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SOME COMMON FIXED POINT THEOREMS FOR $\mathcal{D}$-OPERATOR PAIR WITH APPLICATIONS TO NONLINEAR INTEGRAL EQUATIONS

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Abstract. In this paper, we define $\mathcal{D}$-operator pair for single valued mappings and obtain some common fixed point theorems for this class of maps under relaxed conditions. As application, we discuss the existence of solutions for some nonlinear integral equations that appear in nonlinear analysis and its applications. We also provide some illustrative examples to highlight the realized improvements.

1. Introduction

The study of common fixed points of mappings satisfying certain contractive type conditions has been the focus of various research activity. In 1976, Jungck [1] initiated a study of common fixed points for commuting maps. On the other hand in 1982, Sessa [2] defined weak commutativity and proved common fixed point theorem for weak commuting mappings. In 1986, Jungck [3] defined the compatible maps which is useful for obtaining existence of fixed point theorems and in the study of periodic points. Al-Thagafi and Shahzad [4] defined the concept of occasionally weakly compatible(owc) maps.

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In this paper, we introduce the notion of $\mathcal{D}$-operator pair for single valued maps. This pair contains commuting maps, weakly compatible maps and owc maps as a proper subclass. For this new operator, we prove some common fixed point theorems for a pair of two single valued mappings and a quadruple of single valued mappings without using the (E.A) property or completeness or weak compatibility. In Section 4, we prove the fixed point theorems without using the concept of triangle inequality or symmetry and in Section 5, we prove the results in symmetric space only. The results obtained by us give proper generalization of some important fixed point theorems (see [4]-[11]) and open up a wider scope for the study of common fixed points under contractive type conditions and discuss some illustrative examples. We also see the applications of our results to nonlinear integral equations.

2. Preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of natural numbers, real numbers and complex numbers, respectively and (X, d) denotes a metric space. For $x \in X$ and $A \subseteq X$,

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

Let $f : X \longrightarrow X$ and $g : X \longrightarrow X$. A point $x \in X$ is a fixed point of f if $x = fx$. The set of all fixed points of f is denoted by $F(f)$. A point $x \in X$ is a coincidence point of f and g if $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g. The set of all coincidence points and points of coincidence of f and g are denoted by $C(f, g)$ and $PC(f, g)$, respectively. A point $x \in X$ is a common fixed point of f and g, if $x = fx = gx$. The set of all common fixed points of f and g is denoted by $F(f, g)$. The pair $\{f, g\}$ is called

(1) commuting [1], if $fgx = gfx$, $\forall x \in X$,
(2) weakly commuting [2], if $d(fgx, gfx) \leq d(fx, gx)$, for each $x \in X$,
(3) compatible [3], if $\lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$, for some $t \in X$,
(4) weakly compatible [12], if they commute at there coincidence point i.e. $fgx = gfx$, whenever $fx = gx$, for $x \in X$,
(5) occasionally weakly compatible([4], see also [7])(owc), if $fgu = gfu$, for some $u \in C(f, g)$.
(6) $\mathcal{PD}$-operator pair [11], if there is a point u in X such that $u \in C(f, g)$ and $d(fgu, gfu) \leq diam(PC(f, g))$.

Definition 2.1. Let X be a nonempty set and d be a function $d : X \times X \longrightarrow [0, \infty)$ such that

$$d(x, y) = 0 \text{ iff } x = y, \quad \forall x, y \in X. \tag{2.1}$$

For a space (X, d) satisfying (2.1) and $A \subseteq X$, the diameter of A is defined by

$$\operatorname{diam}(A) = \sup\{\max\{d(x, y), d(y, x)\} : x, y \in A\}.$$

Definition 2.2. A symmetric on a set X is a mapping $d : X \times X \longrightarrow [0, \infty)$ such that

(1) $d(x, y) = 0$ if and only if $x = y$, and
(2) $d(x, y) = d(y, x)$.

A set X, together with a symmetric d is called a symmetric space.

3. $\mathcal{D}$-operator pair

Definition 3.1. Let $f, g : X \longrightarrow X$ be mappings. The pair (f, g) is said to be $\mathcal{D}$-operator pair, if there is a point u in X such that $u \in C(f, g)$ and $d(fgu, gfu) \leq R\ diam(PC(f, g))$, for some $R > 0$.

Example 3.1. Let $X = \mathbb{C}$, define $f, g : X \longrightarrow X$ by

$$f(z) = \begin{cases} e^{|z-1|}, & \text{if } z \neq 0, \\ 2, & \text{if } z = 0, \end{cases} \quad g(z) = \begin{cases} 1 + \sin(\log\ z), & \text{if } z \neq 0, \\ 2, & \text{if } z = 0, \end{cases}$$

Since, $C(f, g) = \{0, 1\}$ and $PC(f, g) = \{1, 2\}$.
One can easily calculate that the pair (f, g) is a $\mathcal{D}$-operator pair.

Remark 3.1. (i) Every $\mathcal{PD}$-operator pair is $\mathcal{D}$-operator pair (for $R = 1$), whereas, the converse need not be true, since if $R > 1$, then $\mathcal{PD}$-operator pair need not be $\mathcal{D}$-operator pair.

(ii) In the following diagram, we can see that every pair of nontrivial weakly compatible and occasionally weakly compatible self mappings are $\mathcal{D}$-operator pair. In addition, $\mathcal{D}$-operator pairs are weaker form of weakly commuting and compatible maps, but the reverse implications need not be true.

$$\begin{tikzcd}
\text{Commuting maps} \arrow[r] & \text{Weakly commuting maps} \arrow[r] & \text{Compatible maps} \arrow[d] \\
\mathcal{D}\text{-operator pair} & & \text{Weakly compatible maps} \arrow[ll] \arrow[dl] \\
& \text{Occasionally weakly compatible maps} \arrow[ul] &
\end{tikzcd}$$

Example 3.2. Let $X = [0, \infty)$. Define $f, g : X \longrightarrow X$ by

$$f(x) = \begin{cases} \cos x, & \text{if } \quad x \neq \frac{1}{2}, \\ \\ \frac{1}{4}, & \text{if } \quad x = \frac{1}{2}, \end{cases} \qquad g(x) = \begin{cases} e^x, & \text{if } \quad x \neq \frac{1}{2}, \\ \\ \frac{1}{4}, & \text{if } \quad x = \frac{1}{2}. \end{cases}$$

Here $C(f, g) = \{0, \frac{1}{2}\}$ and $PC(f, g) = \{1, \frac{1}{4}\}$.
Clearly, (f, g) is $\mathcal{D}$-operator pair, but not commuting, not weakly compatible and not occasionally weakly compatible.

4. Fixed point theorems for $\mathcal{D}$- operator pairs under relaxed conditions

In this section, we prove some fixed point theorems for $\mathcal{D}$-operator pairs on the space (X, d), without imposing the restriction of the triangle inequality or symmetry on d. Let $\phi : R^+ \longrightarrow R^+$ be a nondecreasing function satisfying the condition $\phi(t) < t$, for each $t > 0$. We now prove the following theorem.

Theorem 4.1. *Let X be a nonempty set and $d : X \times X \longrightarrow [0, \infty)$ be a function satisfying the condition* (2.1)*. Suppose (f, g) is $\mathcal{D}$-operator pair and satisfy the condition*

$$d(fx, fy) \leq \phi(\max\{d(gx, gy), d(gx, fy), d(fy, gy)\}), \tag{4.1}$$

for each $x, y \in X$. Then f and g have a unique common fixed point.

Proof. Since, (f, g) is $\mathcal{D}$-operator pair, there is a point u in X and $R > 0$ such that $fu = gu$ and

$$d(fgu, gfu) \leq R \operatorname{diam}(PC(f, g)). \tag{4.2}$$

First, we prove that $PC(f, g)$ is singleton. Suppose w and z be two distinct points in X such that $w = fu = gu$ and $z = fv = gv$, for some $u, v \in C(f, g)$. Then from (4.1), we obtain

$$\begin{aligned} d(w, z) = d(fu, fv) &\leq \phi(\max\{d(gu, gv), d(gu, fv), d(fv, gv)\}), \\ &\leq \phi(\max\{d(fu, fv), d(fu, fv), d(fv, fv)\}), \\ &\leq \phi(d(fu, fv)) < d(w, z). \end{aligned}$$

This is a contradiction. So, $w = z$ i.e., $w = fu = gu = fv = gv = z$.
Thus, $PC(f, g)$ is singleton i.e., $w = fu = gu$ is the unique point of coincidence and $\operatorname{diam}(PC(f, g)) = 0$.
From (4.2), $fgu = gfu$, for some points $u \in C(f, g)$. Now, by (4.1), we have

$$\begin{aligned} d(ffu, fv) = d(ffu, fu) &\leq \phi(\max\{d(gfu, gv), d(gfu, fv), d(fv, gv)\}) \\ &\leq \phi(\max\{d(ffu, fu), d(ffu, fu), 0\}) \\ &\leq \phi(d(ffu, fu)). \end{aligned}$$

This is a contradiction. Hence, $ffu = gfu = fu$, f and g have a common fixed point. Uniqueness is obvious. So, f and g have a unique common fixed point. This completes the proof of the Theorem 4.1. $\square$

Remark 4.1. (i) The above result is a proper generalization of Theorem 4.4 of Pathak and Deepmala [11].

(ii) It is clear to note that in the proof of above result, we need not to assume symmetry and triangle inequality of d. Hence, the above result is a proper generalization of the results due to Aamri and Moutawakil [10], for $\mathcal{D}$-operator pair.

Example 4.1. Let $X = [0, 1]$ and defined $d : X \times X \longrightarrow [0, \infty)$ by

$$d(x, y) = \begin{cases} x^2 - y^2, & \text{if } x \geq y, \\ (y - x)^2, & \text{if } x < y. \end{cases}$$

Define $f, g : X \longrightarrow X$ by $f(x) = x/2$ and $g(x) = x$, for all $x, y \in X$. Suppose $\phi : R^+ \longrightarrow R^+$ is defined as, $\phi(t) = t/2,\ t \geq 0$. Then $\phi(t)$ is a nondecreasing function satisfying the condition $\phi(t) < t$, for each $t > 0$. Here, (f, g) is $\mathcal{D}$-operator pair and d satisfy condition (2.1) and it is not symmetric.
In order to verify the contractive condition (4.1).
In case, $x \neq y$ and $0 \leq x < y \leq 1$, then

$$\begin{aligned} d(fx, fy) &= d(x/2, y/2) = ((y - x)/2)^2 \\ &< (y - x)^2/2 = \phi(y - x)^2 \leq \phi(d(gx, gy)). \end{aligned}$$

In case, $x \neq y$ and $0 \leq y \leq x \leq 1$, then

$$\begin{aligned} d(fx, fy) &= d(x/2, y/2) = (x/2)^2 - (y/2)^2 \\ &< (x^2 - y^2)/2 = \phi(x^2 - y^2) \leq \phi(d(gx, gy)). \end{aligned}$$

Thus all the conditions of the Theorem 4.1 are satisfied and 0 is the unique common fixed point of f and g. Here, one needs to note that d is not a metric as d is not symmetric and $d(0, 1) > d(0, 1/2)+d(1/2, 1)$. Thus, all the available metrical common fixed point theorems cannot be used in the context of this example.

Theorem 4.2. *Let X be a nonempty set and $d : X \times X \longrightarrow [0, \infty)$ be a function satisfying the condition* (2.1)*. Suppose (f, g) is $\mathcal{D}$-operator pair and satisfy the condition*

$$\begin{aligned} d(fx, fy) \leq a\ d(gx, gy) &+ b\ \max\{d(fx, gx), d(fy, gy)\} \\ &+ c\ \max\{d(gx, gy), d(gx, fx), d(gy, fy)\}, \end{aligned} \tag{4.3}$$

for each $x, y \in X$, where $a, b, c > 0$, $a + b + c = 1$ and $a + c < 1$. Then f and g have a unique common fixed point.

Proof. By hypothesis, there is a point $u \in X$ such that $fu = gu$. Suppose that there exists another point $v \in X$ for which $fv = gv$. Then from (4.3),

$$\begin{aligned} d(fu, fv) &\leq a\, d(fu, fv) + b \ \max\{0, 0\} + c \ \max\{d(fu, fv), 0, 0\} \\ &= (a + c)\ d(fu, fv), \end{aligned}$$

since, $(a + c) < 1$, the above inequality implies that $d(fu, fv) = 0$, which in turn implies that $fu = fv$. Therefore, fu is unique, diam $(PC(f, g)) = 0$ and $fgu = gfu$, for some points $u \in C(f, g)$. Now, by (4.3), we get

$$d(ffu, fv) = d(ffu, fu) < (d(ffu, fu)).$$

This is a contradiction. Hence, $ffu = gfu = fu$, f and g have a common fixed point. Uniqueness, is obvious. So, f and g have a unique common fixed point. This completes the proof of the Theorem 4.2. □

Remark 4.2. The above result is a proper generalization of Theorem 2 of Jungck and Rhoades [7] and Theorem 2.3 of Bhatt et al. [6] for $\mathcal{D}$-operator pair.

5. Fixed point theorems for $\mathcal{D}$- operator pairs in symmetric spaces

In this section, we prove some fixed point theorems for $\mathcal{D}$-operator pairs on the space (X, d), without imposing the restriction of the triangle inequality and assuming symmetry on d.

Theorem 5.1. *Let f and g be selfmaps of symmetric space X. Suppose (f, g) is $\mathcal{D}$-operator pair and satisfy the condition:*

$$\begin{aligned} d(fx, fy) \leq h \ \max\{&d(gx, gy), d(fx, gx), d(fy, gy), \\ &d(fx, gy), d(fy, gx)\}, \end{aligned} \tag{5.1}$$

for each $x, y \in X, 0 \leq h < 1$. Then f and g have a unique common fixed point.

Proof. Since, (f, g) is $\mathcal{D}$-operator pair, there exists some points $u \in C(f, g)$ such that

$$d(fgu, gfu) \leq R \ \mathrm{diam}(PC(f, g)), \ \text{for some } R > 0. \tag{5.2}$$

First, we prove that $PC(f, g)$ is singleton. Suppose not, let $w, z \in PC(f, g)$ such that $w = fu = gu$ and $z = fv = gv$, $\forall\ u, v, w, z \in X$. Let $w \neq z$, then from (5.1), we have

$$\begin{aligned} d(w,z) &= d(fu,fv) \\ &\leq h\ \max\{d(gu,gv), d(fu,gu), d(fv,gv), d(fu,gv), d(fv,gu)\} \\ &\leq h\ d(fu,fv) \\ &< d(w,z). \end{aligned}$$

This is a contradiction. So, $w = z$ i.e. $w = fu = gu = fv = gv = z$. Thus, $PC(f,g)$ is singleton, i.e. $w = fu = gu$ is the unique point of coincidence. $\mathrm{diam}(PC(f,g)) = 0$. Form (5.2), $fgu = gfu$, for some points $u \in C(f,g)$. Now, by (5.1), we have

$$\begin{aligned} d(ffu,fv) &= d(ffu,fu) \\ &\leq h\ \max\{d(gfu,gv), d(ffu,gfu), d(fv,gv), \\ &\quad d(ffu,gv), d(fv,gfu)\}, \\ &\leq h\ (\max\{d(ffu,fu), d(fu,ffu)\}), \\ &< d(ffu,fu). \end{aligned}$$

This is a contradiction. Hence, $ffu = gfu = fu$, f and g have a common fixed point. Uniqueness, is obvious from (5.1). So, f and g have a unique common fixed point. This completes the proof of the Theorem 5.1. □

Remark 5.1. Theorem 5.1 contains Theorem 2.1 of Al-Thagafi and Shahzad [4] and Jungck and Hussain [8] as a special case.

Example 5.1. Let $X = \mathbb{N}^2 \cup \{0\}$ and $u = (x_1, y_1),\ v = (x_2, y_2) \in X$.
Define $d : X \times X \longrightarrow [0, \infty)$ by $d(u,v) = (x_2 - x_1)^2 + (y_2 - y_1)^2$.
Define $f, g : X \longrightarrow X$ by

$$f(x,y) = \begin{cases} (x+1, y+2), & \text{if} \quad (x,y) \neq (0,0), \\ (0,0), & \text{if} \quad (x,y) = (0,0), \end{cases}$$

$$g(x,y) = \begin{cases} (2x^2, 3y^2), & \text{if} \quad (x,y) \neq (0,0), \\ (0,0), & \text{if} \quad (x,y) = (0,0). \end{cases}$$

Clearly, $C(f,g) = \{(0,0),(1,1)\}$, $PC(f,g) = \{(0,0),(2,3)\}$. Also, (f,g) is a $\mathcal{D}$-operator pair. In order to verify contractive condition (5.1), if $(x_1,y_1) \neq (0,0)$, $(x_2,y_2) \neq (0,0)$,

$$\begin{aligned} d(fu,fv) &= d((x_1+1, y_1+2), (x_2+1, y_2+2)) \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &\leq (x_2 - x_1)^2 (x_2 + x_1)^2 + (y_2 - y_1)^2 (y_2 + y_1)^2 \\ &\leq \tfrac{1}{2}(4(x_2 - x_1)^2 (x_2 + x_1)^2 + 9(y_2 - y_1)^2 (y_2 + y_1)^2) \\ &= \tfrac{1}{2} d(gu,gv). \end{aligned}$$

If $(x_1, y_1) = (x_2, y_2) = (0, 0)$, contractive condition (5.1) is satisfied obviously. By symmetry of d, other cases follows. Thus, all the conditions of Theorem 5.1 are satisfied. Hence, f and g has a unique fixed point $(0, 0)$ in X. In this example, it is easy to see that d is not a metric, since it is symmetric but not transitive.

Corollary 5.1. *Let f and g be selfmaps of symmetric space X. Suppose (f, g) is a $\mathcal{D}$-operator pair and satisfy the condition:*

$$d(fx, fy) \leq h\ d(gx, gy), \tag{5.3}$$

for each $x, y \in X, 0 \leq h < 1$. Then f and g have a unique common fixed point.

Let ϕ is defined as in Section 4. The proof of the following theorem can be easily obtained by replacing condition (5.1) by condition (5.4) in the proof of Theorem 5.1.

Theorem 5.2. *Let f and g be a selfmaps of symmetric space X. Suppose (f, g) is a $\mathcal{D}$-operator pair and satisfy the condition:*

$$d(fx, fy) \leq \phi(\max\{d(gx, gy), d(gx, fy), d(gy, fx), d(gy, fy))\}), \tag{5.4}$$

for each $x, y \in X$. Then f and g have a unique common fixed point.

Example 5.2. Let $X = [0, 1]$ and defined $d : X \times X \longrightarrow [0, \infty)$ by $d(x, y) = (x - y)^2$. Define $f, g : X \longrightarrow X$ by $f(x) = x/3$ and $g(x) = x$, for all $x, y \in X$. Let $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\phi(t) = t/3$, for $t \geq 0$. Clearly, $\phi(t) < t$ is nondecreasing function. Here (f, g) is a $\mathcal{D}$-operator pair. In order, to verify contractive condition(5.4), we see that
If $x = 0$, $0 < y \leq 1$, then

$$\begin{aligned} d(fx, fy) &= d(0, y/3) = (y/3)^2 \\ &< y^2/3 = \phi(y^2) \leq \phi(d(gx, gy)). \end{aligned}$$

If $0 < x < y \leq 1$, then

$$\begin{aligned} d(fx, fy) &= d(x/3, y/3) = ((x - y)/3)^2 \\ &< (x - y)^2/3 = \phi((x - y)^2) \leq \phi(d(gx, gy)). \end{aligned}$$

By symmetry of d, other cases follows. Thus, all the conditions of Theorem 5.2 are satisfied and 0 is the unique common fixed point of f and g.

Corollary 5.2. *Let f and g be selfmaps of symmetric space X. Suppose (f, g) is a $\mathcal{D}$-operator pair and satisfy the condition:*

$$d(fx, fy) \leq \phi(d(gx, gy)), \tag{5.5}$$

for each $x, y \in X$. Then f and g have a unique common fixed point.

Remark 5.2. Theorem 5.2 and Corollary 5.2 are proper generalization of the results due to Bhatt et al. [6] for $\mathcal{D}$-operator pair and an extension of results due to Hussain et al. [5] for $\mathcal{D}$-operator pair.

The proof of the following theorem can be easily obtained by replacing condition (5.1) by condition (5.6) in the proof of Theorem 5.1.

Theorem 5.3. *Let f and g be a selfmaps of symmetric space X. Suppose (f, g) is a $\mathcal{D}$-operator pair and satisfy the condition:*

$$\begin{aligned} d(fx, fy) \leq h\ \max\{&d(gx, gy), (d(fx, gx) + d(fy, gy))/2, \\ &(d(fx, gy) + d(fy, gx))/2\}, \end{aligned} \tag{5.6}$$

for each $x, y \in X, 0 \leq h < 1$. Then f and g have a unique common fixed point.

Remark 5.3. The above result is a proper extension of the results due to Aamri and Moutawakil [9] for $\mathcal{D}$-operator pair for symmetric space without assuming completeness or weak compatibility or E.A. property or triangle inequality on d.

Theorem 5.4. *Suppose that f, g, S and T are selfmaps of symmetric space X and that (f, S) and (g, T) are $\mathcal{D}$-operator pairs and satisfy the condition:*

$$\begin{aligned} d(fx, gy) \leq \phi(\max\{&d(Sx, Ty), d(fx, Sx), d(gy, Ty), \\ &d(fx, Ty), d(gy, Sx)\}), \end{aligned} \tag{5.7}$$

for each $x, y \in X$. Then f, g, S and T have a unique common fixed point.

Proof. Since the pair (f, S) and (g, T) are $\mathcal{D}$-operator pairs, there exists points $x, y \in X$ such that $fx = Sx$ and $gy = Ty$. Also, $d(fSx, Sfx) \leq R$ diam $(PC(f, S))$ and $d(gTy, Tgy) \leq R$ diam $(PC(g, T))$, for some $R > 0$. We claim that $fx = gy$, otherwise, by (5.7), we have

$$\begin{aligned} d(fx, gy) &\leq \phi(\max\{d(fx, gy), d(fx, fx), d(gy, gy), d(fx, gy), d(gy, fx)\}) \\ &\leq \phi(\max\{d(fx, gy), d(gy, fx)\}) \\ &< d(fx, gy). \end{aligned}$$

This is a contradiction. Thus $fx = Sx = gy = Ty$. Moreover, if there is another point $u \in X$ such that $fu = Su$, then using (5.7), it follows that $fu = Su = gy = Ty$, or $fx = fu$. Hence x is the unique point of coincidence of f and S. Similarly, y is the unique point of coincidence of g and T. Thus, diam $(PC(f, S)) = 0$ and diam $(PC(g, T)) = 0$. Hence $fSx = Sfx$, $gTy = Tgy$. By condition (5.7), we have

$$d(ffx, gy) = d(ffx, fx) < d(ffx, fx).$$

This is a contradiction. Thus, $ffx = Sfx = fx$, i.e., f and S have a common fixed point, which is obviously unique. By similar argument, we can show that g and T have a unique common fixed point. For uniqueness, suppose there are points $v, w \in X$ and $v \neq w$ such that $v = fv = Sv$ and $w = gw = Tw$. By (5.7), we have

$$d(v, w) = d(fv, gw) < d(v, w).$$

This is a contradiction. Hence, f, g, S and T have a unique common fixed point. This completes the proof of the Theorem 5.4. □

Remark 5.4. A special case of Theorem 5.4 is a proper generalization of Theorem 4 and Theorem 5 of Jungck and Rhoades [7] for $\mathcal{D}$-operator pair.

The proof of the following theorem can be easily obtained by replacing condition (5.7) by condition (5.8) in the proof of Theorem 5.4.

Theorem 5.5. *Suppose that f, g, S and T are selfmaps of symmetric space X and that (f, S) and (g, T) are $\mathcal{D}$-operator pairs and satisfy the condition:*

$$\begin{aligned} d(fx, gy) \leq h\ \max\{&d(Sx, Ty), d(Sx, fx), d(Ty, gy), \\ &d(Sx, gy), d(Ty, fx)\}, \end{aligned} \tag{5.8}$$

for each $x, y \in X, 0 \leq h < 1$. Then f, g, S and T have a unique common fixed point.

Remark 5.5. The above result is a proper generalization of results due to Jungck and Rhoades [7] and Bhatt et al. [6].

Corollary 5.3. *Suppose that f, g, S and T are selfmaps of symmetric space X and that (f, S) and (g, T) are $\mathcal{D}$-operator pairs and satisfy the condition:*

$$d(fx, gy) \leq h\ d(Sx, Ty), \tag{5.9}$$

for each $x, y \in X, 0 \leq h < 1$. Then f, g, S and T have a unique common fixed point.

6. Application to Nonlinear Integral Equations

Let $J = [0, T]$ be a bounded interval in $\mathbb{R}$, for some $T \in \mathbb{R}$. Consider the following Voltera-Hammerstein nonlinear integral equation:

$$x(t) = f(t, x(t)) + \int_0^t k(t, s) g(s, x(s)) ds, \tag{6.1}$$

for all $t \in J$, where $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$, $k : J \times J \longrightarrow \mathbb{R}$ and $g : J \times \mathbb{R} \longrightarrow \mathbb{R}$ are measurable both in t and s on $\mathbb{R}$.

By a solution of integral equation (6.1), we mean a function $x \in C(J, \mathbb{R})$, where $C(J, \mathbb{R})$ denote the class of continuous real valued functions on J. Define the standard supremum norm $\|.\|$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)|.$$

By $L^1(J, \mathbb{R})$, we denote the space of Lebesgue integrable functions on J and the norm $\|.\|_{L^1}$ in $L^1(J, \mathbb{R})$ is defined by

$$\|x\|_{L^1} = \int_0^T |x(t)| dt.$$

Assume the following conditions:
(C_1) $\int_0^T \sup_{0 \leq s \leq t} |k(t,s)| dt = M_1 < +\infty$,
(C_2) for all $x \in L^1(J, \mathbb{R})$, $g(s, x(s)) \in L^1(J, \mathbb{R})$, there exists $M_2 > 0$ such that,

$$|g(s, x(s)) - g(s, y(s))| \leq M_2 |x(s) - y(s)|, \forall x, y \in L^1(J, \mathbb{R}), s \in J,$$

(C_3) for all $x, y \in L^1(J, \mathbb{R})$, there exists $M_3 > 0$ such that

$$|f(t, x(t)) - f(t, y(t))| \leq M_3 \|x - y\|,$$

(C_4) define $Ax(t) = f(t, x(t))$ and $Bx(t) = \int_0^t k(t,s) g(s, x(s)) ds$, with

$$|ASx(t) - SAx(t)| \leq R\ diam(PC(A, S)),$$

for some $x(t)$ satisfying $Ax(t) = Sx(t)$, $R > 0$, where $x(t)$ is defined as in (6.1).

Theorem 6.1. *Under the assumption* $(C_1) - (C_4)$ *the nonlinear integral equation* (6.1) *has a unique solution in* $L^1(J, \mathbb{R})$ *with* $\frac{M_3}{1 - M_1 M_2} < 1$.

Proof. Define

$$\begin{aligned} Ax(t) &= f(t, x(t)), && (6.2) \\ Bx(t) &= \int_0^t k(t,s) g(s, x(s)) ds, && (6.3) \\ Sx(t) &= (I - B)x(t) = x(t) - \int_0^t k(t,s) g(s, x(s)) ds, && (6.4) \end{aligned}$$

where I is the identity operator on $L^1(J, \mathbb{R})$ and A, B and S are operators from $L^1(J, \mathbb{R})$ into itself. Clearly, $Ax, Bx \in L^1(J, \mathbb{R})$.
By condition(C_3), we have

$$\|Ax - Ay\| = |f(t, x(t)) - f(t, y(t))| \leq M_3 \|x - y\|. \qquad (6.5)$$

Also,

$$\begin{aligned}
\|Bx - By\| &= \int_0^T |Bx(t) - By(t)|dt, \\
&= \int_0^T \Big| \int_0^t k(t,s)g(s,x(s))ds - \int_0^t k(t,s)g(s,y(s))ds \Big| dt, \\
&\leq \int_0^T [\sup_{0\leq s\leq t} |k(t,s)|dt] \int_0^t |g(s,x(s)) - g(s,y(s))|ds, \\
&\leq M_1M_2 \int_0^T |x(s) - y(s)|ds, \\
&= M_1M_2\|x - y\|_{L^1}. \qquad (6.6)
\end{aligned}$$

Now, by Equation(6.4) and (6.6), we have

$$\begin{aligned}
\|Sx - Sy\| &= \|(I - B)x - (I - B)y\| \\
&= \|(x - y) - (Bx - By)\| \\
&\geq \|x - y\| - \|Bx - By\| \\
&\geq \|x - y\| - M_1M_2\|x - y\| \\
&= (1 - M_1M_2)\|x - y\|. \qquad (6.7)
\end{aligned}$$

From, (6.5) and (6.7), we get

$$\|Ax - Ay\| \leq \frac{M_3}{1 - M_1M_2}\|Sx - Sy\|. \qquad (6.8)$$

Hence, by Corollary 5.1, there exists a unique common fixed point $u \in L^1(J, \mathbb{R})$ such that $Au = Su = u$, and consequently, u is the unique solution of (6.1). □

The following example is a natural realization of Theorem 6.1.

Example 6.1. Consider the following integral equation

$$x(t) = \frac{t}{3(1 + t^2)}x(t) + \int_0^t \sin(2ts)\frac{x(s)}{2 + s}ds \qquad (6.9)$$

for all $s, t \in [0, 1]$.
Comparing the Equation(6.1) with (6.9), we get

$$f(t, x(t)) = \frac{t}{3(1 + t^2)}x(t), \; k(t, s) = \sin(2ts),$$

$$g(s, x(s)) = \frac{x(s)}{2 + s}, \quad t \in [0, 1], \; x \in \mathbb{R}.$$

Clearly,

$$\left|\frac{t}{3(1+t^2)}x(t) - \frac{t}{3(1+t^2)}y(t)\right| \leq \frac{1}{3}|x(t)-y(t)|,$$

and

$$\begin{aligned} |g(s,x(s)) - g(s,y(s))| &= \left|\frac{x(s)}{2+s} - \frac{y(s)}{2+s}\right| \\ &= \left|\frac{1}{2+s}(x(s)-y(s))\right| \\ &\leq \frac{1}{2}|x(s)-y(s)|. \end{aligned}$$

Here, $\sup_{0\leq t\leq 1}|k(t,s)| = 1 < \infty$. Also, $\frac{M_3}{1-M_1M_2} = \frac{1/3}{1-(1/2)} = \frac{2}{3} < 1$. Also, condition$(C_4)$ is satisfied for some $R > 0$. Thus, all the conditions of Theorem 6.1 are satisfied, so it guarantees that there exists a solution of Equation (6.9) in $L^1([0,1],\mathbb{R})$.

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