lambda(\eta, u) + d(z, \mu)}{2}\right) \tag{3.28}$$

for all (η, μ), $(u, z) \in \Gamma \times \Upsilon$ *and the following conditions are satisfied:*

(i) $\alpha((\eta, \mu), (u, z)) \geq 1 \Rightarrow \alpha((F(\eta, \mu), F(\mu, \eta)), (F(u, z), F(z, u))) \geq 1$ *for all* $(\eta, \mu), (z, u) \in \Gamma \times \Upsilon$;

(ii) *There exists* $(\eta_0, \mu_0) \in \Gamma \times \Upsilon$ *such that*

$$\alpha((\eta_0, \mu_0), (F(\mu_0, \eta_0), F(\eta_0, \mu_0))) \geq 1$$

and

$$\alpha((F(\eta_0, \mu_0), F(\mu_0, \eta_0)), (\mu_0, \eta_0)) \geq 1;$$

(iii) F *is continuous.*

Then F *has a coupled fixed point, that is, there exists* $(u, z) \in \Gamma \times \Upsilon$ *such that* $u = F(u, z)$ *and* $z = F(z, u)$.

Proof. We consider the complete bipolar metric space (P, Q, Δ) where $P = \Gamma \times \Upsilon$, $Q = \Upsilon \times \Gamma$ and

$$\Delta((\eta, \mu), (u, z)) = \frac{\Lambda(\eta, u) + \Lambda(z, \mu)}{2}$$

for all $(\eta, \mu) \in P$, $(u, z) \in Q$. Using (3.28), we obtain $\alpha((\eta, \mu), (u, z)) \geq 1$ such that

$$\varphi(\Lambda(F(\eta, \mu), F(u, z)) \leq \gamma(\varphi(\Delta((\eta, \mu), (u, z)))))\varphi(\Delta((\eta, \mu), (u, z))) \tag{3.29}$$

and $\alpha((z, u), (\mu, \eta))$ such that

$$\varphi(\Lambda(F(\eta, \mu), F(u, z)) \leq \gamma(\varphi(\Delta((\eta, \mu), (u, z)))))\varphi(\Delta((\eta, \mu), (u, z))). \tag{3.30}$$

Combining (3.29) and (3.30), we obtain $\delta(\varepsilon, \sigma)$ such that

$$\varphi(\Lambda(F\varepsilon, F\sigma)) \leq \gamma(\varphi(\Delta(\varepsilon, \sigma)))\varphi(\Delta(\varepsilon, \sigma)) \tag{3.31}$$

for all $\varepsilon = (\varepsilon_1, \varepsilon_2) \in P$, $\sigma = (\sigma_1, \sigma_2) \in Q$, where $\delta : P \times Q \to [0, +\infty)$ is the function defined by

$$\delta(\varepsilon, \sigma) = \min\{\alpha((\varepsilon_1, \varepsilon_2), (\sigma_1, \sigma_2)), \alpha((\sigma_2, \sigma_1), (\varepsilon_2, \varepsilon_1))\} \tag{3.32}$$

and $T : (P, Q) \rightrightarrows (P, Q)$ is defined as (2.1). Then T is continuous and δ-φ-contractive covariant mapping. We take $\varepsilon = (\varepsilon_1, \varepsilon_2) \in P$, $\sigma = (\sigma_1, \sigma_2) \in Q$ such that $\delta(\varepsilon, \sigma) \geq 1$. Using conditions (i) and (3.28), we see that $\delta(T\varepsilon, T\sigma) \geq 1$. Hence, T is δ-admissible.

Using conditions (ii) and (3.28), we have that there exists $(\eta_0, \mu_0) \in P$ (or $(\mu_0, \eta_0) \in Q$) such that

$$\delta((\eta_0, \mu_0), T(\eta_0, \mu_0)) \geq 1 \text{ (or } \delta(T(\eta_0, \mu_0), (\mu_0, \eta_0)) \geq 1).$$

Hence, we observe that Theorem 3.3 is satisfied. Then T has a fixed point. Therefore, using Lemma 2.6, this is a coupled fixed point of F. $\square$

Example 3.11. Let $A_n(\mathbb{R})$ and $B_n(\mathbb{R})$ be the sets of all $n \times n$ upper and lower triangular matrices over $\mathbb{R}$, respectively, and let a function $\Lambda : A_n(\mathbb{R}) \times B_n(\mathbb{R}) \to \mathbb{R}^+$ be defined as

$$\Lambda(P, Q) = \sum_{i,j=1}^{n} |p_{ij} - q_{ij}|$$

for all $P = \{a_{ij}\}_{n\times n} \in A_n(\mathbb{R})$ and $Q = \{q_{ij}\}_{n\times n} \in B_n(\mathbb{R})$. Then it is apparent that $(A_n(\mathbb{R}), B_n(\mathbb{R}), \Lambda)$ is a complete bipolar metric space. We take a covariant mapping

$$T : (A_n(\mathbb{R}) \times B_n(\mathbb{R}), B_n(\mathbb{R}) \times A_n(\mathbb{R})) \rightrightarrows (A_n(\mathbb{R}), B_n(\mathbb{R})),$$

which $T(P, Q) = \left\{ \dfrac{p_{ij} + q_{ij}}{4} \right\}_{n\times n}$, where

$$P = \{p_{ij}\}_{n\times n},\ Q = \{q_{ij}\}_{n\times n} \in ((A_n(\mathbb{R}) \times B_n(\mathbb{R})) \cup (B_n(\mathbb{R}) \times A_n(\mathbb{R})). \quad (3.33)$$

It is clear that T is a continuous covariant mapping.

On the other hand, we define $\gamma : [0, +\infty) \to [0, \frac{1}{2})$ by

$$\gamma(s) = \frac{s}{2},$$

$\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(s) = s$$

and $\alpha : (A_n(\mathbb{R}) \times B_n(\mathbb{R})) \times (B_n(\mathbb{R}) \times A_n(\mathbb{R})) \to [0, +\infty)$ by

$$\alpha((P, Q), (K, V)) = \begin{cases} 1, & p_{ij} \geq q_{ij},\ k_{ij} \geq v_{ij} \\ 0, & \text{otherwise} \end{cases}$$

where $P = \{p_{ij}\}_{n\times n}$, $V = \{v_{ij}\}_{n\times n} \in A_n(\mathbb{R})$ and $Q = \{q_{ij}\}_{n\times n}$, $K = \{k_{ij}\}_{n\times n} \in B_n(\mathbb{R})$. Then we obtain

$$
\begin{aligned}
&\varphi(\Lambda(T(P,Q),T(K,V))) \\
&\quad \le \Lambda\left(\left\{\frac{p_{ij}+q_{ij}}{4}\right\}_{n\times n}, \left\{\frac{k_{ij}+v_{ij}}{4}\right\}_{n\times n}\right) \\
&\quad \le \sum_{i,j=1}^{n} \left|\frac{p_{ij}+q_{ij}-k_{ij}-v_{ij}}{4}\right| \\
&\quad \le \sum_{i,j=1}^{n} \left|\frac{p_{ij}-k_{ij}}{4}\right| + \left|\frac{q_{ij}-v_{ij}}{4}\right| \\
&\quad = \frac{1}{2}\left(\frac{\Lambda(P,K)+\Lambda(Q,V)}{2}\right) \\
&\quad = \frac{1}{2}\left(\frac{\Lambda(P,K)+\Lambda(Q,V)}{2}\right)\frac{\Lambda(P,K)+\Lambda(Q,V)}{\Lambda(P,K)+\Lambda(Q,V)} \\
&\quad = \gamma\left(\varphi\left(\frac{\Lambda(P,K)+\Lambda(Q,V)}{2}\right)\right)\varphi\left(\frac{\Lambda(P,K)+\Lambda(Q,V)}{2}\right).
\end{aligned}
$$

Therefore, the conditions (i) and (ii) of Theorem 3.10 are satisfied for $(\eta_0, \mu_0) = (I_n, I_n)$. Hence, T has a coupled fixed point. In particular, the coupled fixed point is $(0_{n\times n}, 0_{n\times n}) \in A_n(\mathbb{R}) \cap B_n(\mathbb{R})$ where $0_{n\times n}$ is the null matrix.

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