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EXISTENCE AND STABILITY RESULTS FOR STOCHASTIC FRACTIONAL NEUTRAL DIFFERENTIAL EQUATIONS WITH GAUSSIAN NOISE AND LÉVY NOISE

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Abstract. In this paper we prove the existence and uniqueness of solution of stochastic fractional neutral differential equations with Gaussian noise or Lévy noise by using the Picard-Lindelöf successive approximation scheme. Further stability results of nonlinear stochastic fractional dynamical system with Gaussian and Lévy noises are established. Examples are provided to illustrate the theoretical results.

1. Introduction

Many stochastic systems do not only depend on the present and past states but also involve derivatives with delays. Besides, many practical systems may

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suffer some jump type stochastic perturbations for example, earthquakes, epidemic and so on. Such problems often arise in the study of two or more simple oscillatory systems with some interconnections between them wherein the connection can be replaced by differential equations involving delays in the highest order derivatives. This becomes the reason to study the neutral differential equations with the addition of Gaussian noise or Lévy noise. Stability of stochastic differential equations is important to avoid possible blow up of solutions. Several authors [4, 5, 6, 8, 11, 12, 14, 19, 20] have studied the existence and stability of neutral stochastic systems.

Recently fractional differential equations are attracted the attention of many researchers due to its applications in many fields [7, 9, 10, 18]. Stochastic fractional differential equations are the natural generalizations of the fractional differential equations. For this equations the delays and in particular neutral delays affect the behavior of the systems. A comprehensive study of fractional neutral differential equations can be found in [1, 15, 16, 17]. The introduction of stochastic nature in these models provide interesting results on the bahaviour of the solutions. Boufoussi and Hajji [2] discussed the successive approximation of neutral functional stochastic differential equations with jumps. Chadha and Bora [3] studied the existence, uniqueness and stability of mild solution for stochastic evolution equations with infinite delay and Poisson jumps. Umamaheswari et al. [22] discussed the existence and stability results for fractional stochastic differential equations with Lévy noise.

In this paper, we derive the stability results of stochastic fractional neutral systems driven by Gaussian and Lévy noises. Successive approximation technique is used to prove the existence results. Uniqueness of such equations is also proved under suitable conditions. Examples are provided to support the theory.

2. Preliminaries

In this section we present a few well-known concepts of fractional and stochastic differential equations.

Definition 2.1. ([21]) (Riemann-Liouville fractional integral). The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \ t > 0, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. ([21]) (Riemann-Liouville fractional derivative). The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, is defined as

$$D_{0+}^{\alpha} f(t) = \left(\frac{d}{dt}\right)^n I_{0+}^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \quad (2.2)$$

where the function $f(t)$ has absolutely continuous derivatives upto order (n - 1).

Definition 2.3. ([21]) (Caputo fractional derivative). The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, is defined as

$$^{C}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad (2.3)$$

where the function $f(t)$ has absolutely continuous derivatives upto order (n - 1).

Definition 2.4. ([13]) (Stochastic Process). A collection $\{X(t)|\ t \geq 0\}$ of random variables is called a stochastic process.

Definition 2.5. ([13]) (Brownian motion). A real-valued stochastic process $W(t)$ is called a Brownian motion or Wiener process if

(1) $W(0) = 0$ a.s.,
(2) $W(t) - W(s)$ is $N(0, t-s)$ for all $t \geq s \geq 0$,
(3) for all times $0 < t_1 < t_2 < \cdots < t_n$, the random variables $W(t_1), W(t_2) - W(t_1), \cdots, W(t_n) - W(t_{n-1})$ are independent (independent increments).

Definition 2.6. ([22]) (Lévy Process). Let $\{X(t)|\ t \geq 0\}$ be a stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Then X is a Lévy process if

(1) $X(0) = 0$ (a.s),
(2) X has an independent and stationary increments,
(3) X is stochastically continuous. That is, for all $a > 0$ and for all $s > 0$ $\lim_{t \to s} P(|X(t) - X(s)| > a) = 0$.

Lemma 2.7. ([13]) (Chebyshev's Inequality). If X is a random variable and $1 \leq p < \infty$, then

$$\mathbb{P}(|X| \geq \lambda) \leq \frac{1}{\lambda^p} \mathbb{E}(|X|^p) \text{ for all } \lambda > 0.$$

Lemma 2.8. ([13]) (Borel Cantelli Lemma). If $\{A_k\} \subset \mathcal{F}$ and $\sum\limits_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$, then

$$\mathbb{P}(\lim_{k\to\infty} \sup\ A_k) = 0.$$

The following theorem is used to prove our main result.

Theorem 2.9. ([13]) (i) *If* $\{X_n\}_{n=1}^{\infty}$ *is a submartingale, then*

$$\mathcal{P}\left(\max_{1\le k\le n} X_k \ge \lambda\right) \le \mathbb{E}(X_n^+),$$

for all $n = 1, 2, \ldots$ *and* $\lambda > 0$.
(ii) *If* $\{X_n\}_{n=1}^{\infty}$ *is a martingale and* $1 < p < \infty$*, then*

$$\mathbb{E}\left(\max_{1\le k\le n} |X_k|^p\right) \le \left(\frac{p}{p-1}\right)^p \mathbb{E}(|X_n|^p),$$

for all $n = 1, 2, \ldots$.

The following lemmas are necessary to obtain the main results. For that, we assume the following hypothesis:

(H): There exist constants $M > 0$, $a > 0$ such that for $t \ge 0$,

$$|E_{\alpha,\beta}(At^{\alpha})| \le Me^{-at},$$

where $0 < \alpha < 1$ and $\beta = 1, 2, \alpha$ and the Mittag Leffler function of an $n \times n$ matrix A is defined as

$$E_{\alpha,\beta}(At) = \sum_{k=0}^{\infty} \frac{(At)^k}{\Gamma(\alpha k + \beta)}, \ (\alpha, \beta > 0).$$

Lemma 2.10. *Assume that the hypothesis (H) holds. Then for any stochastic process* $F : [0, \infty) \to \mathbb{R}^n$ *which is strongly measurable with* $\int_0^T \mathbb{E}(|F(s)|^2)\mathrm{d}s <$ ∞, $0 < T \le \infty$, *the following inequality holds for* $0 < t \le T$

$$\mathbb{E}\left|\int_0^t E_{\alpha,\beta}(A(t-s)^{\alpha})F(s)\mathrm{d}s\right|^2 \le (M^2/a)\int_0^t \exp(-a(t-s))\mathbb{E}(|F(s)|^2)\mathrm{d}s,$$

where $\alpha \in (1/2, 1)$ *and* $\beta = 1, 2, \alpha$.

Proof. Assume that the hypothesis (H) holds; that is there exist constants $M > 0, a > 0$ such that for $t \ge 0$

$$|E_{\alpha,\beta}(At^{\alpha})| \le Me^{-at},$$

where $0 < \alpha < 1$ and $\beta = 1, 2, \alpha$. By the Hölder inequality, we obtain, for $0 < t \leq T$,

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t E_{\alpha,\beta}(A(t-s)^\alpha)F(s)\mathrm{d}s\right|^2 \\
&\leq \mathbb{E}\left(\int_0^t M\exp(-(a/2)(t-s))\exp(-(a/2)(t-s))|F(s)|\mathrm{d}s\right)^2 \\
&\leq \mathbb{E}\left(\int_0^t M\exp(-(a/2)(t-s))\mathrm{d}s\right)^2 \mathbb{E}\left(\int_0^t \exp(-(a/2)(t-s))|F(s)|\mathrm{d}s\right)^2 \\
&\leq (M^2/a)\int_0^t \exp(-a(t-s))\mathbb{E}(|F(s)|^2)\mathrm{d}s,
\end{aligned}$$

which complete the proof of the lemma. □

Lemma 2.11. ([22]) *Assume that the hypothesis (H) holds. Then for any B_t-adapted predictable process $\Phi : [0,\infty) \to \mathbb{R}^n$ with $\int_0^T \mathbb{E}|\Phi(s)|^2\mathrm{d}s < \infty,\ t \geq 0$, the following inequality holds for $0 < t \leq T$*

$$\mathbb{E}\left|\int_0^t E_{\alpha,\beta}(A(t-s)^\alpha)\Phi(s)\mathrm{d}W(s)\right|^2 \leq M^2\int_0^t \exp(-a(t-s))\mathbb{E}|\Phi(s)|^2\mathrm{d}s,$$

where $\alpha \in (1/2, 1)$ and $\beta = 1, 2, \alpha$.

3. Nonlinear equation with Gaussian noise

Consider the stochastic fractional neutral differential equation with Gaussian noise of the form:

$$\left.\begin{aligned}
&{}^C D^\alpha(x(t) - p(t, x(t))) = b(t, x(t)) + \sigma(t, x(t))\frac{\mathrm{d}W(t)}{\mathrm{d}t},\ t \in J = [0, T], \\
&x(0) = x_0,
\end{aligned}\right\} \tag{3.1}$$

where $\alpha \in (1/2, 1)$, $p \in C(J \times \mathbb{R}^n, \mathbb{R}^n)$, $b \in C(J \times \mathbb{R}^n, \mathbb{R}^n)$, $\sigma \in C(J \times \mathbb{R}^n, \mathbb{R}^{n\times m})$ and $W = \{W(t), t \geq 0\}$ is an m-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The integral form of equation (3.1) is given by

$$\begin{aligned}
x(t) =& x_0 - p(0, x_0) + p(t, x(t)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} b(s, x(s))\mathrm{d}s \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s, x(s))\mathrm{d}W(s).
\end{aligned} \tag{3.2}$$

Assume the following conditions:

(H1) (Linear growth condition) $b : J \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : J \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $p : J \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and there exists a constant $K > 0$ such that

$$|b(t,x)|^2 + |\sigma(t,x)|^2 + |p(t,x)|^2 \leq K^2(1 + |x|^2)$$

for all $x \in \mathbb{R}^n$.

(H2) (Lipschitz condition) There exists a constant $L > 0$ such that

$$|b(t,x) - b(t,y)|^2 + |\sigma(t,x) - \sigma(t,y)|^2 + |p(t,x) - p(t,y)|^2 \leq L^2(|x - y|^2)$$

for all $x, y \in \mathbb{R}^n$.

Theorem 3.1. *If the hypotheses* $(H1) - (H2)$ *are satisfied and if* x_0 *is a random variable defined on* $(\Omega, \mathcal{F}, \mathcal{P})$ *and independent of the* σ*-algebra* $\mathcal{F}_s^t \subset \mathcal{F}$ *generated by* $\{W(s), t \geq s \geq 0\}$ *such that* $\mathbb{E}(|x_0|^2) < \infty$ *then the initial value problem* (3.1) *has a unique solution which is* t*-continuous with the property that* $x(t,\omega)$ *is adapted to the filtration* $\mathcal{F}_t^{x_0}$ *generated by* x_0 *and* $\{W(s)(\cdot), s \leq t\}$ *and*

$$\sup_{0\leq t\leq T} \mathbb{E}[|x(t)|^2] < \infty.$$

Proof. First we prove the existence of solution of the initial value problem. For that, define $x^{(0)}(t) = x_0$ and $x^{(k)}(t) = x^{(k)}(t,\omega)$ inductively by

$$\begin{aligned} x^{(k+1)}(t) =& x_0 - p(0,x_0) + p(t, x^{(k)}(s)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} b(s, x^{(k)}(s)) \mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s, x^{(k)}(s)) \mathrm{d}W(s), \end{aligned} \tag{3.3}$$

for $k = 0, 1, 2, \ldots$. If, for fixed $k \geq 0$, the approximation $x^{(k)}(t)$ is $\mathcal{F}_t$-measurable and continuous on J, then it follows from (H1)-(H2) that the integrals in (3.3) are meaningful and that the resulting process $x^{(k+1)}(t)$ is $\mathcal{F}_t$-measurable and continuous on J. As $x^{(0)}(t)$ is obviously $\mathcal{F}_t$-measurable and continuous on J, it follows by induction that so too is each $x^{(k)}(t)$ for $k = 1, 2, \ldots$. It is clear that

$$\sup_{0\leq t\leq T} \mathbb{E}(|x^{(0)}(t)|^2) < \infty.$$

Since x_0 is $\mathcal{F}_t$-measurable with $\mathbb{E}(|x_0|^2) < \infty$, by using the algebraic inequality, the Cauchy-Schwarz inequality, the Itô isometry and the linear growth condition, we obtain from (3.3)

$$\begin{aligned}\mathbb{E}(|x^{(k+1)}(t)|^2) \leq& 4\mathbb{E}[|x_0|^2 + |p(0,x_0)|^2] + 4\mathbb{E}[|p(t,x^{(k)}(s))|^2] \\ &+ \frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left[\int_0^t \left|b(s,x^{(k)}(s))\right|^2 \mathrm{d}s\right] \\ &+ \frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left[\int_0^t \left|\sigma(s,x^{(k)}(s))\right|^2 \mathrm{d}s\right].\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}(|x^{(k+1)}(t)|^2) \leq& 4(1+K^2)\mathbb{E}[|x_0|^2] + 4K^2\mathbb{E}(1+|x^{(k)}(t)|^2) \\ &+ 2K^2\frac{4}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left(\int_0^t \left(1+|x^{(k)}(s)|^2\right)\mathrm{d}s\right),\end{aligned}$$

for $k = 0, 1, 2, \ldots$. By induction, we have

$$\sup_{0\leq t\leq T} \mathbb{E}(|x^{(k)}(t)|^2) \leq C_5 < \infty,$$

for $k = 1, 2, 3, \ldots$ and $C_5 > 0$. Let

$$d^{(k)}(t) = \mathbb{E}(|x^{(k+1)}(t) - x^{(k)}(t)|).$$

We claim that

$$d^{(k)}(t) \leq \frac{(Mt)^{(k+1)}}{(k+1)!}, \text{ for all } k = 0, 1, 2, \ldots,$$

for some constant M depending on K, L and x_0. From equation (3.3) by applying the Schwarz inequality, Itô isometry and the Lipchitz condition we obtain

$$\begin{aligned}d^{(k)}(t) =& \mathbb{E}[|x^{(k+1)}(t) - x^{(k)}(t)|^2] \\ \leq& 3\mathbb{E}(|p(s,x^{(k)}(s) - p(s,x^{(k-1)}(s))|^2) \\ &+ \frac{3}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|b(s,x^{(k)}(s)) - b(s,x^{(k-1)}(s))\right|^2\right]\mathrm{d}s \\ &+ \frac{3}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|\sigma(s,x^{(k)}(s)) - \sigma(s,x^{(k-1)}(s))\right|^2\right]\mathrm{d}s \\ \leq& 3L^2\mathbb{E}\left[\left|x^{(k)}(t) - x^{(k-1)}(t)\right|^2\right] \\ &+ 3\frac{L^2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|x^{(k)}(s) - x^{(k-1)}(s)\right|^2\right]\mathrm{d}s \\ &+ 3\frac{L^2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|x^{(k)}(s) - x^{(k-1)}(s)\right|^2\right]\mathrm{d}s. \qquad (3.4)\end{aligned}$$

By applying again the Schwarz inequality, the Itô isometry together with the growth conditions for $k=0$, one gets

$$\begin{aligned} d^{(0)}(t) &= \mathbb{E}[|x^{(1)}(t)-x^{(0)}(t)|^2] \\ &\leq 3\mathbb{E}(|p(t,x_0)-p(0,x_0)|^2) + \frac{3}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|b(s,x^{(0)}(s))\right|^2\right]\mathrm{d}s \\ &\quad + \frac{3}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left[\left|\sigma(s,x^{(0)}(s))\right|^2\right]\mathrm{d}s \\ &\leq 3L^2t + 2K^2\frac{3}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left(\int_0^t (1+|x_0|^2)\mathrm{d}s\right) \\ &\leq 3L^2t + 2K^2\frac{3}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}(t)(1+\mathbb{E}(|x_0|^2)). \end{aligned} \tag{3.5}$$

Now, for $k=1$, replacing $\mathbb{E}[|x^{(1)}(t)-x^{(0)}(t)|^2]$ in the inequality (3.4) with the value on the right hand side of inequality (3.5) and integrating, we obtain

$$\begin{aligned} \mathbb{E}[|x^{(2)}(t)-x^{(1)}(t)|^2] &\leq 3L^2\mathbb{E}(|x^{(1)}(t)-x^{(0)}(t)|^2) \\ &\quad + 2L^2\frac{3}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}[|x^{(1)}(s)-x^{(0)}(s)|^2]\mathrm{d}s \\ &\leq 3L^2\big[K_1+K^2(1+\mathbb{E}(|x_0|^2))\big]\left(\frac{2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\right)^2\int_0^t s\mathrm{d}s \\ &\leq 3L^2\left[K_1+K^2(1+\mathbb{E}(|x_0|^2))\right]\left(\frac{2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\right)^2 \times \frac{t^2}{2!}, \end{aligned}$$

where $K_1<\infty$. For $k=2$, proceeding as before, we have

$$\mathbb{E}[|x^{(3)}(t)-x^{(2)}(t)|^2] \leq 3L^2\left[K_1+K^2(1+\mathbb{E}(|x_0|^2))\right]\left(\frac{2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\right)^3 \times \frac{t^3}{3!}.$$

Thus by the principle of mathematical induction, we have

$$d^{(k)}(t) = \mathbb{E}[|x^{(k+1)}(t)-x^{(k)}(t)|^2] \leq \frac{BM^{k+1}t^{(k+1)}}{(k+1)!},\ k=0,1,2,\ldots,\ 0\leq t\leq T,$$

where $B=3L^2\left[K_1+K^2(1+\mathbb{E}(|x_0|^2))\right]$ and $M=\left(\frac{2}{(\Gamma(\alpha))^2}\frac{T^{2\alpha-1}}{2\alpha-1}\right)$ are constants depending only on α, T, L^2, K^2 and $\mathbb{E}(|x_0|^2)$.

Note that

$$\begin{aligned}
&\mathbb{E}\left(\max_{0\le t\le T}|x^{(k+1)}(t)-x^{(k)}(t)|^2\right)\\
&\quad\le 3L^2\left(\max_{0\le t\le T}|x^{(k)}(t)-x^{(k-1)}(t)|^2\right)\\
&\quad+3L^2\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left(\max_{0\le t\le T}\int_0^t|x^{(k)}(s)-x^{(k-1)}(s)|^2\mathrm{d}s\right)\\
&\quad+3\mathbb{E}\left(\max_{0\le t\le T}\int_0^t(t-s)^{\alpha-1}|\sigma(s,x^{(k)}(s))-\sigma(s,x^{(k-1)}(s))|^2\mathrm{d}W(s)\right).
\end{aligned}$$

Use of second part of the Theorem 2.9 gives

$$\begin{aligned}
&\mathbb{E}\left(\max_{0\le t\le T}|x^{(k+1)}(t)-x^{(k)}(t)|^2\right)\\
&\quad\le 3L^2\left(\max_{0\le t\le T}|x^{(k)}(t)-x^{(k-1)}(t)|^2\right)\\
&\quad+3L^2\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left(\int_0^T|x^{(k)}(s)-x^{(k-1)}(s)|^2\mathrm{d}s\right)\\
&\quad+12L^2\frac{T^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left(\int_0^T|x^{(k)}(s)-x^{(k-1)}(s)|^2\mathrm{d}s\right)\\
&\quad\le B\frac{M^{k+1}}{(k+1)!}T^{(k+1)}.
\end{aligned}\tag{3.6}$$

Use of Chebyshev's inequality gives

$$\mathcal{P}\left(\max_{0\le t\le T}|x^{(k+1)}(t)-x^{(k)}(t)|^2>\frac{1}{k^2}\right)\le\frac{1}{(1/k^2)^2}\mathbb{E}\left(\max_{0\le t\le T}|x^{(k+1)}(t)-x^{(k)}(t)|^2\right).$$

Using the equation (3.6) and summing up the resultant inequalities, one gets

$$\sum_{k=0}^{\infty}\mathcal{P}\left(\max_{0\le t\le T}|x^{(k+1)}(t)-x^{(k)}(t)|^2>\frac{1}{k^2}\right)\le\sum_{k=0}^{\infty}\frac{BM^{k+1}k^4T^{(k+1)}}{(k+1)!},$$

where the series on the right side converges by ratio test. Hence the series on the left side also converges, so by the Borel-Cantelli lemma, conclude that $\max_{0\le t\le T}\left(|x^{(k+1)}(t)-x^{(k)}(t)|^2\right)$ converges to 0, almost surely, that is, the successive approximations $\{x^{(k)}(t)\}$ converge, almost surely, uniformly on J to a limit $x(t)$ defined by

$$\lim_{n\to\infty}\left(x^{(0)}(t)+\sum_{k=1}^{n}(x^{(k)}(t)-x^{(k-1)}(t))\right)=\lim_{n\to\infty}x^{(n)}(t)=x(t).\tag{3.7}$$

From (3.3), we have

$$\begin{aligned} x(t) =& x_0 - p(0,x_0) + p(t,x(t)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} b(s,x(s))\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s,x(s))\mathrm{d}W(s), \end{aligned} \tag{3.8}$$

for all $t \in J$. This completes the proof of the existence of solution of (3.1).

Next, we have to prove the uniqueness of solution. The uniqueness follows from the Itô isometry and the Lipschitz conditions with $L < 1/2$.

Let $x(t,\omega)$ and $y(t,\omega)$ be solution processes through the initial data $(0,x_0)$ and $(0,y_0)$, respectively, that is, $x(0,\omega) = x_0(\omega)$ and $y(0,\omega) = y_0(\omega)$, $\omega \in \Omega$. Let

$$\begin{aligned} a(s,\omega) &= b\left(s,x(s)\right) - b\left(s,y(s)\right), \\ \gamma(s,\omega) &= \sigma\left(s,x(s)\right) - \sigma\left(s,y(s)\right). \end{aligned}$$

Then, by virtue of the Schwarz inequality and the Itô isometry, we have

$$\begin{aligned} \mathbb{E}[|x(t)-y(t)|^2] \leq\ & 4L_1^2\mathbb{E}[|x_0 - p(0,x_0) - y_0 + p(0,y_0)|^2] \\ &+ \frac{4L_1^2}{(\Gamma(\alpha))^2}\frac{t^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left[\int_0^t |a(s,\omega)|^2\mathrm{d}s\right] \\ &+ \frac{4L_1^2}{(\Gamma(\alpha))^2}\frac{t^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left[\int_0^t |\gamma(s,\omega)|^2\mathrm{d}s\right] \\ \leq\ & 4L_1^2\mathbb{E}[|x_0 - p(0,x_0) - y_0 + p(0,y_0)|^2] \\ &+ 2L_1^2\frac{4L^2}{(\Gamma(\alpha))^2}\frac{t^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}[|x(s)-y(s)|^2]\mathrm{d}s, \end{aligned}$$

with $L_1^2 = \frac{1}{1-4L^2}$ and $L < 1/2$. We define $v(t) = \mathbb{E}[|x(t)-y(t)|^2]$. Then the function v satisfies

$$v(t) \leq F + A\int_0^t v(s)\mathrm{d}s,$$

where $F = 4L_1^2\mathbb{E}[|x_0 - p(0,x_0) - y_0 + p(0,y_0)|^2]$ and $A = 2L_1^2\dfrac{4L^2}{(\Gamma(\alpha))^2}\dfrac{t^{2\alpha-1}}{2\alpha-1}$.

By the application of the Gronwall's inequality, we conclude that

$$v(t) \leq F\exp(At).$$

Now assume that $x_0 = y_0$. Then $F = 0$ and so $v(t) = 0$ for all $t \geq 0$. That is,

$$\mathbb{E}[|x(t)-y(t)|^2] = 0,$$

which gives

$$\int_\Omega |x(t)-y(t)|^2\mathrm{d}\mathcal{P} = 0.$$

This implies that $x(t) = y(t)$ a.s for all $t \in J$. That is,

$$\mathcal{P}\Big\{|x(t,\omega) - y(t,\omega)| = 0 \quad \text{for all} \;\; t \in J\Big\} = 1,$$

that is, the solution is unique. This completes the proof of existence and uniqueness of solution of the given stochastic fractional neutral differential equation (3.1). □

4. Stability of system with Gaussian noise

In this section, we study the exponential asymptotic stability in the quadratic mean of a trivial solution. Consider the stochastic fractional neutral differential equation with Gaussian noise of the form

$$\left.\begin{aligned} {}^{C}D^{\alpha}(x(t) - p(t,x(t))) &= Ax(t) + f(t,x(t)) + \sigma(t,x(t))\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \\ x(0) &= x_0, \end{aligned}\right\} \tag{4.1}$$

where $t \in J = [0,T]$, $\alpha \in (1/2,1)$, $b(t,x(t)) = Ax(t) + f(t,x(t))$, $p \in C(J \times \mathbb{R}^n, \mathbb{R}^n)$, $f \in C(J \times \mathbb{R}^n, \mathbb{R}^n)$, $\sigma \in C(J \times \mathbb{R}^n, \mathbb{R}^{n\times m})$ and $W = \{W(t), t \geq 0\}$ is an m-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. $A \in \mathbb{R}^{n\times n}$ is a diagonal stability matrix. Assume from now on that $b(t,0) = \sigma(t,0) = p(t,0) \equiv 0$ a.e t so that equation (4.1) admits a trivial solution.

Definition 4.1. ([13]) The trivial solution of equation (4.1) is said to be exponentially stable in the quadratic mean if there exist positive constants C, ν such that

$$\mathbb{E}(|x(t)|^2) \leq C\mathbb{E}(|x(0)|^2)\exp(-\nu t), \;\; t \geq 0.$$

Theorem 4.2. *Let the assumptions of Theorem* 3.1 *hold. Then the solution of equation* (4.1) *is exponentially stable in the quadratic mean provided*

$$a > \beta = \beta(a,K,M_1) = \left(5M_1^2(K^2/a(\Gamma(\alpha))^2T + (K^2/a + K^2)\frac{T^{2\alpha-1}}{2\alpha-1}\right).$$

Proof. The integral form of the equation (4.1) is

$$\begin{aligned} x(t) = &E_{\alpha}(At^{\alpha})[x_0 - p(0,x_0)] + p(t,x(t)) \\ &+ \int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha})p(s,x(s))\mathrm{d}s \\ &+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha})f(s,x(s))\mathrm{d}s \\ &+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha})\sigma(s,x(s))\mathrm{d}W(s). \end{aligned} \tag{4.2}$$

By using algebraic inequality and Lemmas 2.10 and 2.11, we get

$$\begin{aligned}\mathbb{E}[|x(t)|^2] \leq & 10M^2 \exp(-at)(1+K^2)\mathbb{E}(|x_0|^2) + 5\mathbb{E}(|p(t,x(t))|^2) \\ & + 5(\Gamma(\alpha))^2 TM^2/a \int_0^t \exp(-a(t-s))\mathbb{E}(|p(s,x(s))|^2)\mathrm{d}s \\ & + 5(M^2/a)\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \exp(-a(t-s))\mathbb{E}|f(s,x(s))|^2\mathrm{d}s \\ & + 5M^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \exp(-a(t-s))\mathbb{E}|\sigma(s,x(s))|^2\mathrm{d}s.\end{aligned}$$

Linear growth assumption when $f(t,0)=\sigma(t,0)=p(t,0)\equiv 0$ a.e t yields

$$\begin{aligned}\exp(at)\mathbb{E}|x(t)|^2 \leq & 10M_1^2(1+K^2)\mathbb{E}(|x_0|^2) \\ & + 5M_1^2\left[K^2/a(\Gamma(\alpha))^2T + (K^2/a+K^2)\frac{T^{2\alpha-1}}{2\alpha-1}\right] \\ & \times \int_0^t \exp(as)\mathbb{E}(|x(s)|^2)\mathrm{d}s\end{aligned}$$

with $M_1 = M^2/(1-5K^2)$ and $K^2 < 1/5$. Applying Gronwall's inequality, we obtain

$$\begin{aligned}\exp(at)\mathbb{E}|x(t)|^2 \leq & 10M_1^2(1+K^2)\mathbb{E}|x_0|^2 \\ & \times \exp\left(5M_1^2(K^2/a(\Gamma(\alpha))^2T + (K^2/a+K^2)\frac{T^{2\alpha-1}}{2\alpha-1}t\right).\end{aligned}$$

Consequently

$$\mathbb{E}|x(t)|^2 \leq C\mathbb{E}|x_0|^2\exp(-\nu t),\ t\geq 0,$$

where $\nu = a-\beta$ and $C = 10M_1^2(1+K^2)$. □

5. Nonlinear equation with Lévy noise

Consider the stochastic fractional neutral differential equation with Lévy noise of the form:

$$\begin{aligned}{}^C D^\alpha(x(t)-p(t,x(t))) &= b(t,x(t)) + \sigma(t,x(t))\frac{\mathrm{d}W(t)}{\mathrm{d}t} + \int_z g(t,x(t),z)\frac{\mathrm{d}\tilde{N}(t,z)}{\mathrm{d}t}, \\ x(0) &= x_0, \qquad (5.1)\end{aligned}$$

where $t\in J$, $\alpha\in(1/2,1)$ and $z\in\mathbb{R}_0^n = \mathbb{R}^n/\{0\}$. Here $b,p : J\times\mathbb{R}^n\to\mathbb{R}^n$, $\sigma : J\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$, $g : J\times\mathbb{R}^n\times\mathbb{R}_0^n\to\mathbb{R}^{n\times l}$ are given functions such that $b(t,x(t))$, $\sigma(t,x(t))$, $p(t,x(t))$ and $g(t,x(t),z)$ are $\mathcal{F}_t$ measurable for all $x\in\mathbb{R}^n$ and $z\in\mathbb{R}_0^n$, $\tilde{N}(\mathrm{d}t,\mathrm{d}z) = N(\mathrm{d}t,\mathrm{d}z)-\nu(\mathrm{d}z)\mathrm{d}t$ which is the l-dimensional compensated jump measure of $\eta(.)$ an independent compensated Poisson random measure and $W = \{W(t), t \geq 0\}$ is an m-dimensional Brownian motion on a complete probability space $\Omega \equiv (\Omega, \mathcal{F}, \mathcal{P})$. The integral form of the equation (5.1) is

$$\begin{aligned} x(t) =& x_0 - p(0, x_0) + p(t, x(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} b(s, x(s)) \mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s, x(s)) \mathrm{d}W(s) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_z g(s, x(s), z) \mathrm{d}\tilde{N}(s, z). \end{aligned} \tag{5.2}$$

Assume that the following conditions hold true:

(H4) $b : J \times \mathbb{R}^n \to \mathbb{R}^n$, $p : J \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : J \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are continuous and there exists constant $K_2 > 0$ such that

$$|b(t,x)|^2 + |\sigma(t,x)|^2 + |p(t,x)|^2 + \int_z |g(t,x,z)|^2 \nu(\mathrm{d}z) \leq K_2^2(1 + |x|^2)$$

for all $x \in \mathbb{R}^n$.

(H5) There exists a constant $L_2 > 0$ such that

$$\begin{aligned} |b(t,x) - b(t,y)|^2 + &|\sigma(t,x) - \sigma(t,y)|^2 + \int_z |g(t,x,z) - g(t,y,z)|^2 \nu(\mathrm{d}z) \\ &+ |p(t,x) - p(t,y)|^2 \leq L_2^2(|x-y|^2) \end{aligned}$$

for all $x, y \in \mathbb{R}^n$.

Theorem 5.1. *If the hypotheses* $(H4) - (H5)$ *are satisfied and if* x_0 *is a random variable defined on* $(\Omega, \mathcal{F}, \mathcal{P})$ *and independent of the* σ*-algebra* $\mathcal{F}_s^t \subset \mathcal{F}$ *generated by* $\{W(s), t \geq s \geq 0\}$ *such that* $\mathbb{E}(|x_0|^2) < \infty$*. Then the initial value problem* (5.1) *has a unique solution which is* t*-continuous with the property that* $x(t, \omega)$ *is adapted to the filtration* $\mathcal{F}_t^{x_0}$ *generated by* x_0 *and* $\{W(s)(\cdot), s \leq t\}$ *and*

$$\sup_{0 \leq t \leq T} \mathbb{E}[|x(t)|^2] < \infty.$$

Proof. To prove uniqueness, let $x(t, \omega)$ and $y(t, \omega)$ be solution processes through the initial data $(0, x_0)$ and $(0, y_0)$, respectively, that is, $x(0, \omega) = x_0(\omega)$ and $y(0, \omega) = y_0(\omega)$, $\omega \in \Omega$. Let

$$\begin{aligned} \gamma_1(s, \omega) &= b\left(s, x(s)\right) - b\left(s, y(s)\right), \\ \gamma_2(s, \omega) &= \sigma\left(s, x(s)\right) - \sigma\left(s, y(s)\right), \\ \gamma_3(s, \omega) &= \int_z g(s, x(s), z) \nu(\mathrm{d}z) - \int_z g(s, y(s), z) \nu(\mathrm{d}z). \end{aligned}$$

By the use of Schwarz inequality and the Itô isometry, we have

$$\begin{aligned}\mathbb{E}[|x(t)-y(t)|^2] &\le 5L_3^2\mathbb{E}[|x_0-p(0,x_0)-y_0+p(0,y_0)|^2]\\ &\quad+\frac{5L_3^2}{(\Gamma(\alpha))^2}\frac{t^{2\alpha-1}}{2\alpha-1}\mathbb{E}\Big[\int_0^t|\gamma_1(s,\omega)|^2\mathrm{d}s\Big]\\ &\quad+\frac{5L_3^2}{(\Gamma(\alpha))^2}\frac{t^{2\alpha-1}}{2\alpha-1}\mathbb{E}\Big[\int_0^t|\gamma_2(s,\omega)|^2\mathrm{d}s\Big]\\ &\quad+\frac{5L_3^2}{(\Gamma(\alpha))^2}\frac{t^{2\alpha-1}}{2\alpha-1}\mathbb{E}\Big[\int_0^t|\gamma_3(s,\omega)|^2\mathrm{d}s\Big]\\ &\le\frac{5L_3^2}{(\Gamma(\alpha))^2}\mathbb{E}[|x_0-y_0|^2]\\ &\quad+5L_2^2\frac{3L_3^2}{(\Gamma(\alpha))^2}\frac{t^{2\alpha-1}}{2\alpha-1}\int_0^t\mathbb{E}[|x(s)-y(s)|^2]\mathrm{d}s,\end{aligned}$$

with $L_3^2=\frac{1}{1-5L_2^2}$ and $L_2^2<1/5$. We define $v(t)=\mathbb{E}[|x(t)-y(t)|^2]$. Then the function v satisfies $v(t)\le F+A\displaystyle\int_0^t v(s)\mathrm{d}s$, where $F=5L_1^2\mathbb{E}[|x_0-p(0,x_0)-y_0+p(0,y_0)|^2]$ and $A=5L^2\dfrac{3L_2^2}{(\Gamma(\alpha))^2}\dfrac{t^{2\alpha-1}}{2\alpha-1}$. By the application of Gronwall's inequality, we conclude that

$$v(t)\le F\exp(At).$$

Now assume that $x_0=y_0$. Then $F=0$ and so $v(t)=0$ for all $t\ge 0$. That is,

$$\mathbb{E}[|x(t)-y(t)|^2]=0,$$

which gives

$$\int_\Omega|x(t)-y(t)|^2\mathrm{d}\mathcal{P}=0.$$

This implies that $x(t)=y(t)$ a.s for all $t\in J$. That is,

$$\mathcal{P}\big\{|x(t,\omega)-y(t,\omega)|=0\quad\text{for all}\quad t\in J\big\}=1,$$

that is, the solution is unique.

Existence of solution of (5.1) is proved as before using Lipschitz condition, growth condition and Cauchy Schwarz inequality by approximation. □

6. Stability of system with Lévy noise

In this section we study the exponentially asymptotic stability in the quadratic mean of a trivial solution. Consider the following stochastic fractional nonlinear system with Lévy noise of the form

$$\begin{aligned} {}^{C}D^{\alpha}(x(t) - p(t, x(t))) &= Ax(t) + f(t, x(t)) + \sigma(t, x(t))\frac{\mathrm{d}W(t)}{\mathrm{d}t} \\ &\quad + \int_z g(t, x(t), z)\frac{\mathrm{d}\tilde{N}(t, z)}{\mathrm{d}t},\ t \in J, \\ x(0) &= x_0, \end{aligned} \tag{6.1}$$

where $\alpha \in (1/2, 1)$, $z \in \mathbb{R}_0^n = \mathbb{R}^n/\{0\}, b(t, x(t)) = Ax(t) + f(t, x(t))$, $f, p \in C(J \times \mathbb{R}^n, \mathbb{R}^n)$, $\sigma \in C(J \times \mathbb{R}^n, \mathbb{R}^{n\times m})$, $g \in C(J \times \mathbb{R}^n \times \mathbb{R}_0^n, \mathbb{R}^{n\times l})$, $\tilde{N}(\mathrm{d}t, \mathrm{d}z)$ is as in (5.1), $A \in \mathbb{R}^{n\times n}$ is a diagonal stability matrix. Assume from now on that $b(t, 0) = \sigma(t, 0) = p(t, 0) \equiv 0$ a.e t so that equation (6.1) admits a trivial solution.

Definition 6.1. ([13]) The trivial solution of equation (6.1) is said to be exponentially stable in the quadratic mean if there exist positive constants C, ν such that

$$\mathbb{E}(|x(t)|^2) \leq C\mathbb{E}(|x_0|^2)\exp(-\nu t),\ \ t \geq 0.$$

Theorem 6.2. *Let the assumptions of Theorem* 5.1 *hold. Then the solution of equation* (6.1) *is exponentially stable in the quadratic mean provided*

$$a > \beta = \beta(a, K, M_1) = 6M_1^2\left[K^2/a(\Gamma(\alpha))^2T + (2K^2/a + K^2)\frac{T^{2\alpha-1}}{2\alpha - 1}\right].$$

Proof. The integral form of the equation (6.1) is

$$\begin{aligned} x(t) =& E_\alpha(At^\alpha)[x_0 - p(0, x_0)] + p(t, x(t)) \\ &+ \int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)p(s, x(s))\mathrm{d}s \\ &+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s, x(s))\mathrm{d}s \\ &+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\sigma(s, x(s))\mathrm{d}W(s) \\ &+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\int_z g(s, x(s), z)\tilde{N}(\mathrm{d}s, \mathrm{d}z). \end{aligned} \tag{6.2}$$

Applying assumption (H) when $f(t, 0) = \sigma(t, 0) = g(t, 0, z) = p(t, 0) \equiv 0$ a.e t yields

$$\begin{aligned}\exp(at)\mathbb{E}(|x(t)|^2) \leq & 12M_1^2(1+K^2)\mathbb{E}(|x_0|^2) \\ & + 6(M_1^2/a)K^2(\Gamma(\alpha))^2 T \int_0^t \exp(as)\mathbb{E}(|x(s)|^2)\mathrm{d}s \\ & + 6M_1^2(2K^2/a+K^2)\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \exp(as)\mathbb{E}(|x(s)|^2)\mathrm{d}s \\ \leq & 12M_1^2(1+K^2)\mathbb{E}|x_0|^2 + M_2\int_0^t \exp(as)\mathbb{E}(|x(s)|^2)\mathrm{d}s,\end{aligned}$$

where $M_1 = M/(1-6K^2)$ with $K^2 < 1/6$ and $M_2 = 6M_1^2\left[K^2/a(\Gamma(\alpha))^2T + (2K^2/a+K^2)\frac{T^{2\alpha-1}}{2\alpha-1}\right]$. Applying Gronwall's inequality, we obtain

$$\exp(at)\mathbb{E}(|x(t)|^2) \leq 12M_1^2(1+K^2)\mathbb{E}(|x_0|^2)\exp(M_2). \tag{6.3}$$

Consequently,

$$\mathbb{E}(|x(t)|^2) \leq C\mathbb{E}(|x_0|^2)\exp(-\nu t),\ t \geq 0, \tag{6.4}$$

where $\nu = a - \beta$ and $C = 12M_1^2(1+K^2)$. □

7. Examples

In this section, we provide two examples to support the theory developed in the previous sections.

Example 7.1. Consider the following stochastic fractional neutral differential equation of the form

$$\left.\begin{aligned} & {}^C D^{0.7}\left(x(t) - \frac{1}{5\sqrt{2}}\cos x\right) = -0.2x(t) + \frac{1}{5\sqrt{2}}\sin x + \frac{1}{5\sqrt{2}}x(t)\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \\ & x(0) = 5, \quad t \in J. \end{aligned}\right\} \tag{7.1}$$

Here $\alpha = 0.7$, $b(t,x(t)) = 0.2x(t) + \frac{1}{5\sqrt{2}}\sin x$, $p(t,x(t)) = \frac{1}{5\sqrt{2}}\cos x$ and $\sigma(t,x(t)) = \frac{1}{5\sqrt{2}}x(t)$. It can be easily seen that $b(t,x(t))$, $p(t,x(t))$ and $\sigma(t,x(t))$ satisfy the assumptions of Theorem 3.1. Hence, by Theorem 3.1, the stochastic fractional differential equation (7.1) has a unique solution. Also the equation (7.1) satisfies the condition of Theorem 4.2. So from Theorem 4.2, the stochastic fractional differential equation with $A = -0.2$ and $f(t,x(t)) = \frac{1}{5\sqrt{2}}\sin x$ is exponentially stable.

Example 7.2. Consider the following stochastic fractional neutral differential equation with Lévy noise of the form

$$
\begin{aligned}
{}^{C}D^{0.6}\left(x(t)-\frac{1}{15}\exp(-x(t))\right) &= -0.5x(t)+\frac{1}{15}\left(\frac{1}{1+t}\right)\frac{\mathrm{d}W(t)}{\mathrm{d}t} \\
&\quad + \int_{\mathbb{R}/\{0\}}\frac{1}{15}tz\frac{d\tilde{N}(t,z)}{dt}, \\
x(0) &= 1. \qquad (7.2)
\end{aligned}
$$

Here $\alpha = 0.6$, $b = 0.5x(t)$, $p(t,x(t)) = \frac{1}{15}\exp(-x(t))$, $\sigma = \frac{1}{15}\left(\frac{1}{1+t}\right)$ and $g = \frac{1}{15}tz$. It can be easily seen that $b(t,x(t))$, $\sigma(t,x(t))$ and $g(t,x(t),z)$ satisfies the hypothesis (H4) and (H5). Hence by the Theorem 5.1 the stochastic fractional differential equation (7.2) has a unique solution. Also the equation (7.2) satisfies the condition of Theorem 6.2. So from Theorem 6.2 the stochastic fractional differential equation with $A = -0.5$ is exponentially stable.

8. Conclusion

Existence and uniqueness results for neutral stochastic fractional differential equations with Gaussian noise and Lévy noise are obtained. The Picard-Lindelöf successive approximation technique is used to obtain the existence results. Further sufficient conditions for the stability of neutral systems with Gaussian noise and Lévy noise are established by deriving the equivalent nonlinear integral equation corresponding to the system. Examples are given to illustrate the theory.

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