Nonlinear Functional Analysis and Applications
Vol. 28, No. 2 (2023), pp. 421-437
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2023.28.02.07
http://nfaa.kyungnam.ac.kr/journal-nfaa

IMPROVED BOUNDS OF POLYNOMIAL INEQUALITIES WITH RESTRICTED ZERO

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Abstract. Let $p(z)$ be a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$. Then Malik [12] obtained the following inequality:

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|.$$

In this paper, we shall first improve as well as generalize the above inequality. Further, we also improve the bounds of two known inequalities obtained by Govil et al. [8].

1. Introduction

Let $p(z)$ be a polynomial of degree n. Then, according to a famous well-known classical result due to Bernstein [3],

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

[0]Received August 10, 2022. Revised October 14, 2022. Accepted October 18, 2022.
[0]2020 Mathematics Subject Classification: 30A10, 30C10, 30C15.
[0]Keywords: Polynomials, derivatives, inequalities.
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Inequality (1.1) is sharp and equality holds if $p(z)$ has all its zeros at the origin. If $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then Erdös conjectured and later Lax [11] proved that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

Inequality (1.2) is best possible and equality holds for $p(z) = a + bz^n$, where $|a| = |b|$.

For the class of polynomials $p(z)$ of degree n not vanishing in $|z| < k$, $k \geq 1$, Malik [12] proved

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.3}$$

The result is best possible and equality holds for $p(z) = (z + k)^n$.

Chan and Malik [6] considered a polynomial of the type $p(z) = a_o + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, and obtained the following extention of inequality (1.3).

Theorem 1.1. ([6]) *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^\mu} \max_{|z|=1} |p(z)|. \tag{1.4}$$

The result is best possible and extremal polynomial is $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, *where* n *is a multiple of* μ.

Next, Bidkham and Dewan [4] generalized inequality (1.3) and obtained

Theorem 1.2. ([4]) *If* $p(z)$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *then for* $1 \leq R \leq k$,

$$\max_{|z|=R} |p'(z)| \leq \frac{n(R+k)^{n-1}}{(1+k)^n} \max_{|z|=1} |p(z)|. \tag{1.5}$$

The result is best possible and equality in (1.5) *holds for* $p(z) = (z + k)^n$.

Aziz and Zargar [2] considered the class of polynomials $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, not vanishing in $|z| < k$, $k \geq 1$ and proved the following extension of Theorem 1.1 and generalization of Theorem 1.2.

Theorem 1.3. ([2]) *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *then for* $0 < r \leq R \leq k$,

$$\max_{|z|=R} |p'(z)| \leq \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}} \max_{|z|=r} |p(z)|. \tag{1.6}$$

The result is best possible and equality in (1.6) *holds for* $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, *where* n *is a multiple of* μ.

For a given polynomial $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ of degree n having no zero in $|z| < k$, $k \geq 1$, it is indeed desirable to know the dependence of

$$\max_{|z|=1} |p'(z)| / \max_{|z|=1} |p(z)|, \tag{1.7}$$

on the coefficients $a_0, a_1, \ldots, a_m$, $1 \leq m \leq n$. It is clear that these coefficients are not quite arbitrary. Govil et al. [8] obtained the following which gives the dependence of (1.7) on a_0, a_1 and a_2.

Theorem 1.4. ([8]) *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu - \lambda^2|} \times \max_{|z|=1} |p(z)|, \tag{1.8}$$

where $\lambda = \frac{k}{n}\frac{a_1}{a_0}$, $\mu = \frac{2k^2}{n(n-1)}\frac{a_2}{a_0}$.

It is really of interest to investigate inequalities in the reversed direction of Bernstein type discussed above and Turán [15] was the first who obtained such inequalities that if $p(z)$ has all its zeros in $z \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.9}$$

The result is sharp and equality holds in (1.9) for the polynomial having all its zeros on $|z| = 1$.

Malik [12] generalized inequality (1.9) by proving that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.10}$$

The result is sharp and extremal polynomial being $p(z) = (z+k)^n$.

For the same class of polynomials, by involving certain co-efficients of the polynomial, Govil et al. [8] obtained the following result.

Theorem 1.5. ([8]) *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k \leq 1$, *then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n\,|a_n| + |a_{n-1}|}{(1+k^2)\,n\,|a_n| + 2\,|a_{n-1}|} \max_{|z|=1} |p(z)|. \tag{1.11}$$

2. Lemmas

The following lemmas are needed for the proofs of the theorems and the corollaries in next section.

Lemma 2.1. ([13]) *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, then*

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^\mu \le 1. \tag{2.1}$$

Lemma 2.2. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then for $0 < r \le R \le k$,*

$$\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1} + R^\mu}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}R^\mu + k^{2\mu}R)} \le \frac{R^{\mu-1}}{R^\mu + k^\mu}. \tag{2.2}$$

Proof. Since $p(z) \ne 0$ in $|z| < k$, $k > 0$, the polynomial $P(z) = p(Rz) \ne 0$ in $|z| < \frac{k}{R}$, $\frac{k}{R} \ge 1$, where $0 < R \le k$. Hence applying Lemma 2.1 to $P(z)$, we get

$$\frac{\mu}{n}\frac{|a_\mu|R^\mu}{|a_0|}\left(\frac{k}{R}\right)^\mu \le 1. \tag{2.3}$$

Now, (2.3) becomes

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^\mu \le 1,$$

which is equivalent to

$$\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1} + R^\mu}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}R^\mu + k^{2\mu}R)} \le \frac{R^{\mu-1}}{R^\mu + k^\mu}.$$

□

Lemma 2.3. ([5]) *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then for $0 < r \le R \le k$,*

$$\exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}t^\mu + k^{2\mu}t)}dt\right\} \le \left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}}. \tag{2.4}$$

Lemma 2.4. ([13]) *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, then*

$$\max_{|z|=1}|p'(z)| \le n\frac{1 + \frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}{1 + k^{\mu+1} + \frac{\mu}{n}|\frac{a_\mu}{a_0}|(k^{\mu+1} + k^{2\mu})}\max_{|z|=1}|p(z)|. \tag{2.5}$$

Inequality (2.5) *is sharp and equality holds for the polynomial* $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, *where* n *is a multiple of* μ.

Lemma 2.5. ([9]) *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *is a polynomial of degree* n *having no zero in* $|z| < k$, $k > 0$, *then for* $0 < r \le R \le k$,

$$\max_{|z|=R} |p(z)| \le \exp\left\{ n \int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}t^\mu + k^{2\mu} t)} dt \right\} \times \max_{|z|=r} |p(z)|. \tag{2.6}$$

Lemma 2.6. ([14]) *If* $p(z)$ *is the polynomial of degree* n *having no zero in* $|z| < k$, $k \ge 1$, *then for* $|z| \le k$, $|\xi| \le k$, *where* ξ *is a real or complex number, we have*

$$(\xi - z)p'(z) + np(z) \ne 0. \tag{2.7}$$

Lemma 2.7. ([8]) *If* $f(z)$ *is analytic and* $|f(z)| \le 1$ *in* $|z| \le 1$, *then for* $|z| \le 1$,

$$|f(z)| \le \frac{(1-|a|)|z|^2 + |bz| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |bz| + (1-|a|)}, \tag{2.8}$$

where $a = f(0)$, $b = f'(0)$. *The example*

$$f(z) = \frac{a + \frac{b}{1+a} z - z^2}{1 - \frac{b}{1+a} z - a z^2},$$

shows that the estimate is sharp.

Lemma 2.8. ([7]) *If* $p(z)$ *is a polynomial of degree* n, *then on* $|z| = 1$,

$$|p'(z)| + |q'(z)| \le n \max_{|z|=1} |p(z)|, \tag{2.9}$$

where

$$q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}.$$

Lemma 2.9. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \ge 1$, *then*

$$1 - \frac{1}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu-\lambda^2|} \ge 0, \tag{2.10}$$

where $\lambda = \frac{k}{n}\frac{a_1}{a_0}$, $\mu = \frac{2k^2}{n(n-1)}\frac{a_2}{a_0}$.

Proof. From Lemma 2.1, we have

$$|\lambda| = \frac{k}{n}\frac{|a_1|}{|a_0|} \le 1.$$

Now,

$$(1-k+k^2+k|\lambda|) - (1+k^2|\lambda|) = k(k-1)(1-|\lambda|) \ge 0$$

or

$$(1-k+k^2+k|\lambda|) \ge (1+k^2|\lambda|),$$

which implies that

$$(1+k)\left[(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|\right] \ge (1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu-\lambda^2|,$$

which is equivalent to

$$1 - \frac{1}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \ge 0.$$

□

Lemma 2.10. ([8]) *If $p(z)$ is a polynomial of degree n, then*

$$\max_{|z|=1}|p'(z)| \ge n\max_{|z|=1}|p(z)| - \max_{|z|=1}|q'(z)|, \tag{2.11}$$

where

$$q(z) = z^n\overline{p\left(\frac{1}{\bar{z}}\right)}.$$

Lemma 2.11. ([10]) *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, then*

$$\frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \le n\frac{1+k|\lambda|}{1+k^2+2k|\lambda|}, \tag{2.12}$$

where $\lambda = \frac{k}{n}\frac{a_1}{a_0}$, $\mu = \frac{2k^2}{n(n-1)}\frac{a_2}{a_0}$.

Lemma 2.11 was conjectured by Govil et al. [8] and later precisely proved by Krishnadas et al [10].

Lemma 2.12. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$, then*

$$\frac{n}{1+k}\frac{(1-|\omega|)(1+k^2|\omega|)+(n-1)k|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+(n-1)k|\Omega-\omega^2|} \ge \frac{n\,|a_n|+|a_{n-1}|}{(1+k^2)\,n\,|a_n|+2\,|a_{n-1}|}, \tag{2.13}$$

where $\omega = \frac{1}{nk}\frac{a_{n-1}}{a_n}$, $\Omega = \frac{2}{n(n-1)k^2}\frac{a_{n-2}}{a_n}$.

Proof. If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$, then $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$ is a polynomial of degree at most n having no zero in $|z| < 1/k$, $1/k \ge 1$. Applying Lemma 2.12 to $q(z)$, we have

$$\frac{n}{1+\frac{1}{k}} \frac{(1-|\omega|)(1+\frac{1}{k^2}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|}{(1-|\omega|)(1-\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|} \le n \frac{1+\frac{1}{k}|\omega|}{1+\frac{1}{k^2}+2\frac{1}{k}|\omega|},$$

which is equivalent to

$$\begin{aligned} & n - \frac{n}{1+\frac{1}{k}} \frac{(1-|\omega|)(1+\frac{1}{k^2}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|}{(1-|\omega|)(1-\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|} \\ & \ge n - n\frac{1+\frac{1}{k}|\omega|}{1+\frac{1}{k^2}+2\frac{1}{k}|\omega|}. \end{aligned}$$

This simplifies to

$$\begin{aligned} & \frac{n}{1+k} \frac{(1-|\omega|)(1+k^2|\omega|)+(n-1)k|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+(n-1)k|\Omega-\omega^2|} \\ & \ge \frac{n\,|a_n|+|a_{n-1}|}{(1+k^2)\,n\,|a_n|+2\,|a_{n-1}|}. \end{aligned}$$

□

3. Main results

In this paper, under the same set of hypotheses, we first obtain an improvement of Theorem 1.3 by involving some of the coefficients of the polynomial $p(z)$. In fact, we obtain

Theorem 3.1. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \ge 1$, *then for* $0 < r \le R \le k$,

$$\begin{aligned} \max_{|z|=R} |p'(z)| \quad \le \quad & n \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1} + R^\mu}{R^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(k^{\mu+1}R^\mu + k^{2\mu}R\right)} \\ & \times \ \exp\left\{ n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(k^{\mu+1}t^\mu+k^{2\mu}t\right)} dt \right\} \\ & \times \ \max_{|z|=r}|p(z)|. \end{aligned} \tag{3.1}$$

Proof. Since $p(z)\neq 0$ in $|z|<k$, $k>0$, the polynomial $P(z)=p(Rz)\neq 0$ in $|z|<\frac{k}{R}$, $\frac{k}{R}\geq 1$, where $0<R\leq k$. Hence applying Lemma 2.4 to $P(z)$, we have

$$\max_{|z|=1}|P'(z)|\leq n\frac{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|R^\mu\left(\frac{k}{R}\right)^{\mu+1}}{1+\left(\frac{k}{R}\right)^{\mu+1}+\frac{\mu}{n}|\frac{a_\mu}{a_0}|R^\mu\left(\frac{k^{\mu+1}}{R^{\mu+1}}+\frac{k^{2\mu}}{R^{2\mu}}\right)}\max_{|z|=1}|P(z)|.$$

This gives

$$R\max_{|z|=R}|p'(z)|\leq n\frac{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|R^\mu\left(\frac{k}{R}\right)^{\mu+1}}{1+\left(\frac{k}{R}\right)^{\mu+1}+\frac{\mu}{n}|\frac{a_\mu}{a_0}|R^\mu\left(\frac{k^{\mu+1}}{R^{\mu+1}}+\frac{k^{2\mu}}{R^{2\mu}}\right)}\max_{|z|=R}|p(z)|,$$

which is equivalent to

$$\begin{aligned}\max_{|z|=R}|p'(z)| &\leq n\frac{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|\frac{k^{\mu+1}}{R}}{R+\frac{k^{\mu+1}}{R^\mu}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(\frac{Rk^{\mu+1}}{R}+\frac{Rk^{2\mu}}{R^\mu}\right)}\max_{|z|=R}|p(z)| \\ &= n\frac{R^\mu+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1}}{R^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(k^{\mu+1}R^\mu+k^{2\mu}R\right)}\max_{|z|=R}|p(z)|.\end{aligned}$$

Using Lemma 2.5 for $\max\limits_{|z|=R}|p(z)|$, we obtain

$$\begin{aligned}\max_{|z|=R}|p'(z)| = n&\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(k^{\mu+1}R^\mu+k^{2\mu}R\right)} \\ &\times\exp\left\{n\int_r^R\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(k^{\mu+1}t^\mu+k^{2\mu}t\right)}dt\right\}\max_{|z|=r}|p(z)|.\end{aligned}$$

This completes the proof of Theorem 3.1. □

Remark 3.2. To show that Theorem 3.1 is in general, an improvement of Theorem 1.3, it is sufficient to show that

$$\begin{aligned}&\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}R^\mu+k^{2\mu}R)} \\ &\quad\times\exp\left\{n\int_r^R\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}t^\mu+k^{2\mu}t)}dt\right\} \\ &\leq\frac{R^{\mu-1}}{R^\mu+k^\mu}\left(\frac{k^\mu+R^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}.\end{aligned}\tag{3.2}$$

By Lemma 2.2, we have

$$\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}R^\mu+k^{2\mu}R)} \le \frac{R^{\mu-1}}{R^\mu+k^\mu}. \tag{3.3}$$

Also by Lemma 2.3, we have

$$\exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}t^\mu+k^{2\mu}t)}dt\right\} \le \left(\frac{k^\mu+R^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}. \tag{3.4}$$

Multiplying inequalities (3.3) and (3.4), we have inequality (3.2).

Remark 3.3. Using inequality (3.2) of Remark 3.2, Theorem 3.1 reduces to Theorem 1.3 which further generalizes Theorem 1.2.

Remark 3.4. Putting $r = 1$, in Theorem 3.1, we have the following generalization of Lemma 2.4 proved by Qazi [13].

Corollary 3.5. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, then for $1 \le R \le k$,*

$$\begin{aligned}\max_{|z|=R}|p'(z)| \quad \le \quad & n\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(k^{\mu+1}R^\mu+k^{2\mu}R\right)} \\ & \times\exp\left\{n\int_1^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(k^{\mu+1}t^\mu+k^{2\mu}t\right)}dt\right\} \\ & \times\max_{|z|=1}|p(z)|. \end{aligned} \tag{3.5}$$

Remark 3.6. If we assign $r = R = 1$ and $\mu = 1$ in Theorem 3.1, we obtain the following inequality proved by Govil et al. [8], which further improves the bound given by inequality (1.3) due to Malik [12].

Corollary 3.7. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, then*

$$\max_{|z|=1}|p'(z)| \le n\frac{1+\frac{1}{n}\left|\frac{a_1}{a_0}\right|k^2}{1+k^2+2\frac{1}{n}\left|\frac{a_1}{a_0}\right|k^2}\max_{|z|=1}|p(z)|. \tag{3.6}$$

Remark 3.8. Using the fact

$$\frac{1}{n}\left|\frac{a_1}{a_0}\right| k \leq 1$$

from Lemma 2.1, inequality (3.6) of Corollary 3.7 reduces to inequality (1.3) proved by Malik [12].

Remark 3.9. Putting $r = R = \mu = k = 1$, inequality (3.1) of Theorem 3.1 reduces to Erdös-Lax inequality (1.2).

Next, we consider polynomials of degree $n \geq 3$ and prove the following theorem which is an improvement of Theorem 1.4 by involving $\min\limits_{|z|=k} |p(z)|$. In fact, we prove

Theorem 3.10. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n \geq 3$ having no zero in $|z| < k$, $k \geq 1$, then*

$$\begin{aligned}\max_{|z|=1} |p'(z)| \leq{} & \frac{n}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu-\lambda^2|} \max_{|z|=1} |p(z)| \\ & - \frac{n}{k^n}\left(1 - \frac{1}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu-\lambda^2|}\right) \\ & \times \min_{|z|=k} |p(z)|, \qquad (3.7)\end{aligned}$$

where $\lambda = \frac{k}{n}\frac{a_1}{a_0}$, $\mu = \frac{2k^2}{n(n-1)}\frac{a_2}{a_0}$. The result is best possible and equality in (3.7) holds for

$$p(z) = a_0 \frac{1}{k^n}(z+k)^{n_1}\left(z^2 + 2kz\frac{na - n_1}{n - n_1} + k^2\right)^{\frac{n-n_1}{3}},$$

where a is an arbitrary real number and n_1 is an integer such that $\frac{n}{3} \leq n_1 \leq n$, $n - n_1$ is even.

Proof. Consider a new polynomial $Q(z) = p(z) + m\alpha z^n$, where α is a real or complex number such that $|\alpha| < \left(\frac{1}{k}\right)^n$, $m = \min\limits_{|z|=k} |p(z)|$.

Now, on $|z| = k$

$$\begin{aligned}|m\alpha z^n| &< m\frac{1}{k^n}k^n \\ &= m \\ &\leq |p(z)|.\end{aligned}$$

Then by Rouche's theorem, $p(z)$ and $Q(z)$ must have same number of zeros in $|z| < k$ and hence $Q(z)$ has no zero in $|z| < k$. And for $|z| < k$, $|\xi| < k$, where

ξ is a real or complex number, by Lemma 2.6, we have

$$nQ(z) + (\xi - z)Q'(z) \neq 0,$$

that is,

$$nQ(z) - zQ'(z) \neq -\xi Q'(z). \tag{3.8}$$

Consequently, for $|z| \leq k$

$$\left| \frac{Q'(z)}{nQ(z) - zQ'(z)} \right| \leq \frac{1}{k}. \tag{3.9}$$

Hence if

$$f(z) = \frac{kQ'(kz)}{nQ(kz) - kzQ'(kz)}, \tag{3.10}$$

then $|f(z)| \leq 1$ for $|z| \leq 1$.

Also

$$f(0) = \frac{ka_1}{na_0} = \lambda \tag{3.11}$$

and

$$f'(0) = (n-1)\left\{ \frac{2k^2 a_2}{n(n-1)a_0} - \left(\frac{ka_1}{na_0} \right)^2 \right\} = (n-1)(\mu - \lambda^2). \tag{3.12}$$

Then for $|z| \leq 1$, we use Lemma 2.7 to conclude that

$$|f(z)| \leq \frac{(1-|\lambda|)|z|^2 + (n-1)|\mu - \lambda^2||z| + |\lambda|(1-|\lambda|)}{|\lambda|(1-|\lambda|)|z|^2 + (n-1)|\mu - \lambda^2||z| + (1-|\lambda|)}.$$

Thus in particular for $|z| = 1$, we have

$$|Q'(z)| \leq \frac{1}{k} \frac{(1-|\lambda|) + (n-1)|\mu - \lambda^2|k + |\lambda|(1-|\lambda|)k^2}{|\lambda|(1-|\lambda|) + (n-1)|\mu - \lambda^2|k + (1-|\lambda|)k^2} |nQ(z) - zQ'(z)|. \tag{3.13}$$

If $q(z) = z^n \overline{Q(\frac{1}{\bar{z}})}$, then on $|z| = 1$, $|nQ(z) - zQ'(z)| = |q'(z)|$. Therefore inequality (3.13) becomes

$$|Q'(z)| \leq \frac{1}{k} \frac{(1-|\lambda|) + (n-1)|\mu - \lambda^2|k + |\lambda|(1-|\lambda|)k^2}{|\lambda|(1-|\lambda|) + (n-1)|\mu - \lambda^2|k + (1-|\lambda|)k^2} |q'(z)|. \tag{3.14}$$

From Lemma 2.8, we have

$$\max_{|z|=1} \big(|Q'(z)| + |q'(z)| \big) \leq n \max_{|z|=1} |Q(z)|. \tag{3.15}$$

Combining inequalities (3.14) and (3.15), we get

$$\max_{|z|=1} |Q'(z)| \leq \frac{n}{1+k} \frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|) + k(n-1)|\mu - \lambda^2|} \max_{|z|=1} |Q(z)|,$$

which is equivalent to

$$\max_{|z|=1}|p'(z)+\alpha mnz^{n-1}| \leq \frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \times \max_{|z|=1}|p(z)+\alpha mz^n|. \tag{3.16}$$

Suppose z_0 on $|z|=1$ is such that

$$\max_{|z|=1}|p'(z)| = |p'(z_0)|. \tag{3.17}$$

Now,

$$|p'(z_0)+n\alpha mz_0^{n-1}| \leq \max_{|z|=1}|p'(z)+n\alpha mz^{n-1}|. \tag{3.18}$$

In the left hand side of inequality (3.18) for suitable choice of the argument of α, we have

$$|p'(z_0)+n\alpha mz_0^{n-1}| = |p'(z_0)|+n|\alpha|m. \tag{3.19}$$

Using (3.19) and (3.17) in inequality (3.18), we have

$$\max_{|z|=1}|p'(z)|+n|\alpha|m \leq \max_{|z|=1}|p'(z)+n\alpha mz^{n-1}|. \tag{3.20}$$

Combining inequalities (3.20) and (3.16), we have

$$\max_{|z|=1}|p'(z)|+n|\alpha|m \leq \frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \times \max_{|z|=1}|p(z)+\alpha mz^n|. \tag{3.21}$$

Again suppose z_1 on $|z|=1$ is such that

$$\begin{aligned}\max_{|z|=1}|p(z)+\alpha mz^n| &= |p(z_1)+\alpha mz_1^n| \\ &\leq |p(z_1)|+|\alpha|m \\ &\leq \max_{|z|=1}|p(z)|+|\alpha|m. \end{aligned}\tag{3.22}$$

Using inequality (3.22) in inequality (3.21), we have

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \times \left\{\max_{|z|=1}|p(z)|+|\alpha|m\right\} - n|\alpha|m,$$

which on taking limit as $|\alpha| \to \frac{1}{k^n}$ becomes

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|} \times \left\{\max_{|z|=1}|p(z)|+\frac{1}{k^n}m\right\}-n\frac{1}{k^n}m,$$

which on simplification gives

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|}\max_{|z|=1}|p(z)| - \frac{n}{k^n}\left\{1-\frac{1}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|}\right\}m.$$

This completes the proof of Theorem 3.10. □

Remark 3.11. To show that Theorem 3.10 is indeed an improvement of Theorem 1.4, it is sufficient to show

$$\left(1-\frac{1}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|}\right) \geq 0. \tag{3.23}$$

From Lemma 2.9, we have inequality (3.23).

Remark 3.12. Using inequality (2.12) of Lemma 2.11, Theorem 3.10 reduces to the following result which improves the bound given by Govil et al. [8, Theorem 1, (10)].

Corollary 3.13. *If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n \geq 3$ having no zero in $|z|<k$, $k \geq 1$, then*

$$\max_{|z|=1}|p'(z)| \leq n\frac{1+k|\lambda|}{1+k^2+2k|\lambda|}\max_{|z|=1}|p(z)| - \frac{n}{k^n}\left(1-\frac{1+k|\lambda|}{1+k^2+2k|\lambda|}\right)\min_{|z|=k}|p(z)|, \tag{3.24}$$

where $\lambda = \frac{k}{n}\frac{a_1}{a_0}$.

Remark 3.14. Using the fact

$$|\lambda| = \frac{1}{n}\left|\frac{a_1}{a_0}\right| k \leq 1$$

from Lemma 2.1, inequality (3.24) of Corollary 3.13 reduces to the following result which is an improvement of inequality (1.3) proved by Malik [12].

Corollary 3.15. *If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n\geq 3$ having no zero in $|z|<k$, $k\geq 1$, then*

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{1+k}\max_{|z|=1}|p(z)|-\frac{n}{k^{n-1}+k^{n}}\min_{|z|=k}|p(z)|. \tag{3.25}$$

Remark 3.16. Putting k=1 in Theorem 3.10, we obtain the following inequality proved by Aziz and Dawood [1], which further improves the bound given by Erdös-Lax inequality (1.2).

Corollary 3.17. *If $p(z)$ is a polynomial of degree $n\geq 3$ having no zero in $|z|<k$, $k\geq 1$, then*

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{2}\left\{\max_{|z|=1}|p(z)|-\min_{|z|=1}|p(z)|\right\}. \tag{3.26}$$

As an application of Theorem 3.10, we obtain the following result which is an improvement of the result proved by Govil et al. [8, Corollary 2, (17)].

Theorem 3.18. *If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$, $a_0\neq 0$, is a polynomial of degree $n\geq 3$ having all its zeros in $|z|\leq k$, $k\leq 1$, then*

$$\max_{|z|=1}|p'(z)|\geq \frac{n}{1+k}\,\frac{(1-|\omega|)(1+k^2|\omega|)+(n-1)k|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+(n-1)k|\Omega-\omega^2|}\times\left\{\max_{|z|=1}|p(z)|+\min_{|z|=k}|p(z)|\right\}, \tag{3.27}$$

where $\omega=\frac{1}{nk}\frac{a_{n-1}}{a_n}$, $\Omega=\frac{2}{n(n-1)k^2}\frac{a_{n-2}}{a_n}$.

Proof. If $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$, $a_0\neq 0$, is a polynomial of degree $n\geq 3$, then $q(z)=z^n\overline{p(1/\overline{z})}$ is also a polynomial of degree $n\geq 3$, then on $|z|=1$,

$$|p(z)|=|q(z)|. \tag{3.28}$$

Also by Lemma 2.10, we have

$$\max_{|z|=1}|p'(z)|\geq n\max_{|z|=1}|p(z)|-\max_{|z|=1}|q'(z)|,$$

that is

$$n\max_{|z|=1}|p(z)|-\max_{|z|=1}|p'(z)|\leq \max_{|z|=1}|q'(z)|. \tag{3.29}$$

If $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then $q(z)$ has no zero in $|z| < 1/k$, $1/k \geq 1$. Hence applying Theorem 3.10 to $q(z)$, we have

$$\max_{|z|=1} |q'(z)| \leq \frac{n}{1+\frac{1}{k}} \frac{(1-|\omega|)(1+\frac{1}{k^2}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|}{(1-|\omega|)(1-\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|} \max_{|z|=1} |q(z)|$$
$$- nk^n \left\{1 - \frac{1}{1+\frac{1}{k}} \frac{(1-|\omega|)(1+\frac{1}{k^2}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|}{(1-|\omega|)(1-\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|}\right\}$$
$$\times \min_{|z|=k} |q(z)|, \tag{3.30}$$

where $\omega = \frac{1}{nk}\frac{a_{n-1}}{a_n}$, $\Omega = \frac{2}{n(n-1)k^2}\frac{a_{n-2}}{a_n}$.
Since $q(z) = z^n\overline{p(1/\overline{z})}$,

$$\begin{aligned}\min_{|z|=\frac{1}{k}} |q(z)| &= \min_{|z|=\frac{1}{k}} |z^n\overline{p(1/\overline{z})}| \\ &= \frac{1}{k^n}\min_{|z|=\frac{1}{k}} |p(1/\overline{z})| \\ &= \frac{1}{k^n}\min_{|z|=k} |p(z)|. \end{aligned} \tag{3.31}$$

Combining inequalities (3.29) and (3.30) and then using (3.28) and (3.31), we obtain

$$n\max_{|z|=1} |p(z)| - \max_{|z|=1} |p'(z)|$$
$$\leq \frac{n}{1+\frac{1}{k}} \frac{(1-|\omega|)(1+\frac{1}{k^2}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|}{(1-|\omega|)(1-\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k}|\omega|)+(n-1)\frac{1}{k}|\Omega-\omega^2|} \max_{|z|=1} |p(z)|$$
$$- \frac{nk^n}{1+k} \frac{(1-|\omega|)(1+k^2|\omega|)+(n-1)k|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+(n-1)k|\Omega-\omega^2|} \frac{1}{k^n} \min_{|z|=k} |p(z)|,$$

which is equivalent to

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \frac{(1-|\omega|)(1+k^2|\omega|)+(n-1)k|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+(n-1)k|\Omega-\omega^2|}$$
$$\times \left\{\max_{|z|=1} |p(z)| + \min_{|z|=k} |p(z)|\right\}. \tag{3.32}$$

Hence, the proof of Theorem 3.18 is complete. □

Remark 3.19. Theorem 3.18 improves upon the result of Govil et al. [8, Corollary 2,(17)] by involving $\min_{|z|=k} |p(z)|$.

So, to show that Theorem 3.18 is an improvement of the result due to Govil et al. [8, Corollary 2, (17)], it is sufficient to show

$$\frac{n}{1+k}\frac{(1-|\omega|)(1+k^2|\omega|)+(n-1)k|\Omega-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+(n-1)k|\Omega-\omega^2|}\geq 0,$$

which is equivalent to show

$$1-|\omega|\geq 0.$$

Applying Lemma 2.1 to $q(z)=z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$, where $p(z)$ is as defined in Theorem 3.18, we have

$$|\omega|=\frac{1}{nk}\frac{|a_{n-1}|}{|a_n|}\leq 1.$$

Remark 3.20. Using inequality (2.13) of Lemma 2.12, Theorem 3.18 reduces to the following result which is an improvement of Theorem 1.5 due to Govil et al. [8].

Corollary 3.21. *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$, $a_0\neq 0$, *is a polynomial of degree* $n\geq 3$ *having all its zeros in* $|z|\leq k$, $k\leq 1$, *then*

$$\max_{|z|=1}|p'(z)|\geq\frac{n\,|a_n|+|a_{n-1}|}{(1+k^2)\,n\,|a_n|+2\,|a_{n-1}|}\left\{\max_{|z|=1}|p(z)|+\min_{|z|=k}|p(z)|\right\}. \tag{3.33}$$

Remark 3.22. Putting $k=1$ in Theorem 3.18, we obtain the following refinement of inequality (1.9) which was proved by Aziz and Dawood [1].

Corollary 3.23. *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$, $a_0\neq 0$, *is a polynomial of degree* $n\geq 3$ *having all its zeros in* $z\leq 1$, *then*

$$\max_{|z|=1}|p'(z)|\geq\frac{n}{2}\left\{\max_{|z|=1}|p(z)|+\min_{|z|=1}|p(z)|\right\}. \tag{3.34}$$

Acknowledgments: We are thankful to NIT, Manipur for providing us financial support. We are also grateful to the referee for his/her useful suggestions.

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