Nonlinear Functional Analysis and Applications
Vol. 28, No. 2 (2023), pp. 383-394
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2023.28.02.04
http://nfaa.kyungnam.ac.kr/journal-nfaa

ω-INTERPOLATIVE CONTRACTIONS IN BIPOLAR METRIC SPACES

Jong Kyu Kim[1], Manoj Kumar[2] and Pankaj[3]

[1]Department of Mathematics Education, Kyungnam University,
Changwon, 51767, Korea
e-mail: jongkyuk@kyungnam.ac.kr

[2]Department of Mathematics, Baba Mastnath University,
Asthal Bohar, Rohtak, Haryana, 124021, India
e-mail: manojantil18@gmail.com

[3]Department of Mathematics, Baba Mastnath University,
Asthal Bohar, Rohtak, Haryana, 124021, India
e-mail: maypankajkumar@gmail.com

Abstract. In this paper, we shall introduce the new notions of ω-orbital admissible mappings, ω-interpolative Kannan type contraction and ω-interpolative Ciric-Reich-Rus type contraction. In the setting of these new contractions, we will prove some fixed point theorems in bipolar metric spaces. Some existing results from literature are also deduced from our main results. Some examples are also provided to illustrate the theorems.

1. Introduction

In 1922, Banach [2] was the first, who introduced the constructive method to obtain a fixed point of a self-map. After that, the scholars introduced a number of generalizations of Banach contraction principle to prove fixed point theorems. In 1968, Kannan [4] proposed a new contraction to prove fixed point theorems. In 1971-72, Riech, Ciric and Rus ([13]-[17]) combined the Kannan and Banach contractions and gave a new contraction. Very recently, in 2018,

[0]Received July 23, 2022. Revised October 18, 2022. Accepted October 27, 2022.
[0]2020 Mathematics Subject Classification: 47H10, 54H25.
[0]Keywords: Fixed point, bipolar metric space, ω-orbital admissible mappings, ω-interpolative Kannan type contraction, ω-interpolative Ciric-Reich-Rus type contraction.
[0]Corresponding author: M. Kumar(manojantil18@gmail.com).

Karapinar et al. introduced interpolative Kannan [5] type and interpolative Ciric-Reich-Rus type [6] contraction in complete metric spaces.

In 2014, Popescu [12] refined the α-admissible mapping and introduced a new type of mapping called ω-orbital admissible mappings. In 2019, Aydi et al. [1] using ω-orbital admissible mappings proved fixed point theorems for interpolative Ciric-Reich-Rus type contraction. There are number of metric spaces like partial, rectangular, G-metric, b-metric, cone etc. present in literature. Recently, in 2016, Mutlu and Gurdal [9] introduced the concept of bipolar metric space (see [7]). Later, Mutlu et al. ([10]-[11]) proved coupled fixed point theorems for multivalued mappings in bipolar metric spaces. In 2021, Gaba et al. [3] introduced the concept of (α, BK)-contraction and in 2022, Murthi [8] proved fixed point theorems for Boyd-Wong type contractions in bipolar metric space.

In this paper, we shall also prove some fixed point theorems for ω-interpolative Kannan type and ω-interpolative Ciric-Reich-Rus type contractions.

To prove our results, we need some basic existing definitions from literature as follows:

Definition 1.1. ([9]) Let X and Y be two non empty sets and $d : X \times Y \to [0, \infty)$ be a map satisfying the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$ for all $(x, y) \in X \times Y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X \cap Y$;
(3) $d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$.

for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then d is called a bipolar metric and (X, Y, d) is called a bipolar metric space.

If $X \cap Y = \phi$ then space is called disjoint otherwise joint. The set X is called a left pole and the set Y is called a right pole of (X, Y, d). The elements of X, Y and $X \cap Y$ are called left, right and central elements, respectively.

Example 1.2. ([10]) Let $X = \{U_m(\mathbb{R}) : U_m(\mathbb{R})$ is an upper triangular matrix of order m over $\mathbb{R}\}$ and Y=$\{V_m(\mathbb{R}) : V_m(\mathbb{R})$ is a lower triangular matrix of order m over $\mathbb{R}$ $\}$. Define $d : X \times Y \to [0, \infty)$ by $d(P, Q) = \sum_{i,j=1}^{m} |p_{ij} - q_{ij}|$ for all P=$(p_{ij})_{m\times m} \in U_m(\mathbb{R})$ and Q=$(q_{ij})_{m\times m} \in V_m(\mathbb{R})$. Then, clearly (X, Y, d) is a bipolar metric space.

Definition 1.3. ([9]) Let (X, Y, d) be a bipolar metric space. Then any sequence $(x_n) \subseteq X$ is called a left sequence and is said to be convergent to right element say y if $d(x_n, y) \to 0$ as $n \to \infty$. Similarly, a right sequence $(y_n) \subseteq Y$ is said to be convergent to a left element say x if $d(x, y_n) \to 0$ as $n \to \infty$.

Definition 1.4. ([9]) Let (X, Y, d) be a bipolar metric space.

(1) A sequence $\{(x_n, y_n)\}$ on $X \times Y$ is called a bisequence on (X, Y, d).
(2) If both the sequences $\{x_n\}$ and $\{y_n\}$ converge, then the bisequence $\{(x_n, y_n)\}$ is said to be convergent. If both the sequences $\{x_n\}$ and $\{y_n\}$ converge to the same point $u \in X \cap Y$ then the bisequence $\{(x_n, y_n)\}$ is called biconvergent.
(3) A bisequence $\{(x_n, y_n)\}$ on (X, Y, d) is said to be a Cauchy bisequence, if for each $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $d(x_n, y_m) < \epsilon$ for all $n, m \geq N$.
(4) A bipolar metric space is said to be complete if every Cauchy bisequence is convergent in this space.

Definition 1.5. ([9]) Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bipolar metric spaces and $T : X_1 \cup Y_1 \to X_2 \cup Y_2$ be a function:

(1) If $TX_1 \subseteq X_2$ and $TY_1 \subseteq Y_2$, then T is called a covariant mapping and is denoted by $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$.
(2) If $TX_1 \subseteq Y_2$ and $TY_1 \subseteq X_2$, then T is called a contravariant mapping and is denoted by $T : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$.

Definition 1.6. ([9]) Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bipolar metric spaces.

(1) A map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left continuous at a point $x_0 \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_2(Tx_0, Ty) < \epsilon$ whenever $d_1(x_0, y) < \delta$.
(2) A map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called right continuous at a point $y_0 \in Y$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_2(Tx, Ty_0) < \epsilon$ whenever $d_1(x, y_0) < \delta$.
(3) A map T is called continuous if it is left continuous at each $x_0 \in X$ and right continuous at each $y_0 \in Y$.
(4) A contravariant map $T : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$ is continuous if and only if T is continuous as covariant map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$.

Definition 1.7. Let Ψ be the family of the functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(1) ψ is non-decreasing,
(2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$,

for all $t > 0$, where $\psi^n(t)$ is a nth iteration of ψ. These functions are known as (c)-comparison functions.

It can be easily verified that if $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.

2. Main results

In this section, firstly we shall introduce the new notions of ω-interpolative Kannan type and ω-interpolative Ciric-Reich-Rus type contractions and prove fixed point theorems for such type of contractions in bipolar metric space.

Definition 2.1. Let $\omega : X \times Y \to [0, \infty)$ be a mapping. A contravariant mapping $T : X \cup Y \rightleftarrows X \cup Y$ is said to be ω-orbital admissible if

$$\omega(x, Tx) \geq 1 \implies \omega(T^2x, Tx) \geq 1 \tag{2.1}$$

and

$$\omega(Ty, y) \geq 1 \implies \omega(Ty, T^2y) \geq 1, \tag{2.2}$$

for all $(x, y) \in X \times Y$.

Definition 2.2. Let (X, Y, d) be a bipolar metric space and $T : X \cup Y \rightleftarrows X \cup Y$ be a contravariant mapping. Then T is said to be a ω-interpolative Kannan type contravariant contraction if there exists $\psi \in \Psi$, $\omega : X \times Y \to [0, \infty)$ and $\alpha \in (0, 1)$ such that

$$\omega(x, y)d(Ty, Tx) \leq \psi([d(x, Tx)]^{\alpha}[d(Ty, y)]^{1-\alpha}), \tag{2.3}$$

for all $(x, y) \in X \times Y$ but $x, y \notin$Fix(T), where Fix(T)=$\{z \in X \cup Y : T(z) = z\}$.

Definition 2.3. Let (X, Y, d) be a bipolar metric space and $T : X \cup Y \rightleftarrows X \cup Y$ be a contravariant mapping. Then T is said to be a ω-interpolative Ciric-Riech-Rus type contravariant contraction if there exists $\psi \in \Psi$, $\omega : X \times Y \to [0, \infty)$ and $\alpha, \beta \in (0, 1)$ such that

$$\omega(x, y)d(Ty, Tx) \leq \psi([d(x, y)]^{\beta}[d(x, Tx)]^{\alpha}[d(Ty, y)]^{1-\alpha-\beta}), \tag{2.4}$$

for all $(x, y) \in X \times Y$ but $x, y \notin$Fix(T).

Theorem 2.4. *Let (X, Y, d) be a bipolar metric space and $T : X \cup Y \rightleftarrows X \cup Y$ be a ω-interpolative Kannan type contravariant contraction satisfying the followings:*

(1) *T is ω-orbital admissible;*
(2) *There exists $x_0 \in X$ such that $\omega(x_0, Tx_0) \geq 1$;*
(3) *T is continuous.*

Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\omega(x_0, Tx_0) \geq 1$. We define a bisequence $\{(x_n, y_n)\}$ as $x_{n+1} = Ty_n$ and $y_n = Tx_n$ for all $n \in \mathbb{N}$. Since T is ω-orbital admissible, from equation (2.1), we obtain that

$$\omega(x_0, y_0) = \omega(x_0, Tx_0) \geq 1.$$

It implies that

$$\omega(T^2x_0, Tx_0) = \omega(x_1, y_0) \geq 1. \tag{2.5}$$

Equation (2.5) implies that

$$\omega(x_1, y_0) = \omega(Ty_0, y_0) \geq 1. \tag{2.6}$$

Using equation (2.2) in equation (2.6), we get

$$\omega(Ty_0, T^2y_0) \geq 1,$$

this implies that

$$\omega(x_1, y_1) \geq 1. \tag{2.7}$$

By repeating the same process, we get

$$\omega(x_n, y_n) \geq 1 \text{ and } \omega(x_{n+1}, y_n) \geq 1. \tag{2.8}$$

Using equation (2.3), (2.8) and Definition 1.7, we find that for $x = x_n$ and $y = y_{n-1}$

$$\begin{aligned} d(x_n, y_n) &= d(Ty_{n-1}, Tx_n) \\ &\leq \omega(x_n, y_{n-1})d(Ty_{n-1}, Tx_n) \\ &\leq \psi([d(x_n, Tx_n)]^\alpha [d(Ty_{n-1}, y_{n-1})]^{1-\alpha}) \\ &= \psi([d(x_n, y_n)]^\alpha [d(x_n, y_{n-1})]^{1-\alpha}) \\ &\leq [d(x_n, y_n)]^\alpha [d(x_n, y_{n-1})]^{1-\alpha}. \end{aligned} \tag{2.9}$$

From equation (2.9), we get

$$[d(x_n, y_n)]^{1-\alpha} \leq [d(x_n, y_{n-1})]^{1-\alpha},$$

this implies that

$$d(x_n, y_n) \leq d(x_n, y_{n-1}). \tag{2.10}$$

Now, by using equation (2.10), we obtain

$$\begin{aligned} [d(x_n, y_n)]^\alpha [d(x_n, y_{n-1})]^{1-\alpha} &\leq [d(x_n, y_{n-1})]^\alpha [d(x_n, y_{n-1})]^{1-\alpha} \\ &= d(x_n, y_{n-1}). \end{aligned} \tag{2.11}$$

Using equation (2.11) in equation (2.9), we get

$$d(x_n, y_n) \leq \psi(d(x_n, y_{n-1})). \tag{2.12}$$

Again, by using equation (2.3), (2.8) and Definition 1.7, we find that for $x = x_n$ and $y = y_n$

$$\begin{aligned} d(x_{n+1}, y_n) &= d(Ty_n, Tx_n) \\ &\leq \omega(x_n, y_n)d(Ty_n, Tx_n) \\ &\leq \psi([d(x_n, Tx_n)]^\alpha [d(Ty_n, y_n)]^{1-\alpha}) \\ &= \psi([d(x_n, y_n)]^\alpha [d(x_{n+1}, y_n)]^{1-\alpha}) \\ &\leq [d(x_n, y_n)]^\alpha [d(x_{n+1}, y_n)]^{1-\alpha}. \end{aligned} \tag{2.13}$$

From equation (2.13), we get

$$[d(x_{n+1}, y_n)]^\alpha \leq [d(x_n, y_n)]^\alpha,$$

this implies that

$$d(x_{n+1}, y_n) \leq d(x_n, y_n). \tag{2.14}$$

Now, by using equation (2.14), we obtain

$$\begin{aligned} [d(x_n, y_n)]^\alpha [d(x_{n+1}, y_n)]^{1-\alpha} &\leq [d(x_n, y_n)]^\alpha [d(x_n, y_n)]^{1-\alpha} \\ &= d(x_n, y_n). \end{aligned} \tag{2.15}$$

Using equation (2.15) in equation (2.13), we get

$$d(x_{n+1}, y_n) \leq \psi(d(x_n, y_n)). \tag{2.16}$$

By induction, from equations (2.12) and (2.16), we obtain

$$d(x_n, y_n) \leq \psi^n(d(x_1, y_0)), d(x_{n+1}, y_n) \leq \psi^{n+1}(d(x_0, y_0)), \tag{2.17}$$

for all $n \in \mathbb{N}$. From the Definition 1.7, it is clear that there exists $\epsilon > 0$ and $N(\epsilon) \in \mathbb{N}$ such that

$$\Sigma_{n \geq N(\epsilon)} \psi^n(d(x_1, y_0)) \leq \frac{\epsilon}{2} \text{ and } \Sigma_{n \geq N(\epsilon)} \psi^{n+1}(d(x_0, y_0)) \leq \frac{\epsilon}{2}. \tag{2.18}$$

Now, for all $n, m \in \mathbb{N}$ with $m > n > N(\epsilon)$, by using property (3) of Definition 1.1, we get

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_{n+1}) \\ &\quad + \cdots + d(x_m, y_{m-1}) + d(x_m, y_m) \\ &= \sum_{k=n}^{m} d(x_k, y_k) + \sum_{k=n}^{m-1} d(x_{k+1}, y_k). \end{aligned}$$

Using equation (2.17), we get

$$\begin{aligned} d(x_n, y_m) &\leq \sum_{k=n}^{m} d(x_1, y_0) + \sum_{k=n}^{m-1} d(x_0, y_0) \\ &\leq \Sigma_{n \geq N(\epsilon)} \psi^n(d(x_1, y_0)) + \Sigma_{n \geq N(\epsilon)} \psi^{n+1}(d(x_0, y_0)). \end{aligned}$$

By using equation (2.18), we get

$$d(x_n, y_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{2.19}$$

On the other hand, for all $n, m \in \mathbb{N}$ with $n > m > N(\epsilon)$, by using property (3) of Definition 1.1, we get

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_{n-1}) + d(x_{n-1}, y_{n-1}) + d(x_{n-1}, y_{n-2}) \\ &\quad + \cdots + d(x_m, y_{m-1}) + d(x_m, y_m) \\ &= \sum_{k=m}^{n-1} d(x_k, y_k) + \sum_{k=m}^{n} d(x_k, y_{k-1}). \end{aligned}$$

Using equation (2.17), we get

$$\begin{aligned} d(x_n, y_m) &\leq \sum_{k=n}^{m} d(x_1, y_0) + \sum_{k=n}^{m-1} d(x_0, y_0), \\ &\leq \Sigma_{n \geq N(\epsilon)} \psi^n(d(x_1, y_0)) + \Sigma_{n \geq N(\epsilon)} \psi^{n+1}(d(x_0, y_0)). \end{aligned}$$

By using equation (2.18), we get

$$d(x_n, y_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{2.20}$$

Therefore $\{(x_n, y_n)\}$ is a Cauchy bisequence. Since (X, Y, d) is a complete bipolar metric space, so $\{(x_n, y_n)\}$ is biconvergent. That is, there exists $w \in X \cap Y$ such that $x_n \to w$ and $y_n \to w$ as $n \to \infty$. Since T is continuous, $x_n \to w$ implies that $y_n = Tx_n \to w$ and combining this with $y_n \to w$ gives $Tw = w$. Thus w is the fixed point of T. □

In the next theorem we omit continuity.

Theorem 2.5. *Let (X, Y, d) be a bipolar metric space and $T : X \cup Y \rightleftarrows X \cup Y$ be a ω-interpolative Kannan type contravariant contraction satisfying the followings:*

(1) *T is ω-orbital admissible;*
(2) *There exists $x_0 \in X$ such that $\omega(x_0, Tx_0) \geq 1$;*
(3) *If $\{(x_n, y_n)\}$ is a bisequence such that $\omega(x_n, y_n) \geq 1$ for all n and $y_n \to w \in X \cap Y$ as $n \to \infty$, there exists $\{x_{n(k)}\}$ in $\{x_n\}$ such that $\omega(x_{n(k)}, w) \geq 1$ for all $k \geq 1$.*

Then T has a fixed point.

Proof. Following the proof of Theorem 2.4, we obtain that $\{(x_n, y_n)\}$ is a Cauchy bisequence. Since, (X, Y, d) is a complete bipolar metric space, so $\{(x_n, y_n)\}$ is biconvergent. That is, there exists $w \in X \cap Y$ such that $x_n \to w$ and $y_n \to w$ as $n \to \infty$. Let $Tw \neq w$. Then $d(Tw, w) > 0$. From equation (2.8) and condition (3), we get $\omega(x_{n(k)}, w) \geq 1$ for all $k \geq 1$. Since $d(x_{n(k)}, w) \to 0$, $d(x_{n(k)}, y_{n(k)}) \to 0$ as $k \to \infty$ and $d(Tw, w) > 0$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$

$$d(x_{n(k)}, w) < d(Tw, w) \text{ and } d(x_{n(k)}, y_{n(k)}) < d(Tw, w). \tag{2.21}$$

Using above, equations (2.3) and (2.21), we get

$$\begin{aligned} d(Tw, y_{n(k)}) &= d(Tw, Tx_{n(k)}) \\ &\leq \omega(x_{n(k)}, w) d(Tw, Tx_{n(k)}) \\ &\leq \psi([d(x_{n(k)}, y_{n(k)})]^{\alpha} [d(Tw, w)]^{1-\alpha}) \\ &\leq \psi(d(Tw, w)) \\ &\leq d(Tw, w). \end{aligned}$$

Taking $k \to \infty$, we get $d(Tw, w) < d(Tw, w)$ which is a contradiction. So, $Tw = w$. Thus w is the fixed point of T. □

Theorem 2.6. *Let (X, Y, d) be a bipolar metric space and $T : X \cup Y \rightleftarrows X \cup Y$ be a ω-interpolative Ciric-Riech-Rus type contravariant contraction satisfying the followings:*

(1) *T is ω-orbital admissible;*
(2) *There exists $x_0 \in X$ such that $\omega(x_0, Tx_0) \geq 1$;*
(3) *T is continuous.*

Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\omega(x_0, Tx_0) \geq 1$. We define a bisequence as $x_{n+1} = Ty_n$ and $y_n = Tx_n$ for all $n \in \mathbb{N}$. Since T is ω-orbital admissible, by the proof of Theorem 2.4, we get

$$\omega(x_n, y_n) \geq 1, \;\; \omega(x_{n+1}, y_n) \geq 1. \tag{2.22}$$

Using equation (2.4) and (2.22), we find that for $x = x_n$ and $y = y_{n-1}$,

$$\begin{aligned} d(x_n, y_n) &= d(Ty_{n-1}, Tx_n) \\ &\leq \omega(x_n, y_{n-1}) d(Ty_{n-1}, Tx_n) \\ &\leq \psi([d(x_n, y_{n-1})]^{\beta} [d(x_n, Tx_n)]^{\alpha} [d(Ty_{n-1}, y_{n-1})]^{1-\alpha-\beta}) \\ &= \psi([d(x_n, y_n)]^{\alpha} [d(x_n, y_{n-1})]^{1-\alpha}) \\ &\leq [d(x_n, y_n)]^{\alpha} [d(x_n, y_{n-1})]^{1-\alpha}. \end{aligned} \tag{2.23}$$

From equation (2.23), we get

$$[d(x_n, y_n)]^{1-\alpha} \leq [d(x_n, y_{n-1})]^{1-\alpha},$$

this implies that

$$d(x_n, y_n) \leq d(x_n, y_{n-1}). \tag{2.24}$$

Now, by using equation (2.24), we obtain

$$\begin{aligned} [d(x_n, y_n)]^{\alpha} [d(x_n, y_{n-1})]^{1-\alpha} &\leq [d(x_n, y_{n-1})]^{\alpha} [d(x_n, y_{n-1})]^{1-\alpha} \\ &= d(x_n, y_{n-1}). \end{aligned} \tag{2.25}$$

Using equation (2.25) in equation (2.23), we get

$$d(x_n, y_n) \leq \psi(d(x_n, y_{n-1})). \tag{2.26}$$

Again, by using equation (2.4), (2.22) and Definition 1.7, we find that for $x = x_n$ and $y = y_n$

$$\begin{aligned} d(x_{n+1}, y_n) &= d(Ty_n, Tx_n) \\ &\leq \omega(x_n, y_n) d(Ty_n, Tx_n) \\ &\leq \psi([d(x_n, y_n)]^\beta [d(x_n, Tx_n)]^\alpha [d(Ty_n, y_n)]^{1-\alpha}) \\ &= \psi([d(x_n, y_n)]^{\alpha+\beta} [d(x_{n+1}, y_n)]^{1-\alpha-\beta}) \\ &\leq [d(x_n, y_n)]^{\alpha+\beta} [d(x_{n+1}, y_n)]^{1-\alpha-\beta}. \end{aligned} \tag{2.27}$$

From equation (2.27), we get

$$[d(x_{n+1}, y_n)]^{\alpha+\beta} \leq [d(x_n, y_n)]^{\alpha+\beta},$$

this implies that

$$d(x_{n+1}, y_n) \leq d(x_n, y_n). \tag{2.28}$$

Now, by using equation (2.28), we obtain

$$\begin{aligned} [d(x_n, y_n)]^{\alpha+\beta} [d(x_{n+1}, y_n)]^{1-\alpha-\beta} &\leq [d(x_n, y_n)]^{\alpha+\beta} [d(x_n, y_n)]^{1-\alpha-\beta} \\ &= d(x_n, y_n). \end{aligned} \tag{2.29}$$

Using equation (2.29) in equation (2.27), we get

$$d(x_{n+1}, y_n) \leq \psi(d(x_n, y_n)). \tag{2.30}$$

By induction, from equations (2.26) and (2.30), we obtain

$$d(x_n, y_n) \leq \psi^n(d(x_1, y_0)) \quad \text{and} \quad d(x_{n+1}, y_n) \leq \psi^{n+1}(d(x_0, y_0)), \tag{2.31}$$

for all $n \in \mathbb{N}$. From the Definition 1.7, it is clear that there exists $\epsilon > 0$ and $N(\epsilon) \in \mathbb{N}$ such that

$$\Sigma_{n \geq N(\epsilon)} \psi^n(d(x_1, y_0)) \leq \frac{\epsilon}{2} \quad \text{and} \quad \Sigma_{n \geq N(\epsilon)} \psi^{n+1}(d(x_0, y_0)) \leq \frac{\epsilon}{2}. \tag{2.32}$$

Now, following the same steps like in Theorem 2.4 after equation (2.18), clearly, we have

$$d(x_n, y_m) < \epsilon, \tag{2.33}$$

for all $n, m \in \mathbb{N}$ with $n, m > N(\epsilon)$. Therefore $\{(x_n, y_n)\}$ is a Cauchy bisequence. Since (X, Y, d) is a complete bipolar metric space, so $\{(x_n, y_n)\}$ is biconvergent. That is, there exists $w \in X \cap Y$ such that $x_n \to w$ and $y_n \to w$ as $n \to \infty$. Since T is continuous, $x_n \to w$ implies that $y_n = Tx_n \to w$ and combining this with $y_n \to w$ gives $Tw = w$. Thus w is the fixed point of T. □

In the next theorem we also omit continuity.

Theorem 2.7. *Let (X, Y, d) be a bipolar metric space and $T : X \cup Y \rightleftarrows X \cup Y$ be a ω-interpolative Ciric-Riech-Rus type contravariant contraction satisfying the followings:*

(1) *T is ω-orbital admissible;*
(2) *There exists $x_0 \in X$ such that $\omega(x_0, Tx_0) \geq 1$;*
(3) *If (x_n, y_n) is a bisequence such that $\omega(x_n, y_n) \geq 1$ for all n and $y_n \to w \in X \cap Y$ as $n \to \infty$, there exists $\{x_{n(k)}\}$ in $\{x_n\}$ such that $\omega(x_{n(k)}, w) \geq 1$ for all $k \geq 1$.*

Then T has a fixed point.

Proof. From the proof of Theorem 2.6, we obtain that $\{(x_n, y_n)\}$ is a Cauchy bisequence. Since, (X, Y, d) is a complete bipolar metric space, $\{(x_n, y_n)\}$ is biconvergent. That is, there exists $w \in X \cap Y$ such that $x_n \to w$ and $y_n \to w$ as $n \to \infty$. Let $Tw \neq w$. Then $d(Tw, w) > 0$. From equation (2.22) and condition (3), we get $\omega(x_{n(k)}, w) \geq 1$ for all $k \geq 1$. Since $d(x_{n(k)}, w) \to 0$, $d(x_{n(k)}, y_{n(k)}) \to 0$ as $k \to \infty$ and $d(Tw, w) > 0$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$

$$d(x_{n(k)}, w) < d(Tw, w) \ \ and \ \ d(x_{n(k)}, y_{n(k)}) < d(Tw, w). \tag{2.34}$$

Using equations (2.4) and (2.34), we get

$$\begin{aligned} d(Tw, y_{n(k)}) &= d(Tw, Tx_{n(k)}) \\ &\leq \omega(x_{n(k)}, w) d(Tw, Tx_{n(k)}) \\ &\leq \psi([d(x_{n(k)}, w)]^{\beta} [d(x_{n(k)}, y_{n(k)})]^{\alpha} [d(Tw, w)]^{1-\alpha-\beta}) \\ &\leq \psi(d(Tw, w)) \\ &\leq d(Tw, w). \end{aligned}$$

Taking $k \to \infty$, we get $d(Tw, w) < d(Tw, w)$ which is a contradiction. So, $Tw = w$. Thus w is the fixed point of T. □

Example 2.8. Let $X = \{1, 2, 3\}$, $Y = \{2, 3, 4\}$, and $d(x, y) = |x - y|$, for all $(x, y) \in X \times Y$. Then it is clear that (X, Y, d) is a complete bipolar metric space. Define $T : X \cup Y \rightleftarrows X \cup Y$ by $Tz = 3$ for all $z \in X \cup Y$. So T is continuous. Taking $\omega(x, y) = 1$ for all $(x, y) \in X \times Y$, then it is easy to verify that T is ω-orbital admissible, and $\psi(t) = \frac{t}{2}$ for all $t \in [0, \infty)$. One can see easily that equation (2.3) holds for the above setup. So, all the conditions of Theorem 2.4 are satisfied. Thus T has a fixed point. Clearly, 3 is the fixed point of T. Hence Theorem 2.4 is verified.

Example 2.9. Let $X = \{1, 2, 3\}$ and $Y = \{2, 3, 4\}$, and $d(x, y) = |x - y|$, for all $(x, y) \in X \times Y$. Clearly (X, Y, d) is a complete bipolar metric space. Define $T : X \cup Y \rightleftarrows X \cup Y$ by $Tz = 2$ for all $z \in X \cup Y$. Then T is continuous.

Taking $\omega(x,y) = 1$ for all $(x,y) \in X \times Y$, then it is easy to verify that T is ω-orbital admissible, and $\psi(t) = \frac{t}{2}$ for all $t \in [0,\infty)$. One can see easily that equation (2.4) holds for the above setup. So, all the conditions of Theorem 2.6 are satisfied. Thus T has a fixed point. Clearly, 2 is the fixed point of T. Hence Theorem 2.6 is verified.

Some results of the literature can be easily deduced from our main results as follows:

Corollary 2.10. ([6]) *Let (X,Y,d) be a complete bipolar metric space and $T : X \cup Y \rightleftarrows X \cup Y$ be a contravariant mapping. If there exists $\lambda \in [0,1)$, $\alpha \in (0,1)$ such that*

$$d(Ty,Tx) \leq \lambda([d(x,Tx)]^{\alpha}[d(Ty,y)]^{1-\alpha}),$$

for all $(x,y) \in X \times Y$ but $x,y \notin Fix(T)$, where $Fix(T) = \{z \in X \cup Y : Tz = z\}$, then T has a fixed point.

Proof. In Theorem 2.4, by taking $\omega(x,y)$=1 and $\psi(t) = \lambda t$ one can get the proof easily. □

Corollary 2.11. ([5]) *Let (X,Y,d) be a complete bipolar metric space and $T : X \cup Y \rightleftarrows X \cup Y$ be a contravariant mapping. If there exists $\lambda \in [0,1)$, $\alpha,\beta \in (0,1)$ such that*

$$d(Ty,Tx) \leq \lambda([d(x,y)]^{\beta}[d(x,Tx)]^{\alpha}[d(Ty,y)]^{1-\alpha-\beta}),$$

for all $(x,y) \in X \times Y$ but $x,y \notin Fix(T)$, then T has a fixed point.

Proof. In Theorem 2.6, by taking $\omega(x,y)$=1 and $\psi(t) = \lambda t$ one can get the proof easily. □

Corollary 2.12. *Let (X,Y,d) be a complete bipolar metric space and $T : X \cup Y \rightleftarrows X \cup Y$ be a contravariant mapping then if there exists $\psi \in \Psi$, $\alpha \in (0,1)$ such that*

$$d(Ty,Tx) \leq \psi([d(x,Tx)]^{\alpha}[d(Ty,y)]^{1-\alpha}),$$

for all $(x,y) \in X \times Y$ but $x,y \notin Fix(T)$, then T has a fixed point.

Proof. In Theorem 2.4, by taking $\omega(x,y)$=1 one can get the proof easily. □

Corollary 2.13. *Let (X,Y,d) be a complete bipolar metric space and $T : X \cup Y \rightleftarrows X \cup Y$ be a contravariant mapping. If there exists $\psi \in \Psi$, $\alpha,\beta \in (0,1)$ such that*

$$d(Ty,Tx) \leq \psi([d(x,y)]^{\beta}[d(x,Tx)]^{\alpha}[d(Ty,y)]^{1-\alpha-\beta}),$$

for all $(x,y) \in X \times Y$ but $x,y \notin Fix(T)$, then T has a fixed point.

Proof. Taking $\omega(x,y)$=1 in Theorem 2.6 and obtain the proof. □

References

[1] H. Aydi, E. Karapinar and Antonio Francisco Roldan Lopez de Hierro, *ω- interpolative Ciric-Reich-Rus type contraction*, Mathematics, **7**(1) (2019):57.

[2] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrals*, Fund. Math., **3**(1) (1922), 133-181.

[3] Y.U. Gaba, M. Aphane and H. Aydi, *(α, BK)-Contractions in bipolar metric spaces*, J. Math., (2021).

[4] E. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., **60** (1968), 71-76.

[5] E. Karapinar, R.P. Agarwal and H. Aydi, *Interpolative Riech-Rus-Ciric type contractions on partial metric spaces*, Mathematics, **6** (2018):256.

[6] E. Karapinar, *Revisiting the Kannan type contraction via interpolation*, Adv. Theory Nonlinear Anal. Appl., **2** (2018), 85-87.

[7] D. Kitkuan, A. Padcharoen, J.K. Kim and W.H. Lim, *On α-Geraghty contractive mappings in bipolar metric spaces,* Nonlinear Funct. Anal. Appl., **28**(1) (2023), 295-309.

[8] P.P. Murthy, Z. Mitrovic, C.P. Dhuri and S. Radenovic, *The common fixed point theorems in bipolar metric space*, Gulf J. Math., **12**(2) (2022), 31-38.

[9] A. Mutlu and U. Gurdal, *Bipolar metric spaces and some fixed point theorems*, J. Nonlinear Sci. Appl., **9**(9) (2016), 5362-5373.

[10] A. Mutlu, U. Gurdal and K. Ozkan, *Fixed point results for α-ψ-contractive mappings in bipolar metric spaces*, J. ineq. Special Funct., **11**(1) (2020), 64-75.

[11] A. Mutlu, U. Gurdal and K. Ozkan,, *Fixed point theorems for multivalued mappings on bipolar metric spaces*, Fixed Point Theory, **21**(1) (2020), 271-280.

[12] O. Popescu, *Some new fixed point theorems for α-Geraghty contractive type maps in metric spaces*, Fixed Point Theory Appl., **2014** (2014): 190.

[13] S. Reich, *Fixed point of contractive functions*, Boll. Un. Math. Ital., **4** (1972), 26-42.

[14] S. Riech, *Kannan fixed point theorem*, Boll. Un. Math. Ital., **4** (1971), 1-11.

[15] S. Riech, *Some remarks concerning contraction mappings*, Can. Math. Bull., **14** (1971), 121-124.

[16] I.A. Rus, *Generalized contractions and applications*, Cluj University Press: Clui Napoca Romania, 2001.

[17] I.A. Rus, *Principles and applications of the fixed point theory*, Editura Dacia: Clui Napoca Romania, 1979.