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CAPUTO DELAYED FRACTIONAL DIFFERENTIAL EQUATIONS BY SADIK TRANSFORM

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Abstract. In this article, we are interested in studying the fractional Sadik Transform and a combination of the method of steps that will be applied together to find accurate solutions or approximations to homogeneous and non-homogeneous delayed fractional differential equations with constant-coefficient and possible extension to time-dependent delays. The results show that the process is correct, exact, and easy to do for solving delayed fractional differential equations near the origin. Finally, we provide several examples to illustrate the applicability of this method.

1. Introduction

The theory of delayed differential equations is important due to its demonstrated applications in various physical problems of science and engineering.

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Inclusion of delay in the delayed differential equation seems to be opening new possibilities of applications, especially in the field of engineering, chemistry, physics, and finance [2, 5, 14, 9, 10, 15, 24]. Fractional calculus transacts with study the integration and differentiation operators of fractional-order over real or complex domains and some of their applications are in the area of control theory of dynamical systems, fluid flow, diffusive transport akin to diffusion, electrical networks, electrochemistry of corrosion, viscoelasticity, etc.

A major development has occurred in fractional differential equations (FDEs) in recent years, we refer the reader with references see [1, 3, 4, 7, 10, 18, 19, 25]. On the other hand, the recent development of FDEs with Sadik Transform (ST) and the theoretical analysis can be seen in [17, 20, 21, 22, 23]. For instance, Shaikh in [20], proposed a new integral transform that known as ST. This transform is a unification of some famous transforms such as Laplace, Sumudu, Kamal, Laplace-Carson, and Elzaki transform. Further, the author proved that above mentioned transforms are particular cases of ST. Shaikh in [22], obtained transfer function of dynamical system in control theory using ST. Moreover, the author solved some applications in control theory by ST. However, exact solutions of delayed fractional differential equations (DFDEs) are not known for most. Therefore different numerical techniques [13, 16, 26] have been advanced and used to find approximate solutions, but sometimes applying complicated calculations and algorithms. The suggested combination of the method of steps and ST conquer such snags by proceeding simple and easily applicable ways for solving DFDEs near the origin. ST has the strength to handle the problems in extreme simplified way.

In this article, we introduce new results of solution to the problem (2.1)-(2.3) involving Caputo delayed fractional operator with ST. Moreover, we use the method of steps and ST to analyize our results. To the best of our knowledge, the Caputo delayed functional differential equations involving ST have not yet been investigated and developed till the present day. Using presented methods, we are able to find the unique exact solution to a well posed initial value problem.

The rest of article is organized as follows. In the next section, we introduce the problem statement. In the third section, we begin by summarizing the forms Caputo type fractional derivative, and we also present the background material and important lemmas which are related to our work. In the sequel. The fourth section contains exact solutions or approximations to homogeneous and non-homogeneous DFDEs to the problem (2.1)-(2.3) by means combination of the method of steps and ST. In the last section, we present some illustrative examples.

2. Problem statement

In the article, we employ the ST of fractional order and a combination of the method of steps to solving certain classes of DFDEs of second order in the form:

$$\mu_1\mathcal{D}^2u(\kappa)+\mu_2\,{}^C\mathcal{D}^{\rho}_{0+}u(\kappa)+\mu_3u(\kappa-\nu)+\mu_4u(\kappa)=g(\kappa),\quad \kappa>0, \tag{2.1}$$

with initial function

$$u(\kappa)=\varphi(\kappa),\quad \kappa\in[-\nu,0] \tag{2.2}$$

and initial conditions

$$u(0)=\ell_1,\quad u'(0)=\ell_2, \tag{2.3}$$

where g is a continuous function on $[0,\kappa]$, $\nu>0,\mu_1,\mu_2,\mu_3,\mu_4,\ell_1,\ell_2$ are constants, $\varphi(\kappa)$ is continuous on $[-\nu,0]$ and ${}^C\mathcal{D}^{\rho}_{0+}$ $(1<\rho<2)$ denotes the Caputo fractional derivative of order ρ. Equation (2.1) covers numerous equations with applications, e.g. the composite fractional oscillation equation [8], delayed model of growth in cell populations [12], or the delayed model of Bagley-Torvik equation [6].

3. Preliminaries

In this section, we introduce some basic definitions of fractional calculus theory and properties of the ST which are used throughout this manuscript.

Definition 3.1. ([10]) The Caputo derivative of fractional order ρ $(n-1<\rho<n\in\mathbb{N})$ is given by

$${}^C\mathcal{D}^{\rho}_{0+}u(\kappa)=\frac{1}{\Gamma(n-\rho)}\int_0^{\kappa}(\kappa-\nu)^{n-\rho-1}u^{(n)}(\nu)d\nu,$$

where the function $u(\kappa)$ has absolutely continuous derivatives up to order $(n-1)$. In particular, if $0<\rho<1$, we have

$${}^C\mathcal{D}^{\rho}_{0+}u(\kappa)=\frac{1}{\Gamma(1-\rho)}\int_0^{\kappa}(\kappa-\nu)^{-\rho}u'(\nu)d\nu.$$

Definition 3.2. A function u on $0\le\kappa<1$ is said to be exponentially bounded of order $\varrho_2\in\mathbb{R}$ if it satisfies an inequality of the form

$$u(\kappa)\le Me^{\varrho_2\kappa},$$

for some real constant $M>0$.

Definition 3.3. ([20]) (Sadik transform) Assume that u is piecewise continuous on the interval $[0,A]$ which is exponentially bounded function of order a, for any real constant a, and some positive constants K and M. Then the ST of $u(\kappa)$ is defined by

$$U(v,\sigma_1,\sigma_2)=\mathcal{S}[u(\kappa)]=\frac{1}{v^{\sigma_2}}\int_0^\infty e^{-\kappa v^{\sigma_1}}u(\kappa)d\kappa,$$

where v is complex variable, σ_1 is any non zero real number, and σ_2 is any real number.

Definition 3.4. ([14]) (Mittag-Leffer function) Let $\varrho_1,\varrho_2\in\mathbb{C}$, $Re(\varrho_1)>0$, $Re(\varrho_2)>0$. Then the Mittag-Leffer function of one variable is given

$$E_{\varrho_1}(\kappa)=\sum_{\lambda_1=0}^{\infty}\frac{\kappa^{\lambda_1}}{\Gamma(\varrho_1\lambda_1+1)}.$$

The Mittag-Leffer function of two variables is given by

$$E_{\varrho_1,\varrho_2}(\kappa)=\sum_{\lambda_1=0}^{\infty}\frac{\kappa^{\lambda_1}}{\Gamma(\varrho_1\lambda_1+\varrho_2)}.$$

Property 3.5. ([20]) *Let $U(v,\sigma_1,\sigma_2)$ be a ST of $u(\kappa)$, that is, $\mathcal{S}[u(\kappa)]=U(v,\sigma_1,\sigma_2)$. Then*

(1) *If $u(\kappa)=1$, then $\mathcal{S}[1]=\frac{1}{v^{\sigma_1+\sigma_2}}$.*
(2) *If $u(\kappa)=\kappa^n$, then $\mathcal{S}[\kappa^n]=\frac{n!}{v^{n\sigma_1+(\sigma_1+\sigma_2)}}$.*

Lemma 3.6. ([17]) *Let $n-1<\gamma<n$, $(n=[\gamma]+1)$ and $u(\kappa),u'(\kappa),u''(\kappa),\dots,u^{(n-1)}(\kappa)$ be continuous on $[0,\infty)$ and of exponential order, while ${}^{C}\mathcal{D}^{\gamma}_{0+}u(\kappa)$ is piecewise continuous on $[0,\infty)$. Then ST of Caputo fractional derivative of order γ of function u is given by*

$$\mathcal{S}[{}^{C}\mathcal{D}^{\rho}_{0+}u(\kappa)]=v^{\rho\sigma_1}U(v,\sigma_1,\sigma_2)-\sum_{\lambda_1=0}^{n-1}v^{(\rho-n+\lambda_1)\sigma_1-\sigma_2}u^{(n-1-\lambda_1)}(0^+),$$

in particular for $n=2$,

$$\mathcal{S}[{}^{C}\mathcal{D}^{\rho}_{0+}u(\kappa)]=v^{\rho\sigma_1}U(v,\sigma_1,\sigma_2)-v^{(\rho-2)\sigma_1-\sigma_2}u^{(1)}(0^+)-v^{(\rho-1)\sigma_1-\sigma_2}u(0^+).$$

Lemma 3.7. ([17]) *Let $u(\kappa)=\kappa^{\varrho_1 m+\varrho_2-1}E^{(m)}_{\varrho_1,\varrho_2}(\pm a\kappa^{\varrho_1})$. Then the ST of $u(\kappa)$ is given by*

$$\frac{1}{v^{\sigma_2}}\int_0^\infty e^{-v^{\sigma_1}\kappa}\kappa^{\varrho_1 m+\varrho_2-1}E^{(m)}_{\varrho_1,\varrho_2}(\pm a\kappa^{\varrho_1})d\kappa=\frac{m!v^{\sigma_1\varrho_1-(\sigma_1\varrho_2+\sigma_2)}}{(v^{\sigma_1\varrho_1}\mp a)^{m+1}},$$

where $\sigma_1,\sigma_2\in\mathbb{C},\mathcal{R}e(\varrho_1)>0,\mathcal{R}e(\varrho_2)>0,\mathcal{R}e(v)>|a|^{\frac{1}{\mathcal{R}e(\sigma_1\varrho_1)}}$ and $E^{(m)}_{\varrho_1,\varrho_2}(\kappa)=\frac{d^m}{d\kappa^m}E_{\varrho_1,\varrho_2}(\kappa)$.

4. Main results

In this partition, we present the exact solutions or approximations to homogeneous and non-homogeneous Caputo DFDEs to the problem (2.1)-(2.3) involving ST. To prove that, we need the following auxiliary theorems.

Theorem 4.1. *For $\rho, \gamma > 0$, $\varrho_1 \in \mathbb{R}$ and $v^{\sigma_1\rho} > |\varrho_1|$, the following inverse ST formula holds:*

$$\mathcal{S}^{-1}\left[\frac{v^{\sigma_1(\rho-\gamma)}}{v^{\sigma_1\rho}-\varrho_1}\right] = \kappa^{\sigma_1\gamma-1}E_{\sigma_1\rho,\sigma_1\gamma}(-\varrho_1\kappa^{\sigma_1\rho}).$$

Proof.

$$\begin{aligned}
\mathcal{S}^{-1}\left[\frac{v^{\sigma_1(\rho-\gamma)}}{v^{\sigma_1\rho}-\varrho_1}\right] &= \mathcal{S}^{-1}\left[\frac{1}{v^{\sigma_1\gamma}(1+\frac{\varrho_1}{v^{\sigma_1\rho}})}\right] \\
&= \mathcal{S}^{-1}\left[\frac{1}{v^{\sigma_1\gamma}}\sum_{n=0}^{\infty}\left(\frac{-\varrho_1}{v^{\sigma_1\rho}}\right)^n\right] \\
&= \mathcal{S}^{-1}\left[\sum_{n=0}^{\infty}\frac{(-\varrho_1)^n\Gamma(n(\sigma_1\rho)+\sigma_1\gamma)}{\Gamma(n(\sigma_1\rho)+\sigma_1\gamma)v^{n(\sigma_1\rho)+\sigma_1\gamma}}\right],
\end{aligned}$$

by using Property 3.5 (2), we get

$$\begin{aligned}
\mathcal{S}^{-1}\left[\frac{v^{\sigma_1(\rho-\gamma)}}{v^{\sigma_1\rho}-\varrho_1}\right] &= \sum_{n=0}^{\infty}\frac{(-\varrho_1)^n}{\Gamma(n(\sigma_1\rho)+\sigma_1\gamma)}\mathcal{S}^{-1}\left(\frac{\Gamma(n(\sigma_1\rho)+\sigma_1\gamma)}{v^{n(\sigma_1\rho)+\sigma_1\gamma}}\right) \\
&= \sum_{n=0}^{\infty}\frac{(-\varrho_1)^n}{\Gamma(n(\sigma_1\rho)+\sigma_1\gamma)}\kappa^{n(\sigma_1\rho)+\sigma_1\gamma-1} \\
&= \kappa^{\sigma_1\gamma-1}\sum_{n=0}^{\infty}\frac{(-\varrho_1\kappa^{\sigma_1\rho})^n}{\Gamma(n(\sigma_1\rho)+\sigma_1\gamma)} \\
&= \kappa^{\sigma_1\gamma-1}E_{\sigma_1\rho,\sigma_1\gamma}(-\varrho_1\kappa^{\sigma_1\rho}).
\end{aligned}$$

□

Theorem 4.2. *For $0 < \rho \le 2$, $\gamma < 2, \mu_1, \mu_2 \in \mathbb{R}$, $v^{\sigma_1(2-\rho)} > |\mu_1|$, $|v^{\sigma_1(2-\rho)} + \mu_1| > |\mu_2|v^{-\sigma_1\rho}$, the following inverse ST formula is valid:*

$$\begin{aligned}
&\mathcal{S}^{-1}\left[\frac{v^{\sigma_1\gamma}}{v^{2\sigma_1}+\mu_1v^{\sigma_1\rho}+\mu_2}\right] \\
&= \kappa^{2\sigma_1-\sigma_1\gamma-1}\sum_{\lambda_1=0}^{\infty}\sum_{\lambda_2=0}^{\infty}\frac{(-\mu_1)^{\lambda_2}(-\mu_2)^{\lambda_1}\binom{\lambda_1+\lambda_2}{\lambda_2}\kappa^{\sigma_1[(2-\rho)\lambda_2+2\lambda_1]}}{\Gamma((2-\rho)\sigma_1\lambda_2)+(2+2\lambda_1-\gamma)\sigma_1}.
\end{aligned}$$

Proof.

$$\begin{aligned}
&\frac{v^{\sigma_1\gamma}}{v^{2\sigma_1}+\mu_1 v^{\sigma_1\rho}+\mu_2}\\
&=\frac{v^{\sigma_1(\gamma-\rho)}}{v^{\sigma_1(2-\rho)}+\mu_1+\mu_2 v^{-\sigma_1\rho}}\\
&=\frac{v^{\sigma_1(\gamma-\rho)}}{v^{\sigma_1(2-\rho)}+\mu_1}\frac{1}{1+\frac{\mu_2 v^{-\sigma_1\rho}}{v^{\sigma_1(2-\rho)}+\mu_1}}\\
&=\frac{v^{\sigma_1(\gamma-\rho)}}{v^{\sigma_1(2-\rho)}+\mu_1}\sum_{\lambda_1=0}^{\infty}(-\mu_2)^{\lambda_1}\left(\frac{v^{-\sigma_1\rho}}{v^{\sigma_1(2-\rho)}+\mu_1}\right)^{\lambda_1}\\
&=v^{\sigma_1\gamma}\sum_{\lambda_1=0}^{\infty}\frac{(-\mu_2)^{\lambda_1}v^{-\sigma_1(\lambda_1\rho+\rho)}}{(v^{\sigma_1(2-\rho)}+\mu_1)^{\lambda_1+1}}\\
&=v^{\sigma_1\gamma}\sum_{\lambda_1=0}^{\infty}\frac{(-\mu_2)^{\lambda_1}v^{-\sigma_1(\lambda_1\rho+\rho)}}{(v^{\sigma_1(2-\rho)(\lambda_1+1)})(1+\mu_1 v^{\sigma_1(\rho-2)})^{\lambda_1+1}}\\
&=v^{\sigma_1\gamma}\sum_{\lambda_1=0}^{\infty}\frac{(-\mu_2)^{\lambda_1}v^{-\sigma_1(2\lambda_1+2)}}{(1+\mu_1 v^{\sigma_1(\rho-2)})^{\lambda_1+1}}\\
&=\left|\frac{1}{(1+v^{\sigma_1})^{\lambda_1+1}}=\sum_{\lambda_2=0}^{\infty}\binom{\lambda_1+\lambda_2}{\lambda_2}(-v^{\sigma_1})^{\lambda_2}\right|\\
&=v^{\sigma_1\gamma}\sum_{\lambda_1=0}^{\infty}(-\mu_2)^{\lambda_1}v^{-\sigma_1(2\lambda_1+2)}\sum_{\lambda_2=0}^{\infty}\binom{\lambda_1+\lambda_2}{\lambda_2}(-\mu_1 v^{\sigma_1(\rho-2)})^{\lambda_2}\\
&=v^{\sigma_1\gamma}\sum_{\lambda_1=0}^{\infty}(-\mu_2)^{\lambda_1}\sum_{\lambda_2=0}^{\infty}\binom{\lambda_1+\lambda_2}{\lambda_2}(-\mu_1)^{\lambda_2}v^{\sigma_1[(\rho-2)-2\lambda_1-2+\lambda_2]}.
\end{aligned}$$

Applying the Property 3.5 (2), we deduce

$$\begin{aligned}
&\mathcal{S}^{-1}\left[v^{\sigma_1\gamma}\sum_{\lambda_1=0}^{\infty}(-\mu_2)^{\lambda_1}\sum_{\lambda_2=0}^{\infty}\binom{\lambda_1+\lambda_2}{\lambda_2}(-\mu_1)^{\lambda_2}v^{\sigma_1[(\rho-2)-2\lambda_1-2+\lambda_2]}\right]\\
&=\kappa^{2\sigma_1-\sigma_1\gamma-1}\sum_{\lambda_1=0}^{\infty}\sum_{\lambda_2=0}^{\infty}\frac{(-\mu_1)^{\lambda_2}(-\mu_2)^{\lambda_1}\binom{\lambda_1+\lambda_2}{\lambda_2}\kappa^{\sigma_1[(2-\rho)\lambda_2+2\lambda_1]}}{\Gamma((2-\rho)\sigma_1\lambda_2)+(2+2\lambda_1-\gamma)\sigma_1}.
\end{aligned}$$

□

Now, we can characterize the proposed algorithm. First, we apply the method of steps ([15], or [11]) to Cauchy problem (2.1)-(2.3). Then we deduce

FDE without delay:

$$\mu_1\mathcal{D}^2u(\kappa)+\mu_2\ {}^{C}\mathcal{D}^{\rho}_{0+}u(\kappa)+\mu_4u(\kappa)=g(\kappa)-\mu_3\varphi(\kappa-\nu),\quad \kappa>0. \tag{4.1}$$

Applying Lemma 3.6 to equation (4.1), we obtain

$$\begin{aligned}U(v,\sigma_1,\sigma_2)=&\frac{G(v,\sigma_1,\sigma_2)-\mu_3\Phi(v,\sigma_1,\sigma_2)}{A}\\&+\frac{\ell_1(\mu_1v^{\sigma_1-\sigma_2}+\mu_2v^{\sigma_1(\rho-1)-\sigma_2})}{A}\\&+\frac{\ell_2(\mu_1v^{-\sigma_2}+\mu_2v^{\sigma_1(\rho-2)-\sigma_2})}{A},\end{aligned}$$

$$\begin{aligned}u(\kappa)=&\mathcal{S}^{-1}\Big[\frac{G(v,\sigma_1,\sigma_2)-\mu_3\Phi(v,\sigma_1,\sigma_2)}{A}\\&+\frac{\ell_1(\mu_1v^{\sigma_1-\sigma_2}+\mu_2v^{\sigma_1(\rho-1)-\sigma_2})}{A}\\&+\frac{\ell_2(\mu_1v^{-\sigma_2}+\mu_2v^{\sigma_1(\rho-2)-\sigma_2})}{A}\Big].\end{aligned}$$

where $A=\mu_1v^{2\sigma_1}+\mu_2v^{\sigma_1\rho}+\mu_4$.

Practical application will be clarified in different examples.

Example 4.3. Consider delayed equation of Bagley-Torvik type in the type

$$\begin{cases}\mathcal{D}^2u(\kappa)+\ {}^{C}\mathcal{D}^{\rho}_{0+}u(\kappa)+u(\kappa-1)+u(\kappa)=1+2\kappa,\quad \kappa\in[0,1],\\ u(\kappa)=\varphi(\kappa)=1+\kappa,\quad \kappa\in[-1,0],\\ u(0)=u'(0)=1.\end{cases} \tag{4.2}$$

Applying the method of steps, we get

$$\mathcal{D}^2u(\kappa)+\ {}^{C}\mathcal{D}^{\rho}_{0+}u(\kappa)+u(\kappa)=1+\kappa,\quad \kappa\in[0,1]. \tag{4.3}$$

Using the ST to equation (4.3), we have

$$\begin{aligned}&(v^{2\sigma_1}+v^{\frac{3}{2}\sigma_1}+1)U(v,\sigma_1,\sigma_2)\\&=\frac{1}{v^{\sigma_1+\sigma_2}}+\frac{1}{v^{2\sigma_1+\sigma_2}}+v^{-\sigma_2}+v^{\frac{-1}{2}\sigma_1-\sigma_2}+v^{\sigma_1-\sigma_2}+v^{\frac{1}{2}\sigma_1-\sigma_2}\\&=\frac{1}{v^{\sigma_1+\sigma_2}}(v^{2\sigma_1}+v^{\frac{3}{2}\sigma_1}+1)+\frac{1}{v^{2\sigma_1+\sigma_2}}(v^{2\sigma_1}+v^{\frac{3}{2}\sigma_1}+1),\end{aligned}$$

which implies

$$U(v,\sigma_1,\sigma_2)=\frac{1}{v^{\sigma_1+\sigma_2}}+\frac{1}{v^{2\sigma_1+\sigma_2}}.$$

Thus,

$$\begin{aligned} u(\kappa) = \mathcal{S}^{-1}\left[U(v,\sigma_1,\sigma_2)\right] &= \mathcal{S}^{-1}\left[\frac{1}{v^{\sigma_1+\sigma_2}}\right] + \mathcal{S}^{-1}\left[\frac{1}{v^{2\sigma_1+\sigma_2}}\right] \\ &= 1+\kappa, \quad \kappa \in [-1,1]. \end{aligned} \tag{4.4}$$

Example 4.4. Consider the following delayed problem

$$\begin{cases} {}^{C}\mathcal{D}^{\rho}_{0+}u(\kappa) + u(\kappa-2) + u(\kappa) = \kappa^2 - 4\kappa + 5, \quad 1 < \rho \leq 2, \\ u(\kappa) = \varphi(\kappa) = 1 + 2\kappa, \quad \kappa \in [-2,0], \\ u(0) = 1, u'(0) = 2. \end{cases} \tag{4.5}$$

Applying the method of steps, we have

$${}^{C}\mathcal{D}^{\rho}_{0+}u(\kappa) + u(\kappa) = 0, \quad \kappa \in [0,2]. \tag{4.6}$$

Using the ST to equation (4.6), we get

$$v^{\rho\sigma_1}U(v,\sigma_1,\sigma_2) - v^{(\rho-2)\sigma_1-\sigma_2}u^{(1)}(0^+) - v^{(\rho-1)\sigma_1-\sigma_2}u(0^+) + U(v,\sigma_1,\sigma_2) = 0,$$

$$U(v,\sigma_1,\sigma_2) = \frac{v^{\sigma_1\rho-(\sigma_1+\sigma_2)}}{v^{\rho\sigma_1}+1} + 2\frac{v^{\sigma_1\rho-(2\sigma_1+\sigma_2)}}{v^{\rho\sigma_1}+1}.$$

Applying Theorem 4.1, we deduce exact solution of the equation (4.5) in the form:

$$u(\kappa) = \begin{cases} \kappa^{\sigma_1+\sigma_2-1}E_{\sigma_1\rho,\sigma_1+\sigma_2}(-\kappa^{\sigma_1\rho}) + 2\kappa^{2\sigma_1+\sigma_2-1}E_{\sigma_1\rho,2\sigma_1+\sigma_2}(-\kappa^{\sigma_1\rho}), \kappa \in [0,2], \\ 1+2\kappa, \quad \kappa \in [-2,0]. \end{cases} \tag{4.7}$$

Example 4.5. Consider a delayed version of the composite fractional oscillation equation

$$\begin{cases} \mathcal{D}^2u(\kappa) - {}^{C}\mathcal{D}^{\rho}_{0+}u(\kappa) + 2u(\kappa-1) - u(\kappa) = 2\kappa^2 - 3\kappa + 2, \quad (1 < \rho \leq 2), \\ u(\kappa) = \varphi(\kappa) = \kappa^2, \quad \kappa \in [-1,0], \\ u(0) = u'(0) = 0. \end{cases} \tag{4.8}$$

Applying the method of steps, we get

$$\mathcal{D}^2u(\kappa) - {}^{C}\mathcal{D}^{\rho}_{0+}u(\kappa) - u(\kappa) = \kappa, \quad \kappa \in [0,1]. \tag{4.9}$$

Using the ST to equation (4.9) and using Theorem 4.2, we have

$$(v^{2\sigma_1} - v^{\rho\sigma_1} - 1)U(v,\sigma_1,\sigma_2) = \frac{1}{v^{2\sigma_1+\sigma_2}}.$$

That is,

$$U(v,\sigma_1,\sigma_2) = \frac{v^{-2\sigma_1-\sigma_2)}}{v^{2\sigma_1} - v^{\rho\sigma_1} - 1}.$$

Therefore we have,

$$u(\kappa) = \mathcal{S}^{-1}\left[\frac{\upsilon^{-2\sigma_1-\sigma_2)}}{\upsilon^{2\sigma_1}-\upsilon^{\rho\sigma_1}-1}\right].$$

Hence, we get

$$u(\kappa) = \begin{cases} \kappa^{4\sigma_1+\sigma_2-1}\sum_{\lambda_1=0}^{\infty}\sum_{\lambda_2=0}^{\infty}\frac{\binom{\lambda_1+\lambda_2}{\lambda_2}\kappa^{\sigma_1[(2-\rho)\lambda_2+2\lambda_1]}}{\Gamma((2-\rho)\sigma_1\lambda_2)+(2\lambda_1+4)\sigma_1)}, & \kappa\in[0,1], \\ \kappa^2, \quad \kappa\in[-1,0]. \end{cases} \tag{4.10}$$

Remark 4.6. (i) In the Example 4.3, the problem (4.2) has same solution (4.4) in case using Laplace transform, Sumudu transform, or ST.

(ii) In the Example 4.4, if $\sigma_1 = 1$, $\sigma_2 = 0$, then the relation (4.7) reduces to

$$(\kappa) = \begin{cases} E_{\rho,1}(-\kappa^{\rho}) + 2\kappa E_{\rho,2}(-\kappa^{\rho}), & \kappa\in[0,2], \\ 1+2\kappa, & \kappa\in[-2,0], \end{cases} \tag{4.11}$$

which is a solution of (4.5) by using Laplace transform.

(iii) In the Example 4.5, if $\sigma_1 = 1$, $\sigma_2 = 0$, then the relation (4.10) reduces to

$$u(\kappa) = \begin{cases} \kappa^{3}\sum_{\lambda_1=0}^{\infty}\sum_{\lambda_2=0}^{\infty}\frac{\binom{\lambda_1+\lambda_2}{\lambda_2}\kappa^{[(2-\rho)\lambda_2+2\lambda_1]}}{\Gamma((2-\rho)\lambda_2)+(2\lambda_1+4))}, & \kappa\in[0,1], \\ \kappa^2, \quad \kappa\in[-1,0], \end{cases}$$

which is a solution of the problem (4.8) by using Laplace transform.

(iv) In general, the ST reduces to Laplace transform if $\sigma_1 = 1$, $\sigma_2 = 0$.

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