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COMMON FIXED POINT THEOREMS FOR COMPATIBLE AND SUBSEQUENTIALLY CONTINUOUS MAPPINGS IN MENGER SPACES

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Abstract. In this paper, we utilize the notions of compatibility and subsequentially continuity (alternately subcompatibility and reciprocally continuity) and prove common fixed point theorems in Menger spaces satisfying implicit relation. Some illustrative examples are also given which demonstrate the validity of our main results. We also present an integral-type common fixed point theorem for four mappings in Menger space.

1. Introduction

Professor Karl Menger [25] introduced the notion of a probabilistic metric space (briefly, PM-space) in 1942. The idea of Menger was to use distribution functions instead of non-negative real numbers as values of the metric. The notion of PM-space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. In fact the study such spaces received an impetus with the pioneering work of Schweizer and Sklar [36]. Fixed point theory is one of the most fruitful and effective tools in mathematics which has many applications within as well as outside mathematics (see [6, 11, 13]).

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In 1986, Jungck [16] introduced the notion of compatible mappings for a pair of self mappings in metric space. Most of the common fixed point theorems for contraction mappings invariably require a compatibility condition besides assuming continuity of at least one of the mappings. Pant [28] noticed these criteria for fixed points of contraction mappings and introduced a new continuity condition, known as reciprocal continuity and obtained a common fixed point theorem by using the compatibility in metric spaces. He also showed that in the setting of common fixed point theorems for compatible mappings satisfying contraction conditions, the notion of reciprocal continuity is weaker than the continuity of one of the mappings. Further, Jungck and Rhoades [17] termed a pair of self mappings to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. In 2008, Al-Thagafi and Shahzad [1] introduced the concept of occasionally weakly compatible mappings in metric spaces which is the most general among all the commutativity concepts. Recently, Đoric et al. [12] showed that the condition of occasionally weak compatibility reduces to weak compatibility in the presence of a unique point of coincidence (or a unique common fixed point) of the given pair of self mappings. Thus, no generalization can be obtained by replacing weak compatibility with occasionally weakly compatibility. Thereafter, Bouhadjera and Godet-Thobie [3] introduced two new notions namely subsequential continuity and subcompatibility which are weaker than reciprocal continuity and compatibility respectively. Imdad et al. [14] improved the results of Bouhadjera and Godet-Thobie [3] and showed that these results can easily recovered by replacing subcompatibility with compatibility or subsequential continuity with reciprocally continuity. Several interesting and elegant results have been obtained by various authors on various spaces (see [4, 8, 15, 19, 27, 38]).

In metric fixed point theory, implicit relations are utilized to cover several contraction conditions in one go rather than proving separate theorem for each contraction condition. In 1999, Popa [35] used implicit relations rather than contraction conditions to prove fixed point theorems in metric spaces whose strength lies in its unifying power as an implicit function can cover several contraction conditions at the same time. This fact is evident from examples furnished in Popa [35]. Many researchers proved a number of fixed point theorems in different settings employing implicit relations (see [2, 7, 9, 10, 14], [20]-[24], [29]-[34]).

The purpose of this paper is to prove common fixed point theorems using the notions of compatibility and subsequentially continuity (alternately subcompatibility and reciprocally continuity) satisfying implicit relation in Menger spaces. Our results never require the conditions on completeness (or closedness) of the underlying space (or subspaces) and continuity in respect of any one of the involved mappings. The results of this paper are generalizations and refinement of several results recently appeared in the literature. Inspired by the work of Branciari [5], we present an integral-type fixed point theorem in Menger space.

2. Preliminaries

Definition 2.1. ([36]) A mapping $\triangle : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (briefly, t-norm) if the following conditions are satisfied: for all $a, b, c, d \in [0,1]$

(1) $\triangle(a,1) = a$ for all $a \in [0,1]$,
(2) $\triangle(a,b) = \triangle(b,a)$,
(3) $\triangle(a,b) \leq \triangle(c,d)$ for $a \leq c$, $b \leq d$,
(4) $\triangle(\triangle(a,b),c) = \triangle(a,\triangle(b,c))$.

Examples of continuous t-norms are $\triangle(a,b) = ab$ and $\triangle(a,b) = \min\{a,b\}$.

Definition 2.2. ([36]) A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

We denote by $\Im$ the set of all distribution functions while H always denotes the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

If X is a non-empty set, $\mathcal{F} : X \times X \to \Im$ is called a probabilistic distance on X and the value of $\mathcal{F}$ at $(x,y) \in X \times X$ is represented by $F_{x,y}$.

Definition 2.3. ([36]) The ordered pair $(X, \mathcal{F})$ is called a PM-space if X is a nonempty set and $\mathcal{F}$ is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$

(1) $F_{x,y}(t) = H(t) \Leftrightarrow x = y$,
(2) $F_{x,y}(t) = F_{y,x}(t)$,
(3) if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t+s) = 1$.

Definition 2.4. ([36]) A Menger space is a triplet $(X, \mathcal{F}, \triangle)$ where $(X, \mathcal{F})$ is a PM-space and t-norm $\triangle$ is such that the inequality

$$F_{x,z}(t+s) \geq \triangle\left(F_{x,y}(t), F_{y,z}(s)\right),$$

holds for all $x, y, z \in X$ and all $t, s > 0$.

Every metric space (X,d) can be realized as a PM-space by taking $\mathcal{F} : X \times X \to \Im$ defined by $F_{x,y}(t) = H(t - d(x,y))$ for all $x, y \in X$.

Definition 2.5. ([26]) A pair (A, S) of self mappings defined on a Menger space $(X, \mathcal{F}, \triangle)$ is said to be compatible if and only if $F_{ASx_n,SAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that Ax_n, $Sx_n \to z$ for some $z \in X$ as $n \to \infty$.

Definition 2.6. ([37]) A pair (A, S) of self mappings defined on a non-empty set X is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $Az = Sz$ some $z \in X$, then $ASz = SAz$.

Remark 2.7. Two compatible self mappings are weakly compatible, however the converse is not true in general (see [37, Example 1]).

Definition 2.8. ([18]) A pair (A, S) of self mappings defined on a non-empty set X is occasionally weakly compatible iff there is a point $x \in X$ which is a coincidence point of A and S at which A and S commute.

The following definition is on the lines of Bouhadjera and Godet-Thobie [3].

Definition 2.9. A pair (A, S) of self mappings defined on a Menger space $(X, \mathcal{F}, \triangle)$ is said to be subcompatible iff there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$ and $\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = 1$, for all $t > 0$.

Remark 2.10. Two occasionally weakly compatible mappings are subcompatible, however the converse is not true in general (see [4, Example 1.2]).

Definition 2.11. ([23]) A pair (A, S) of self mappings defined on a Menger space $(X, \mathcal{F}, \triangle)$ is said to be reciprocally continuous if for a sequence $\{x_n\}$ in X, $\lim_{n\to\infty} ASx_n = Az$ and $\lim_{n\to\infty} SAx_n = Sz$, whenever

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

Remark 2.12. ([28]) If two self mappings are continuous, then they are obviously reciprocally continuous but converse is not true. Moreover, in the setting of common fixed point theorems for compatible pair of self mappings satisfying contractive conditions, continuity of one of the mappings implies their reciprocal continuity but not conversely.

The notion of subsequentially continuous mappings in Menger spaces is as follows:

Definition 2.13. A pair (A, S) of self mappings defined on a Menger space $(X, \mathcal{F}, \triangle)$ is called subsequentially continuous iff there exists a sequence $\{x_n\}$ in X such that,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az$ and $\lim_{n\to\infty} SAx_n = Sz$.

Remark 2.14. If two self mappings are continuous or reciprocally continuous, then they are naturally subsequentially continuous. However, there exist subsequentially continuous pair of maps which are neither continuous nor reciprocally continuous (see [4, Example 1.4]).

3. Implicit Relation

In 2008, Imdad and Ali [14] used the following implicit relations for the existence of a common fixed point due to Popa [35].

Let Ψ denote the family of all continuous functions $\varphi : [0,1]^4 \to \mathbb{R}$ satisfying the following conditions:

(φ-1) For every $u > 0, v \geq 0$ with $\varphi(u, v, u, v) \geq 0$ or $\varphi(u, v, v, u) \geq 0$ we have $u > v$.

(φ-2) $\varphi(u, u, 1, 1) < 0$ for all $u > 0$.

Example 3.1. *Define* $\varphi : [0,1]^4 \to \mathbb{R}$ *as* $\varphi(t_1, t_2, t_3, t_4) = t_1 - \phi(\min\{t_2, t_3, t_4\})$, *where* $\phi : [0,1] \to [0,1]$ *is a continuous function such that* $\phi(s) > s$ *for* $0 < s < 1$.

Example 3.2. *Define* $\varphi : [0,1]^4 \to \mathbb{R}$ *as* $\varphi(t_1, t_2, t_3, t_4) = t_1 - k\min\{t_2, t_3, t_4\}$, *where* $k > 1$.

Example 3.3. *Define* $\varphi : [0,1]^4 \to \mathbb{R}$ *as* $\varphi(t_1, t_2, t_3, t_4) = t_1 - kt_2 - \min\{t_3, t_4\}$, *where* $k > 0$.

Example 3.4. *Define* $\varphi : [0,1]^4 \to \mathbb{R}$ *as* $\varphi(t_1, t_2, t_3, t_4) = t_1 - at_2 - bt_3 - ct_4$, *where* $a > 1$ *and* $b, c \geq 0$ $(b, c \neq 1)$.

Example 3.5. *Define* $\varphi : [0,1]^4 \to \mathbb{R}$ *as* $\varphi(t_1, t_2, t_3, t_4) = t_1 - at_2 - b(t_3 + t_4)$, *where* $a > 1$ *and* $0 \leq b < 1$.

Example 3.6. *Define* $\varphi : [0,1]^4 \to \mathbb{R}$ *as* $\varphi(t_1, t_2, t_3, t_4) = t_1^3 - kt_2t_3t_4$, *where* $k > 1$.

4. Main Results

Theorem 4.1. *Let* $P_1, P_2, \ldots, P_{2n}, A$ *and* B *be self mappings of a Menger space* $(X, \mathcal{F}, \triangle)$, *where* $\triangle$ *is a continuous t-norm. If the pairs* $(A, P_1P_3 \ldots P_{2n-1})$ *and* $(B, P_2P_4 \ldots P_{2n})$ *are compatible and subsequentially continuous, then*

(1) *the pair* $(A, P_1P_3 \ldots P_{2n-1})$ *has a coincidence point,*
(2) *the pair* $(B, P_2P_4 \ldots P_{2n})$ *has a coincidence point,*
(3) *there exists* $\varphi \in \Psi$ *such that*

$$\varphi\begin{pmatrix} F_{Ax,By}(t), F_{P_1P_3\ldots P_{2n-1}x, P_2P_4\ldots P_{2n}y}(t), \\ F_{Ax,P_1P_3\ldots P_{2n-1}x}(t), F_{By,P_2P_4\ldots P_{2n}y}(t) \end{pmatrix} \geq 0, \tag{4.1}$$

holds for all $x, y \in X$ *and* $t > 0$,
(4) *suppose that*

$P_1(P_3 \ldots P_{2n-1}) = (P_3 \ldots P_{2n-1})P_1$,
$P_1P_3(P_5 \ldots P_{2n-1}) = (P_5 \ldots P_{2n-1})P_1P_3$,
$\vdots$
$P_1 \ldots P_{2n-3}(P_{2n-1}) = (P_{2n-1})P_1 \ldots P_{2n-3}$,
$A(P_3 \ldots P_{2n-1}) = (P_3 \ldots P_{2n-1})A$,
$A(P_5 \ldots P_{2n-1}) = (P_5 \ldots P_{2n-1})A$,
$\vdots$
$AP_{2n-1} = P_{2n-1}A$,
similarly,
$P_2(P_4 \ldots P_{2n}) = (P_4 \ldots P_{2n})P_2$,
$P_2P_4(P_6 \ldots P_{2n}) = (P_6 \ldots P_{2n})P_2P_4$,
$\vdots$
$P_2 \ldots P_{2n-2}(P_{2n}) = (P_{2n})P_2 \ldots P_{2n-2}$,
$B(P_4 \ldots P_{2n}) = (P_4 \ldots P_{2n})B$,
$B(P_6 \ldots P_{2n}) = (P_6 \ldots P_{2n})B$,
$\vdots$
$BP_{2n} = P_{2n}B$.

Then $P_1, P_2, \ldots, P_{2n}, A$ *and* B *have a unique common fixed point in* X.

Proof. Since the pair $(A, P_1P_3 \ldots P_{2n-1})$ (also $(B, P_2P_4 \ldots P_{2n})$) is subsequentially continuous and compatible mappings, therefore there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} P_1P_3 \ldots P_{2n-1}x_n = z,$$

for some $z \in X$, and

$$\lim_{n\to\infty} F_{AP_1P_3\ldots P_{2n-1}x_n, P_1P_3\ldots P_{2n-1}Ax_n}(t) = F_{Az, P_1P_3\ldots P_{2n-1}z}(t) = 1,$$

for all $t > 0$ then $Az = P_1P_3\ldots P_{2n-1}z$, whereas in respect of the pair $(B, P_2P_4\ldots P_{2n})$, there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} P_2P_4\ldots P_{2n}y_n = w,$$

for some $w \in X$, and

$$\lim_{n\to\infty} F_{BP_2P_4\ldots P_{2n}y_n, P_2P_4\ldots P_{2n}By_n}(t) = F_{Bw, P_2P_4\ldots P_{2n}w}(t) = 1,$$

for all $t > 0$ then $Bw = P_2P_4\ldots P_{2n}w$. Hence z is a coincidence point of the pair $(A, P_1P_3\ldots P_{2n-1})$ whereas w is a coincidence point of the pair $(B, P_2P_4\ldots P_{2n})$.

Now we assert that $z = w$. By putting $x = x_n$ and $y = y_n$ in inequality (4.1), we have

$$\varphi\left(\begin{array}{c} F_{Ax_n,By_n}(t), F_{P_1P_3\ldots P_{2n-1}x_n, P_2P_4\ldots P_{2n}y_n}(t), \\ F_{Ax_n, P_1P_3\ldots P_{2n-1}x_n}(t), F_{By_n, P_2P_4\ldots P_{2n}y_n}(t) \end{array}\right) \geq 0.$$

Taking the limit as $n \to \infty$, we get

$$\varphi\left(F_{z,w}(t), F_{z,w}(t), F_{z,z}(t), F_{w,w}(t)\right) \geq 0,$$

and so

$$\varphi\left(F_{z,w}(t), F_{z,w}(t), 1, 1\right) \geq 0,$$

which contradicts (φ-2). Hence $z = w$. Now we show that $Az = z$ then by putting $x = z$ and $y = y_n$ in inequality (4.1), we get

$$\varphi\left(\begin{array}{c} F_{Az,By_n}(t), F_{P_1P_3\ldots P_{2n-1}z, P_2P_4\ldots P_{2n}y_n}(t), \\ F_{Az, P_1P_3\ldots P_{2n-1}z}(t), F_{By_n, P_2P_4\ldots P_{2n}y_n}(t) \end{array}\right) \geq 0.$$

Taking the limit as $n \to \infty$, we have

$$\varphi\left(F_{Az,z}(t), F_{P_1P_3\ldots P_{2n-1}z,w}(t), F_{Az, P_1P_3\ldots P_{2n-1}z}(t), F_{w,w}(t)\right) \geq 0,$$

and so

$$\varphi\left(F_{Az,z}(t), F_{Az,z}(t), F_{Az,Az}(t), F_{z,z}(t)\right) \geq 0,$$

or, equivalently,

$$\varphi\left(F_{Az,z}(t), F_{Az,z}(t), 1, 1\right) \geq 0,$$

which contradicts (φ-2). Hence $Az = z$. Therefore, $Az = P_1P_3\ldots P_{2n-1}z = z$. Now we assert that $Bz = z$, then by putting $x = x_n$ and $y = z$ in inequality (4.1), we have

$$\varphi\left(\begin{array}{c} F_{Ax_n,Bz}(t), F_{P_1P_3\ldots P_{2n-1}x_n, P_2P_4\ldots P_{2n}z}(t), \\ F_{Ax_n, P_1P_3\ldots P_{2n-1}x_n}(t), F_{Bz, P_2P_4\ldots P_{2n}z}(t) \end{array}\right) \geq 0.$$

Taking the limit as $n \to \infty$, we get

$$\varphi\left(F_{z,Bz}(t), F_{z,P_2P_4\ldots P_{2n}z}(t), F_{z,z}(t), F_{Bz,P_2P_4\ldots P_{2n}z}(t)\right) \geq 0,$$

and so

$$\varphi\left(F_{z,Bz}(t), F_{z,Bz}(t), F_{z,z}(t), F_{Bz,Bz}(t)\right) \geq 0,$$

or, equivalently,

$$\varphi\left(F_{z,Bz}(t), F_{z,Bz}(t), 1, 1\right) \geq 0,$$

which contradicts (φ-2). Hence $Bz = z$. Therefore, $Bz = P_2P_4\ldots P_{2n}z = z$. Now we prove that z is the common fixed point of all the component mappings. By putting $x = P_3\ldots P_{2n-1}z, y = z, P_1' = P_1P_3\ldots P_{2n-1}$ and $P_2' = P_2P_4\ldots P_{2n}$ in inequality (4.1), we get

$$\varphi\begin{pmatrix} F_{AP_3\ldots P_{2n-1}z,Bz}(t), F_{P_1'P_3\ldots P_{2n-1}z,P_2P_4\ldots P_{2n}z}(t), \\ F_{AP_3\ldots P_{2n-1}z,P_1'P_3\ldots P_{2n-1}z}(t), F_{Bz,P_2P_4\ldots P_{2n}z}(t) \end{pmatrix} \geq 0,$$

and so

$$\varphi\begin{pmatrix} F_{P_3\ldots P_{2n-1}z,z}(t), F_{P_3\ldots P_{2n-1}z,z}(t), \\ F_{P_3\ldots P_{2n-1}z,P_3\ldots P_{2n-1}z}(t), F_{z,z}(t) \end{pmatrix} \geq 0.$$

It implies,

$$\varphi\left(F_{P_3\ldots P_{2n-1}z,z}(t), F_{P_3\ldots P_{2n-1}z,z}(t), 1, 1\right) \geq 0,$$

which contradicts (φ-2). Hence $F_{P_3\ldots P_{2n-1}z,z}(t) = 1$. Thus $(P_3\ldots P_{2n-1})z = z$. That is $P_1z = P_1(P_3\ldots P_{2n-1}z) = z$. Continuing this procedure, we get $Az = P_1z = P_3z = \ldots = P_{2n-1}z = z$. In the same manner, taking $x = z, y = P_4\ldots P_{2n}z, P_1' = P_1P_3\ldots P_{2n-1}$ and $P_2' = P_2P_4\ldots P_{2n}$ in inequality (4.1), we get $z = P_4\ldots P_{2n}z$. Hence, $P_2z = z$. Continuing this procedure, we get $Bz = P_2z = P_4z = \ldots = P_{2n}z = z$. That is z is the common fixed point of $P_1, P_2, \ldots, P_{2n}, A$ and B.

The uniqueness of common fixed point is an easy consequence of inequality (4.1). □

Theorem 4.2. *Let $P_1, P_2, \ldots, P_{2n}$, A and B be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. If the pairs $(A, P_1P_3\ldots P_{2n-1})$ and $(B, P_2P_4\ldots P_{2n})$ are subcompatible and reciprocally continuous, then*

1. *the pair $(A, P_1P_3\ldots P_{2n-1})$ has a coincidence point,*
2. *the pair $(B, P_2P_4\ldots P_{2n})$ has a coincidence point,*
3. *further, the mappings $P_1, P_2, \ldots, P_{2n}, A$ and B have a unique common fixed point in X provided the involved mappings satisfy the inequality* (4.1) *and condition* (4) *of Theorem* 4.1.

Proof. Since the pair $(A, P_1P_3 \dots P_{2n-1})$ (also $(B, P_2P_4 \dots P_{2n})$) is subcompatible and reciprocally continuous, therefore there exists a sequences $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} P_1P_3 \dots P_{2n-1}x_n = z,$$

for some $z \in X$, and

$$\lim_{n\to\infty} F_{AP_1P_3\dots P_{2n-1}x_n, P_1P_3\dots P_{2n-1}Ax_n}(t) = \lim_{n\to\infty} F_{Az, P_1P_3\dots P_{2n-1}z}(t) = 1,$$

for all $t > 0$, whereas in respect of the pair (B, T), there exists a sequence $\{y_n\}$ in X with

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} P_2P_4 \dots P_{2n}y_n = w,$$

for some $w \in X$, and

$$\lim_{n\to\infty} F_{BP_2P_4\dots P_{2n}y_n, P_2P_4\dots P_{2n}By_n}(t) = \lim_{n\to\infty} F_{Bz, P_2P_4\dots P_{2n}z}(t) = 1,$$

for all $t > 0$. Therefore, $Az = P_1P_3 \dots P_{2n-1}z$ and $Bw = P_2P_4 \dots P_{2n}w$ i.e. z is a coincidence point of the pair $(A, P_1P_3 \dots P_{2n-1})$ whereas w is a coincidence point of the pair $(B, P_2P_4 \dots P_{2n})$. The rest of the proof can be completed from the proof of Theorem 4.1. □

Remark 4.3. Notice that the conclusion of Theorem 4.1 remains valid if we replace compatibility with subcompatibility and subsequential continuity with reciprocally continuity, besides retaining the rest of the hypothesis (see [14]).

The following theorem is a slight generalization of Theorem 4.1.

Theorem 4.4. *Let $\{T_\alpha\}_{\alpha\in J}$ and $\{P_i\}_{i=1}^{2n}$ be two families of self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. Suppose that there exists a fixed $\beta \in J$ such that the pairs $(T_\alpha, P_1P_3 \dots P_{2n-1})$ and $(T_\beta, P_2P_4 \dots P_{2n})$ are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair $(T_\alpha, P_1P_3 \dots P_{2n-1})$ has a coincidence point,*
(2) *the pair $(T_\beta, P_2P_4 \dots P_{2n})$ has a coincidence point,*
(3) *there exists $\varphi \in \Psi$ such that*

$$\varphi\begin{pmatrix} F_{T_\alpha x, T_\beta y}(t), F_{P_1P_3\dots P_{2n-1}x, P_2P_4\dots P_{2n}y}(t), \\ F_{T_\alpha x, P_1P_3\dots P_{2n-1}x}(t), F_{T_\beta y, P_2P_4\dots P_{2n}y}(t) \end{pmatrix} \geq 0, \tag{4.2}$$

holds for all $x, y \in X$ and $t > 0$,
(4) *suppose that*
$P_1(P_3 \dots P_{2n-1}) = (P_3 \dots P_{2n-1})P_1$,
$P_1P_3(P_5 \dots P_{2n-1}) = (P_5 \dots P_{2n-1})P_1P_3$,
$\vdots$
$P_1 \dots P_{2n-3}(P_{2n-1}) = (P_{2n-1})P_1 \dots P_{2n-3}$,

$$\begin{aligned}
&T_\alpha(P_3 \dots P_{2n-1}) = (P_3 \dots P_{2n-1})T_\alpha,\\
&T_\alpha(P_5 \dots P_{2n-1}) = (P_5 \dots P_{2n-1})T_\alpha,\\
&\vdots\\
&T_\alpha P_{2n-1} = P_{2n-1}T_\alpha,\\
&\textit{similarly,}\\
&P_2(P_4 \dots P_{2n}) = (P_4 \dots P_{2n})P_2,\\
&P_2P_4(P_6 \dots P_{2n}) = (P_6 \dots P_{2n})P_2P_4,\\
&\vdots\\
&P_2 \dots P_{2n-2}(P_{2n}) = (P_{2n})P_2 \dots P_{2n-2},\\
&T_\beta(P_4 \dots P_{2n}) = (P_4 \dots P_{2n})T_\beta,\\
&T_\beta(P_6 \dots P_{2n}) = (P_6 \dots P_{2n})T_\beta,\\
&\vdots\\
&T_\beta P_{2n} = P_{2n}T_\beta.
\end{aligned}$$

Then all $\{P_i\}$ *and* $\{T_\alpha\}$ *have a unique common fixed point in* X*.*

Proof. Let T_{α_0} be a fixed element in $\{T_\alpha\}_{\alpha\in J}$. By Theorem 4.1 with $A = T_{\alpha_0}$ and $B = T_\beta$ it follows that there exists some $z \in X$ such that

$$T_\beta z = T_{\alpha_0}z = P_1P_3 \dots P_{2n-1}z = P_2P_4 \dots P_{2n}z = z.$$

Let $\alpha \in J$ be arbitrary. Then applying inequality (4.2), we obtain

$$\varphi\begin{pmatrix} F_{T_\alpha z, T_\beta z}(t), F_{P_1P_3\dots P_{2n-1}z, P_2P_4\dots P_{2n}z}(t), \\ F_{T_\alpha z, P_1P_3\dots P_{2n-1}z}(t), F_{T_\beta z, P_2P_4\dots P_{2n}z}(t) \end{pmatrix} \geq 0,$$

and so

$$\varphi\left(F_{T_\alpha z,z}(t), F_{z,z}(t), F_{T_\alpha z,z}(t), F_{z,z}(t)\right) \geq 0,$$

or, equivalently,

$$\varphi\left(F_{T_\alpha z,z}(t), 1, F_{T_\alpha z,z}(t), 1\right) \geq 0,$$

yielding thereby, $F_{T_\alpha z,z}(t) > 1$, a contradiction. Hence $F_{T_\alpha z,z}(t) = 1$. Thus, $T_\alpha z = z$ for each $\alpha \in J$. Since inequality (4.2) implies the uniqueness of the common fixed point, Theorem 4.4 is proved. □

Corollary 4.5. *Let* A, B, S *and* T *be self mappings of a Menger space* $(X, \mathcal{F}, \triangle)$*, where* $\triangle$ *is a continuous t-norm. If the pairs* (A, S) *and* (B, T) *are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair* (A, S) *has a coincidence point,*
(2) *the pair* (B, T) *has a coincidence point,*

(3) *there exists* $\varphi \in \Psi$ *such that*

$$\varphi\left(F_{Ax,By}(t), F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t)\right) \geq 0 \tag{4.3}$$

holds for all $x, y \in X$ *and* $t > 0$.

Then A, B, S *and* T *have a unique common fixed point in* X.

Proof. By setting $P_1P_3 \ldots P_{2n-1} = S$ and $P_2P_4 \ldots P_{2n} = T$ in Theorem 4.1, the proof easily follows. □

Alternately, by setting $S = T$ in Corollary 4.5, we can also derive yet another corollary for three mappings which runs as follows.

Corollary 4.6. *Let* A, B *and* S *be self mappings of a Menger space* $(X, \mathcal{F}, \triangle)$, *where* $\triangle$ *is a continuous t-norm. If the pairs* (A, S) *and* (B, S) *are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair* (A, S) *has a coincidence point,*
(2) *the pair* (B, S) *has a coincidence point,*
(3) *there exists* $\varphi \in \Psi$ *such that*

$$\varphi\left(F_{Ax,By}(t), F_{Sx,Sy}(t), F_{Ax,Sx}(t), F_{By,Sy}(t)\right) \geq 0, \tag{4.4}$$

holds for all $x, y \in X$ *and* $t > 0$.

Then A, B *and* S *have a unique common fixed point in* X.

On taking $A = B$ and $S = T$ in Corollary 4.5 then we get the interesting result.

Corollary 4.7. *Let* A *and* S *be self mappings of a Menger space* $(X, \mathcal{F}, \triangle)$, *where* $\triangle$ *is a continuous t-norm. If the pair* (A, S) *is compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair* (A, S) *has a coincidence point,*
(2) *there exists* $\varphi \in \Psi$ *such that*

$$\varphi\left(F_{Ax,Ay}(t), F_{Sx,Sy}(t), F_{Ax,Sx}(t), F_{Ay,Sy}(t)\right) \geq 0, \tag{4.5}$$

holds for all $x, y \in X$ *and* $t > 0$.

Then A *and* S *have a unique common fixed point in* X.

Remark 4.8. The conclusions of Theorem 4.1, Theorem 4.4, Corollary 4.5, Corollary 4.6, Corollary 4.7 remain true if the corresponding inequalities are replaced in such a manner as defined in Example 3.1-3.6.

Example 4.9. *Let* $X = [0, \infty)$ *and* d *be the usual metric on* X *and for each* $t \in [0, 1]$ *define*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$. Clearly $(X, \mathcal{F}, \triangle)$ be a Menger space. Let A and S be self mappings on X defined as

$$A(X) = \begin{cases} \frac{x}{4}, & \text{if } x \in [0,1]; \\ 5x-4, & \text{if } x \in (1,\infty), \end{cases} \quad S(X) = \begin{cases} \frac{x}{5}, & \text{if } x \in [0,1]; \\ 4x-3, & \text{if } x \in (1,\infty). \end{cases}$$

Consider a sequence $\{x_n\} = \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ in X. Then

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} \left(\frac{1}{4n}\right) = 0 = \lim_{n\to\infty} \left(\frac{1}{5n}\right) = \lim_{n\to\infty} S(x_n).$$

Next,

$$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(\frac{1}{5n}\right) = \lim_{n\to\infty} \left(\frac{1}{20n}\right) = 0 = A(0),$$

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(\frac{1}{4n}\right) = \lim_{n\to\infty} \left(\frac{1}{20n}\right) = 0 = S(0),$$

and

$$\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = 1,$$

for all $t > 0$. Consider another sequence $\{x_n\} = \left\{1 + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ in X. Then

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} \left(5 + \frac{5}{n} - 4\right) = 1 = \lim_{n\to\infty} \left(4 + \frac{4}{n} - 3\right) = \lim_{n\to\infty} S(x_n).$$

Also,

$$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(1 + \frac{4}{n}\right) = \lim_{n\to\infty} \left(5 + \frac{20}{n} - 4\right) = 1 \neq A(1),$$

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(1 + \frac{5}{n}\right) = \lim_{n\to\infty} \left(4 + \frac{20}{n} - 3\right) = 1 \neq S(1),$$

but $\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = 1$. Thus, the pair (A, S) is compatible as well as subsequentially continuous but not reciprocally continuous. Further, we can easily verify inequality (4.5) *by defining φ as in Example* 3.1 *and choosing $\phi(t) = \sqrt{t}$ for all $t \in (0,1)$. Therefore all the conditions of Corollary* 4.7 *are satisfied. Here, 0 is a coincidence as well as unique common fixed point of the pair (A, S). It is noted that this example cannot be covered by those fixed point theorems which involve compatibility and reciprocal continuity both or by involving conditions on completeness (or closedness) of underlying space (or subspaces). Also, in this example neither X is complete nor any subspace $A(X) = \left[0, \frac{1}{4}\right] \cup (1,\infty)$ and $S(X) = \left[0, \frac{1}{5}\right] \cup (1,\infty)$ are closed. It is noted that this example cannot be covered by those fixed point theorems which involve compatibility and reciprocal continuity both (e.g.* [20, 22, 23, 28]*).*

Example 4.10. *Let $X = \mathbb{R}$ (set of real numbers) and d be the usual metric on X and for each $t \in [0,1]$ define*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$. Clearly $(X, \mathcal{F}, \triangle)$ be a Menger space. Let A and S be self mappings on X defined as

$$A(X) = \begin{cases} \frac{x}{4}, & \text{if } x \in (-\infty, 1); \\ 5x - 4, & \text{if } x \in [1, \infty), \end{cases} \quad S(X) = \begin{cases} x + 3, & \text{if } x \in (-\infty, 1); \\ 4x - 3, & \text{if } x \in [1, \infty). \end{cases}$$

Consider a sequence $\{x_n\} = \{1 + \frac{1}{n}\}_{n \in \mathbb{N}}$ in X. Then

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} \left(5 + \frac{5}{n} - 4\right) = 1 = \lim_{n\to\infty} \left(4 + \frac{4}{n} - 3\right) = \lim_{n\to\infty} S(x_n).$$

Also,

$$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(1 + \frac{4}{n}\right) = \lim_{n\to\infty} \left(5 + \frac{20}{n} - 4\right) = 1 = A(1),$$

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(1 + \frac{5}{n}\right) = \lim_{n\to\infty} \left(4 + \frac{20}{n} - 3\right) = 1 = S(1),$$

and

$$\lim_{n\to\infty} F_{ASx_n, SAx_n}(t) = 1,$$

for all $t > 0$. Consider another sequence $\{x_n\} = \{\frac{1}{n} - 4\}_{n \in \mathbb{N}}$ in X. Then

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} \left(\frac{1}{4n} - 1\right) = -1 = \lim_{n\to\infty} \left(\frac{1}{n} - 4 + 3\right) = \lim_{n\to\infty} S(x_n).$$

Next,

$$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(\frac{1}{n} - 1\right) = \lim_{n\to\infty} \left(\frac{1}{4n} - \frac{1}{4}\right) = -\frac{1}{4} = A(-1),$$

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(\frac{1}{4n} - 1\right) = \lim_{n\to\infty} \left(\frac{1}{4n} - 1 + 3\right) = 2 = S(-1),$$

and $\lim_{n\to\infty} F_{ASx_n, SAx_n}(t) \neq 1$. Thus, the pair (A, S) is reciprocally continuous as well as subcompatible but not compatible. Further, we can easily verify inequality (4.5) *by defining φ as in Example* 3.1 *and choosing $\phi(t) = \sqrt{t}$ for all $t \in (0, 1)$. Therefore all the conditions of Corollary* 4.7 *are satisfied. Thus 1 is a coincidence as well as unique common fixed point of the pair (A, S). It is also noted that this example too cannot be covered by those fixed point theorems which involve compatibility and reciprocal continuity both (e.g.* [20, 22, 23, 28]*).*

5. Corresponding results in integral analogue

In this section, we present an integral-type fixed point theorem in the following form:

Corollary 5.1. *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. Assume that there exist a Lebesgue integrable function $\phi : \mathbb{R} \to \mathbb{R}$ and a function $\varphi \in \Psi$ such that*

$$\int_0^{\varphi(u,1,u,1)} \phi(s)ds \geq 0, \quad \int_0^{\varphi(u,1,1,u)} \phi(s)ds \geq 0, \quad \int_0^{\varphi(u,u,1,1)} \phi(s)ds \geq 0, \quad (5.1)$$

implies $u = 1$. Suppose that the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous). If

$$\int_0^{\varphi\left(F_{Ax,By}(t), F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t)\right)} \phi(s)ds \geq 0, \quad (5.2)$$

holds for all $x, y \in X$ and $t > 0$, then the pairs (A, S) and (B, T) have a coincidence point each and a unique common fixed point in X.

Proof. The proof easily follows from Corollary 4.5. But due to paucity of the space, we did not include the entire details of proof. □

Remark 5.2. The results similar to Corollary 5.1 can be outlined in respect of Theorem 4.1, Theorem 4.2, Corollary 4.6 and Corollary 4.7 (also in view of Examples 3.1-3.6) which generalize the well known integral-type fixed point theorems contained in [5].

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