Nonlinear Functional Analysis and Applications
Vol. 17, No. 1 (2012), pp. 99-108

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

FORCED OSCILLATIONS OF SECOND ORDER NONLINEAR FDE WITH IMPULSES

Chaolong Zhang[1], Fengjian Yang[2] and Jianfu Yang[3]

[1]Computational Science School,
Zhongkai University of Agriculture and Engineering
Guangzhou, 510225, P.R.China

[2]Computational Science School,
Zhongkai University of Agriculture and Engineering
Guangzhou, 510225, P.R.China
e-mail: yangfj88@sina.com

[3]Computational Science School,
Zhongkai University of Agriculture and Engineering
Guangzhou, 510225, P.R.China

Abstract. A kind of second order impulses nonlinear FDE with forcing term is studied in this paper. By means of Kartsatos technique, we reduce it to a second order nonlinear impulsive homogeneous equation. Several criteria on the oscillations of solutions are given under some suitable nonlinear impulse functions. At last, an example can be illustrated our results.

1. Introduction

Based on the oscillatory behavior of the forcing term, Wong[1] obtained oscillation criteria for the linear nonhomogeneous equation

$$(r(t)x^{'}(t))^{'} + p(t)x(t) = q(t).$$

[0]Received March 16, 2011. Revised December 21, 2011.
[0]2000 Mathematics Subject Classification: 34A37, 34K06, 34K11, 34K25.
[0]Keywords: Second order, impulses, FDE, forced oscillation.
[0]This project was supported by the National Natural Science Foundation of China (No.10971231,31071560) and the NSF of Guangdong Province of China (No.10151022501000004, 8151027501000053).

Nsar[2] considered the forced super-linear differential equation

$$x^{''}(t)+p(t)\mid x(t)\mid^{\alpha-1}x(t)=q(t)$$

and established some oscillation criteria.

Yang[3] considered the forced nonlinear differential equation

$$(p(t)x^{'}(t))^{'}+q(t)f(x(t))=g(t)$$

and established some oscillation criteria.

The first order impulsive delay differential equation with forcing term can be seen [4], [5]. Forced oscillation of second order super linear differential equation with impulses can be seen [6], [7], [8].

In the paper, by means of Kartsatos technique and Riccati technique, forced oscillation of second order FDE with non-linear impulses is studied. Several criteria on the oscillations of solutions are given. We find some suitable nonlinear impulse functions such that all the solutions of the equation are oscillatory. At last, we give an example to demonstrate our results.

2. Main results

We consider the following systems with forcing term:

$$\begin{cases} x^{''}(t)+p(t)f(x(t),x(t-\tau))=q(t), & t\geq 0, t\neq t_k, \\ x(t_k^+)=g_k(x(t_k)), x^{'}(t_k^+)=h_k(x^{'}(t_k)), & k=1,2\cdots, \end{cases} \tag{2.1}$$

$0<t_1<t_2<\cdots<t_k<\cdots$, $\lim_{k\to\infty}t_k=+\infty, k=1,2\cdots, t_{k+1}-t_k>\tau$, $f(u,v)$ is continuous on $(-\infty,+\infty)\times(-\infty,+\infty)$ $uf(u,v)>0(uv>0), \mid f(u,v)\mid\geq|\varphi(v)|, \tau>0$, $p(t)>0$ is not always equal to 0, $v\varphi(v)>0(v\neq 0), \varphi^{'}(v)\geq 0$, and $q(t)$ is continuous in $[0,+\infty)$. $g_k(x), h_k(x)$ are continuous in $(-\infty,+\infty)$, and there exist positive numbers a_k^*, a_k, b_k^*, b_k

$$a_k^*\leq\frac{g_k(x)}{x}\leq a_k,\ \ b_k^*\leq\frac{h_k(x)}{x}\leq b_k.$$

The initial condition

$$x(t)=\phi(t), t\in[t_0-\tau,t_0], \phi\in PC([t_0-\tau,t_0],R).$$

Definition 2.1. A function $x:[t_0-\tau,t_0+\alpha)\to R(\alpha>0)$ is said to be a solution of (2.1) if

(i) $x(t)$ is continuous on $[t_0,t_0+\alpha)\setminus\{t_k,k\in N\}$.

(ii) $x(t)=\phi(t), t\in[t_0-\tau,t_0], x(t_0^+)=x_0, x^{'}(t_0^+)=x_0^{'}$.

(iii) $x(t)$ satisfies the first equality of (2.1) on $[t_0,t_0+\alpha)\setminus\{t_k,k\in N\}$.

(iv) $x(t), x^{'}(t)$ has two-side limits and left continuous at points t_k, and $x(t_k^+)=g_k(x(t_k)), x^{'}(t_k^+)=h_k(x^{'}(t_k))$.

If $z(t) \in C^2[0,+\infty), z''(t) = q(t)$, there exists two constants p_1, p_2 and two sequences $\{t_i'\}, \{t_i''\}$, $\lim_{i\to+\infty} t_i' = \lim_{i\to+\infty} t_i'' = +\infty$ such that $z(t_i') = p_1 \le z(t) \le p_2 = z(t_i'')$.

If the equation (2.1) has an eventually positive solution $x(t)$. Without loss of generality, $x(t-\tau) > 0, t \ge t_0 + \tau$. Let $y(t) = x(t) - z(t) + p_1$, by the equation (2.1), we have

$$\begin{cases} y''(t) + p(t)\varphi(y(t-\tau)) \le 0, \quad t \ge 0, t \ne t_k, \\ a_k^* y(t_k) + c_k^* \le y(t_k^+) \le a_k y(t_k) + c_k, b_k^* \le \frac{y'(t_k^+)+z'(t_k)}{x'(t_k)} \le b_k, \end{cases} \tag{2.2}$$

where $c_k^* = (a_k^* - 1)(z(t_k) - p_1), c_k = (a_k - 1)(z(t_k) - p_1)$.

If the equation (2.1) has an eventually negative solution $x(t)$. Without loss of generality, $x(t-\tau) < 0, t \ge t_0 + \tau$. Let $y(t) = x(t) - z(t) + p_2$, by the equation (2.1), we have

$$\begin{cases} y''(t) + p(t)\varphi(y(t-\tau)) \ge 0, \quad t \ge 0, t \ne t_k, \\ a_k y(t_k) + d_k \le y(t_k^+) \le a_k^* y(t_k) + d_k^*, b_k^* \le \frac{y'(t_k^+)+z'(t_k)}{x'(t_k)} \le b_k, \end{cases} \tag{2.3}$$

where $d_k^* = (a_k^* - 1)(z(t_k) - p_2), d_k = (a_k - 1)(z(t_k) - p_2)$.

Definition 2.2. A solution of (2.1) is said to be non-oscillatory if it is eventually positive or eventually negative.Otherwise,this solution is said to be oscillatory.

Using the method of steps, one can show that the initial problem has a solution for any $\tau > 0$. By the same method in [9], one can get sufficient conditions that can guarantee the solution of (2.1) exists on $[t_0, +\infty)$. In the following, we always assume the solutions of (2.1) exists on $[t_0, +\infty)$.

Lemma 2.3. *Assume that*

(A_0) *$m \in PC^1[R_+, R]$ and $m(t)$ is left-continuous at $t_k, k = 1, 2, \cdots$,*

(A_1) *For $k = 1, 2, \cdots, t \ge t_0$,*

$$\begin{aligned} m'(t) &\le p(t)m(t) + q(t), \qquad t \ne t_k, \\ m(t_k^+) &\le d_k m(t_k) + b_k, \end{aligned}$$

where $q, p \in PC^1[R_+, R], d_k \ge 0$ and b_k are constants. Then

$$\begin{aligned} m(t) \ &\le m(t_0) \prod_{t_0<t_k<t} d_k exp(\int_{t_0}^t p(s)ds) \\ &+ \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} d_j exp(\int_{t_k}^t p(s)ds) \right) b_k \\ &+ \int_{t_0}^t \prod_{s<t_k<t} d_k exp(\int_s^t p(\sigma)d\sigma) q(s)ds, \qquad t \ge t_0. \end{aligned}$$

The proof of the Lemma 2.3 can be seen in [10, Theorem 1.4.1].

Lemma 2.4. *If $x(t)$ is an eventually positive solution of the equation* (2.1), *there exists a constant k_0, such that $z'(t_k) = 0$, for $k \geq k_0$, and*

$$(H_1) : (t_1 - t_0) + \frac{b_1^*}{a_1}(t_2 - t_1) + \cdots + \frac{b_1^* b_2^* \cdots b_n^*}{a_1 a_2 \cdots a_n}(t_{n+1} - t_n) + \cdots = +\infty,$$

$$(H_2) : \frac{\mid c_1 \mid}{a_1} + \frac{\mid c_2 \mid}{a_1 a_2} + \frac{\mid c_3 \mid}{a_1 a_2 a_3} + \cdots + \frac{\mid c_n \mid}{a_1 a_2 \cdots a_n} + \cdots < +\infty,$$

then for the equation (2.2), $y'(t_k) > 0, y'(t) > 0, t \in (t_k, t_{k+1}], t_k > T > t_{k_0}$.

Proof. Without loss of generality, let $x(t-\tau) > 0, t \geq t_0 + \tau$, by $y(t) = x(t) - z(t) + p_1$, then

$$y''(t) = -p(t)f(x(t), \quad x(t-\tau)) \leq 0,$$

we have $y''(t) \leq 0$ and is not always equal to 0.

At first, we claim $y'(t_k) > 0, t_k > t_{k_0}$. If it is not true, there exists a $t_j \geq t_{k_0}$, $y'(t_j) \leq 0$, so $b_j y'(t_j) \leq y'(t_j^+) \leq b_j^* y'(t_j) \leq 0$. By $y''(t) \leq 0$ and is not always equal to 0, we have $y'(t) < y'(t_j^+) \leq 0, t \in (t_j, t_{j+1}]$. Especially, we have $y'(t_{j+1}) \leq y'(t) < y'(t_j^+) \leq 0$. Let $y'(t_{j+1}) = -\alpha, (\alpha > 0)$, then $y'(t_{j+1}^+) \leq b_{j+1}^* y'(t_{j+1}) = -b_{j+1}^* \alpha$,

$$y'(t) \leq -b_{j+1}^* \alpha, \quad t \in (t_{j+1}, t_{j+2}]. \tag{2.4}$$

Especially, we have $y'(t_{j+2}^+) \leq b_{j+2}^* y'(t_{j+2}) \leq -b_{j+1}^* b_{j+2}^* \alpha$. By induction, for $t \in (t_{j+m}, t_{j+m+1}]$

$$y'(t) \leq -b_{j+1}^* b_{j+2}^* \cdots b_{j+m}^* \alpha. \tag{2.5}$$

Especially, we have

$$y'(t_{j+m+1}^+) = b_{j+m+1}^* y'(t_{j+m+1}) \leq -b_{j+1}^* b_{j+2}^* \cdots b_{j+m}^* b_{j+m+1}^* \alpha.$$

Integrating the inequality (2.4), from t_{j+1} to t_{j+2}, we have

$$y(t_{j+2}) \leq y(t_{j+1}^+) - b_{j+1}^* \alpha (t_{j+2} - t_{j+1}).$$

Let

$$\begin{aligned} &y(t_{j+m}) \\ &\leq a_{j+2} a_{j+3} \cdots a_{j+m-1} [y(t_{j+1}^+) - b_{j+1}^* \alpha (t_{j+2} - t_{j+1}) \\ &\quad -b_{j+1}^* \alpha \tfrac{b_{j+2}^*}{a_{j+2}} (t_{j+3} - t_{j+2}) - \cdots - b_{j+1}^* \alpha \tfrac{b_{j+2}^* \cdots b_{j+m-1}^*}{a_{j+2} \cdots a_{j+m-1}} (t_{j+m} - t_{j+m-1}) \\ &\quad + \tfrac{c_{j+2}}{a_{j+2}} + \tfrac{c_{j+3}}{a_{j+2} a_{j+3}} + \cdots + \tfrac{c_{j+m-1}}{a_{j+2} a_{j+3} \cdots a_{j+m-1}}]. \end{aligned} \tag{2.6}$$

Integrating the inequality (2.5), from t_{j+m} to t_{j+m+1}, we have

$$\begin{aligned}
&y(t_{j+m+1})\\
&\le y(t^+_{j+m}) - b^*_{j+1}b^*_{j+2}\cdots b^*_{j+m}\alpha(t_{j+m+1}-t_{j+m})\\
&\le a_{j+2}a_{j+3}\cdots a_{j+m-1}a_{j+m}[y(t^+_{j+1}) - b^*_{j+1}\alpha(t_{j+2}-t_{j+1})\\
&\quad -b^*_{j+1}\alpha\tfrac{b^*_{j+2}}{a_{j+2}}(t_{j+3}-t_{j+2}) - \cdots - b^*_{j+1}\alpha\tfrac{b^*_{j+2}\cdots b^*_{j+m}}{a_{j+2}\cdots a_{j+m}}(t_{j+m+1}-t_{j+m})\\
&\quad +\tfrac{c_{j+2}}{a_{j+2}} + \tfrac{c_{j+3}}{a_{j+2}a_{j+3}} + \cdots + \tfrac{c_{j+m}}{a_{j+2}a_{j+3}\cdots a_{j+m}}].
\end{aligned} \tag{2.7}$$

By induction, for all $m \in N$, we have the the inequality (2.7) holds. By the conditions $(H_1), (H_2)$, when $m \to +\infty$, we have $y(t_{j+m+1}) \to -\infty$. On the other hand, $y(t) = x(t) - z(t) + p_1$, when $t \to +\infty$, $x(t) \to -\infty$, it contradicts $x(t) > 0$.

So, there exists a $T \ge t_{k_0}$, for all $t_k \ge T$, we have $y^{'}(t_k) > 0$, and by $y^{'}(t)$ is monotone function, we have $y^{'}(t) > 0, t \ge T$. The proof of the Lemma 2.4 is completed. □

Lemma 2.5. *If $x(t)$ is an eventually positive solution of the equation* (2.1)*, the conditions $(H_1), (H_2)$ hold, and there exists a k_0, such that $a^*_k \ge 1, z^{'}(t_k) = 0, k \ge k_0$, then for the equation* (2.2)*, $y(t) > 0, t_k > T > t_{k_0}$.*

Proof. Without loss of generality, let $x(t-\tau) > 0, t \ge t_0 + \tau$.

I) If there exists a $t_j \ge t_0 + \tau$, such that $y(t^+_j) \ge 0$. By the conditions $(H_1), (H_2)$ and the Lemma 2.4, we have $y^{'}(t) > 0, t \in (t_j, t_{j+1}]$, so

$$y(t_{j+1}) > y(t^+_j) \ge 0, y(t^+_{j+1}) \ge a^*_{j+1}y(t_{j+1}) + c^*_{j+1} \ge a^*_{j+1}y(t_{j+1}).$$

By induction, there exists a $T \ge t_{k_0}$, we have $y(t) > 0$, for $t > T$.

II) If all $t_j \ge t_{k_0}$, we have $y(t^+_j) < 0$, i.e. $y(t_j) \le \frac{y(t^+_j)-c^*_j}{a^*_j} < 0$. By $y(t)$ is monotonically increasing in $t \in (t_j, t_{j+1}]$,

$$y(t) < y(t_{j+1}) < 0, \quad t \in (t_j, t_{j+1}]. \tag{2.8}$$

On the other hand, in $(t_k, t_{k+1}], t_k \ge t_j$, we take a sequence $\{t'_n\}$, then $x(t^{'}_n) = y(t^{'}_n) + z(t^{'}_n) - p_1 = y(t^{'}_n) < 0$, it contradicts $x(t) > 0$. So $y(t) > 0, t \ge T$. Summing up the above consideration, we have $y(t) > 0$, for $t \ge T$. The proof of the Lemma 2.5 is completed. □

Lemma 2.6. *If $x(t)$ is an eventually negative solution of the equation* (2.1)*, there exists an constant k_0, such that $z^{'}(t_k) = 0$, for $k \ge k_0$, and*

$$(H_1) : (t_1 - t_0) + \frac{b^*_1}{a_1}(t_2 - t_1) + \cdots + \frac{b^*_1 b^*_2 \cdots b^*_n}{a_1 a_2 \cdots a_n}(t_{n+1} - t_n) + \cdots = +\infty,$$

$$(H_2)^{'}: \frac{|\, d_1 \,|}{a_1} + \frac{|\, d_2 \,|}{a_1 a_2} + \frac{|\, d_3 \,|}{a_1 a_2 a_3} + \cdots + \frac{|\, d_n \,|}{a_1 a_2 \cdots a_n} + \cdots < +\infty,$$

then for the equation (2.3), $y^{'}(t_k) < 0, y^{'}(t) < 0, t \in (t_k, t_{k+1}], t_k > T > t_{k_0}$.

Lemma 2.7. *If $x(t)$ is an eventually negative solution of the equation* (2.1), *the conditions $(H_1), (H_2)^{'}$ hold, and there exists a k_0, such that $a_k^* \geq 1, z^{'}(t_k) = 0, k \geq k_0$, then for the equation* (2.3), $y(t) < 0, t_k > T > t_{k_0}$.

Remark 2.8. If $x(t)$ is an eventually negative solution of (2.1), by the conditions $(H_1), (H_2)^{'}$, the proof of the Lemma 2.6 and Lemma 2.7 are similar to Lemma 2.4 and Lemma 2.5, so it is omitted here.

Theorem 2.9. *Suppose the conditions $(H_1), (H_2)$ hold, if there exists a constant k_0, such that $a_k^* \geq 1, z^{'}(t_k) = 0, k \geq k_0, \varphi^{'}(v) \geq c > 0$ and $F(t) > 0$ is continuous on* $[0, \infty)$,

$$(H_3): \lim_{t \to +\infty} exp\left(\int_{t_1}^{t} (-cF(s))ds\right) < +\infty,$$

$$(H_4): \lim_{t \to +\infty} \int_{t_1}^{t} \left(\prod_{t_1 < t_{0,k} < s} \frac{1}{\theta_{0,k}} exp\left(\int_{s}^{t} (-cF(\tau))d\tau\right) \left[p(s) - \frac{cF^2(s)}{4}\right] \right) ds = +\infty,$$

where

$$\theta_{0,k} = \begin{cases} b_k, & t_{0,k} = t_k; \\ 1, & t_{0,k} = t_k + \tau; \end{cases}$$

then every solution of (2.1) *is oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of (2.1). Without loss of generality, let $x(t-\tau) > 0 (t \geq t_0 + \tau)$. By (2.2) and the Lemma 2.4, Lemma 2.5, we have

$$y(t) > 0, y^{'}(t) > 0, y^{''}(t) \leq 0, \quad t \geq T \geq t_{k_0} \geq t_0 + \tau. \tag{2.9}$$

Without loss of generality, $T = t_0 + \tau$,

$$u(t) = \frac{y^{'}(t)}{\varphi(y(t-\tau))},$$

by (2.9), we have $u(t) > 0, t \geq t_0 + \tau, t \neq t_k, t \neq t_k + \tau$.

$$\begin{aligned} u^{'}(t) &= \frac{y^{''}(t)}{\varphi(y(t-\tau))} - \frac{y^{'}(t)\varphi^{'}(y(t-\tau))y^{'}(t-\tau)}{\varphi^2(y(t-\tau))} \\ &\leq -p(t) - c[u(t)]^2 \\ &= -[p(t) - \frac{cF^2(t)}{4}] - [cu^2(t) + \frac{cF^2(t)}{4}] \\ &\leq -[p(t) - \frac{cF^2(t)}{4}] - cu(t)F(t), \end{aligned}$$

$$u(t_k^+) = \frac{y'(t_k^+)}{\varphi(y(t_k^+ - \tau))} \leq b_k u(t_k), \tag{2.10}$$

$$u(t_k^+ + \tau) = \frac{y'(t_k^+ + \tau)}{\varphi(y(t_k^+))} \leq \frac{y'(t_k^+ + \tau)}{\varphi(a_k y(t_k))} \leq \frac{y'(t_k + \tau)}{\varphi(y(t_k))} \leq u(t_k + \tau), \tag{2.11}$$

so, we have

$$\begin{cases} u'(t) \leq -[p(t) - \frac{cF^2(t)}{4}] - cu(t)F(t), t \neq t_{0,k}, k = 1, 2, \cdots, \\ u(t_{0,k}^+) \leq \theta_{0,k} u(t_{0,k}). \end{cases} \tag{2.12}$$

where $t_1 = t_{0,1} < t_{0,2} = t_1 + \tau < \cdots < t_{0,2m-1} = t_m < t_{0,2m} = t_m + \tau < \cdots$, $\lim\limits_{m\to+\infty} t_{0,m} = +\infty$,

$$\theta_{0,k} = \begin{cases} b_k, & t_{0,k} = t_k; \\ 1, & t_{0,k} = t_k + \tau. \end{cases}$$

By (2.12) and the Lemma 2.3, we have

$$\begin{aligned} u(t) \quad \leq & \prod_{t_1 < t_{0,k} < t} \theta_{0,k} \Bigg[u(t_1^+) exp(\textstyle\int_{t_1}^t (-cF(s))ds) \\ & - \textstyle\int_{t_1}^t \Bigg(\prod\limits_{t_1 < t_{0,k} < s} \frac{1}{\theta_{0,k}} exp(\int_s^t (-cF(\tau))d\tau)[p(s) - \frac{cF^2(s)}{4}] \Bigg) ds \Bigg]. \end{aligned} \tag{2.13}$$

when $t \to +\infty$, by the condition of the Theorem 2.9, we have $u(t) \to -\infty$, this contradicts $u(t) > 0$.

If $x(t-\tau) < 0 (t \geq t_0 + \tau)$, similar to the above method, and the condition $(H_1), (H_2)', (H_3), (H_4)$ hold, we can get a contradiction. So every solution of (2.1) is oscillatory. The proof of Theorem 2.9 is completed. □

Corollary 2.10. *Suppose the conditions* $(H_1), (H_2)$ *hold, if there exists a constant* k_0*, such that* $a_k \geq 1, z'(t_k) = 0, k \geq k_0, F(t) = 0$ *and*

$$\lim_{t\to+\infty} \int_{t_0+\tau}^t \prod_{t_0+\tau < t_{0,k} < s} \frac{1}{\theta_{0,k}} p(s) ds = +\infty,$$

then every solution of (2.1) *is oscillatory.*

Theorem 2.11. *Suppose the conditions* $(H_1), (H_2)$ *hold, if there exists a constant* k_0*, such that* $a_k^* \geq 1, z'(t_k) = 0, k \geq k_0, \varphi'(v) \geq c > 0, \varphi(ab) = \varphi(a)\varphi(b)$ *and* $F(t) > 0$ *is continuous on* $[0, \infty)$*,*

$$(H_3)' : \lim_{t\to+\infty} exp\left(\int_{t_1}^t (-cF(s))ds\right) < +\infty,$$

$$(H_4)^{'} : \lim_{t\to+\infty}\int_{t_1}^{t}\Bigg(\prod_{t_1<t_{0,k}<s}\frac{1}{\omega_{0,k}}exp\Big(\int_s^t(-cF(\tau))d\tau\Big)\Big[p(s)-\frac{cF^2(s)}{4}\Big]\Bigg)\,ds=+\infty,$$

where

$$\omega_{0,k}=\begin{cases} b_k, & t_{0,k}=t_k;\\ \frac{1}{\varphi(a_k)}, & t_{0,k}=t_k+\tau;\end{cases}$$

then every solution of (2.1) *is oscillatory.*

Proof. Similar to the proof of the Theorem 2.9, let $y(t)=x(t)-z(t)+p_1$, $u(t)=\frac{y^{'}(t)}{\varphi(y(t-\tau))}$, we also obtain $u(t)>0, t\geq t_{k_0}$. So, we have

$$\begin{cases} u^{'}(t)\leq-[p(t)-\frac{cF^2(t)}{4}]-cu(t)F(t), t\neq t_{0,k}, k=1,2,\cdots,\\ u(t_{0,k}^{+})\leq\omega_{0,k}u(t_{0,k}),\end{cases}\tag{2.14}$$

where $t_1=t_{0,1}<t_{0,2}=t_1+\tau<\cdots<t_{0,2m-1}=t_m<t_{0,2m}=t_m+\tau<\cdots$, $\lim_{m\to+\infty}t_{0,m}=+\infty$,

$$\omega_{0,k}=\begin{cases} b_k, & t_{0,k}=t_k;\\ \frac{1}{\varphi(a_k)}, & t_{0,k}=t_k+\tau.\end{cases}$$

By (2.14) and the Lemma 2.3, we have

$$\begin{aligned} u(t) \leq & \prod_{t_1<t_{0,k}<t}\omega_{0,k}\Bigg[u(t_1^{+})exp(\int_{t_1}^{t}(-cF(s))ds)\\ & -\int_{t_1}^{t}\Bigg(\prod_{t_1<t_{0,k}<s}\frac{1}{\omega_{0,k}}exp(\int_s^t(-cF(\tau))d\tau)[p(s)-\frac{cF^2(s)}{4}]\Bigg)ds\Bigg],\end{aligned}\tag{2.15}$$

when $t\to+\infty$, by the condition of the Theorem 2.11, we have $u(t)\to-\infty$, this contradicts $u(t)>0$.

If $x(t-\tau)<0(t\geq t_0+\tau)$, similar to the above method, and the condition $(H_1),(H_2)^{'},(H_3)^{'},(H_4)^{'}$ hold, we can get a contradiction. So every solution of (2.1) is oscillatory. The proof of Theorem 2.11 is completed. □

Corollary 2.12. *Suppose the conditions* $(H_1),(H_2)$ *hold, if there exists a constant* k_0*, such that* $a_k\geq1, z^{'}(t_k)=0, k\geq k_0, F(t)=0$ *and*

$$\lim_{t\to+\infty}\int_{t_0+\tau}^{t}\prod_{t_0+\tau<t_{0,k}<s}\frac{1}{\omega_{0,k}}p(s)ds=+\infty,$$

then every solution of (2.1) *is oscillatory.*

3. Example

Example 3.1. *Consider*

$$\begin{cases} x^{''}(t) + t^2(1 + x^2(t))x(t-1) = \sin t, \quad t > 0, t \neq k\pi, k = 1, 2, \cdots, \\ x((k\pi + \frac{\pi}{2})^+) = \frac{k+1}{k} x(k\pi + \frac{\pi}{2}), x^{'}((k\pi + \frac{\pi}{2})^+) = \frac{k+1}{k} x^{'}(k\pi + \frac{\pi}{2}). \end{cases} \tag{3.1}$$

In fact, $p(t) = t^2, q(t) = \sin t, a_k^* = a_k = \frac{k+1}{k} \geq 1, b_k^* = b_k = \frac{k+1}{k}, t_0 = \frac{\pi}{2}, \tau = 1, f(x(t), x(t-1)) = (1+x^2(t))x(t-1), \varphi(x(t-1)) = x(t-1), \varphi^{'}(x(t-1)) = 1 = c, t_k = k\pi + \frac{\pi}{2}, t_{k+1} - t_k = \pi > 1$. Let $F(t) = \frac{1}{t}, z(t) = -\sin t, z^{'}(k\pi + \frac{\pi}{2}) = -\cos(k\pi + \frac{\pi}{2}) = 0, p_1 = -1, p_2 = 1,\ c_k = (\frac{k+1}{k} - 1)(0 - (-1)) = \frac{1}{k}, d_k = (\frac{k+1}{k} - 1)(0 - 1) = -\frac{1}{k}$,

$$\begin{aligned} &(t_1 - t_0) + \tfrac{b_1^*}{a_1}(t_2 - t_1) + \cdots + \tfrac{b_1^* b_2^* \cdots b_n^*}{a_1 a_2 \cdots a_n}(t_{n+1} - t_n) + \cdots \\ &\quad = \pi + \pi + \cdots + \pi + \cdots = +\infty, \end{aligned}$$

$$\begin{aligned} &\tfrac{|c_1|}{a_1} + \tfrac{|c_2|}{a_1 a_2} + \tfrac{|c_3|}{a_1 a_2 a_3} + \cdots + \tfrac{|c_n|}{a_1 a_2 \cdots a_n} + \cdots \\ &\quad = \tfrac{1}{1\cdot 2} + \tfrac{1}{2\cdot 3} + \tfrac{1}{3\cdot 4} + \cdots + \tfrac{1}{n\cdot(n+1)} = 1 - \tfrac{1}{n+1} < +\infty, \end{aligned}$$

$$\begin{aligned} &\tfrac{|d_1|}{a_1} + \tfrac{|d_2|}{a_1 a_2} + \tfrac{|d_3|}{a_1 a_2 a_3} + \cdots + \tfrac{|d_n|}{a_1 a_2 \cdots a_n} + \cdots \\ &\quad = \tfrac{1}{1\cdot 2} + \tfrac{1}{2\cdot 3} + \tfrac{1}{3\cdot 4} + \cdots + \tfrac{1}{n\cdot(n+1)} = 1 - \tfrac{1}{n+1} < +\infty. \end{aligned}$$

So the condition $(H_1), (H_2), (H_2)^{'}$ hold. For the condition $(H_3), (H_4)$ we have

$$\lim_{t\to+\infty} exp\left(\int_{t_1}^{t} -\frac{1}{s} ds\right) = \lim_{t\to+\infty} \frac{t_1}{t} = 0,$$

$$\begin{aligned} &\textstyle\int_{t_1}^{t} \prod\limits_{t_1 < t_{0,k} < s} \frac{1}{\theta_{0,k}} \frac{s}{t}(s^2 - \frac{1}{4s^2}) ds = \frac{1}{t} \int_{t_1}^{t} \prod\limits_{t_1 < t_{0,k} < s} \frac{1}{\theta_{0,k}} (s^3 - \frac{1}{4s}) ds \\ &\textstyle\geq \frac{1}{t} \int_{t_1}^{t} \prod\limits_{t_1 < t_{0,k} < s} \frac{1}{\theta_{0,k}} s^2 ds \geq \frac{1}{t} \int_{t_1}^{t} s ds = \frac{1}{2}(t - \frac{t_1^2}{t}) \to +\infty, \quad (t \to +\infty). \end{aligned}$$

Therefore, the conditions of Theorem 2.9 hold. So every solution of (3.1) is oscillatory.

References

[1] J. S. W. Wong, *Oscillation criteria for a forced second order linear differential equation*, J. Math. Anal. Appl., 231 (1999), 235-240.

[2] A. H. Nsar, *Sufficient conditions for the oscillation of forced super-linear second order differential equations with oscillatory potential*, Proc. Amer. Math. Soc., 126 (1998), 123-125.

[3] Qigui Yang, *Interval oscillation criteria for a forced second order nonlinear ordinary differential equations with oscillatory potential*, Applied Mathematics and Computation, 135 (2003), 49-64.

[4] D. D. Bainov, M. B. Dimitrova and A. B. Dishliev, *Oscillating Solutions of Nonlinear Impulsive Differential Equations with a Deviating Argument*, NOte di Mathematica, 1, (1995), 45-54.

[5] Zhang Chaolong, Yang Jianfu and Guo Chengjun, *Oscillations of Impulses Delay Differential Equations With Forcing Term*, Ann. of Diff. Eqs., 22:1 (2006), 95-100.

[6] Xiuxiang Liu and Zhiting Xu, *Oscillation of a forced super-linear second order differential equation with impulses*, Computers and Mathematics with Applications, 53 (2007), 1740-1749.

[7] A. Özbekler and A. Zafer, *Forced oscillation of super-half-linear impulsive differential equations*, Computers and Mathematics with Applications, 54 (2007), 785-792.

[8] Chaolong Zhang, Weizhen Feng, Jianfu Yang and Mugen Huang, *Oscillations of second order impulses nonlinear FDE with forcing term*, Applied Mathematics and Computation, 198 (2008), 271-279.

[9] Li-zhi Wen and Yong-chao Chen, *Razumikhin type theorems for fumctional differential equations with impulses*, Dynamics of Continuous, Discrete and Impulsive Systems, 6 (1999), 389-400.

[10] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov. *Theory of Impulsive Differential Equations*, Singapore, New Jersey, London: World Scientific, 1989.