Nonlinear Functional Analysis and Applications
Vol. 28, No. 2 (2023), pp. 449-471
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2023.28.02.09
http://nfaa.kyungnam.ac.kr/journal-nfaa

SEMI-ANALYTICAL SOLUTION TO A COUPLED LINEAR INCOMMENSURATE SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

Iqbal M. Batiha[1], Nashat Alamarat[2], Shameseddin Alshorm[3], O. Y. Ababneh[4] and Shaher Momani[5]

[1]Department of Mathematics, Al Zaytoonah University of Jordan,
Queen Alia Airport St 594, Amman 11733, Jordan
Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE
e-mail: i.batiha@zuj.edu.jo

[2]Department of Mathematics, Faculty of Science and Technology,
Irbid National University, 2600 Irbid, Jordan
e-mail: nashatalamarat444@gmail.com

[3]Department of Mathematics, Al Zaytoonah University of Jordan,
Queen Alia Airport St 594, Amman 11733, Jordan
e-mail: alshormanshams@gmail.com

[4]Department of Mathematics, Zarqa University, Zarqa, Jordan
e-mail: osababneh@zu.edu.jo

[5] Department of Mathematics, The University of Jordan, Amman 11942, Jordan,
Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE
e-mail: s.momani@ajman.ac.ae

Abstract. In this paper, we study a linear system of homogeneous commensurate /incommensurate fractional-order differential equations by developing a new semi-analytical scheme. In particular, by decoupling the system into two fractional-order differential equations, so that the first equation of order $(\delta+\gamma)$, while the second equation depends on the solution for the first equation, we have solved the under consideration system, where $0<\delta,\gamma\leq 1$. With the help of using the Adomian decomposition method (ADM), we obtain the general solution. The efficiency of this method is verified by solving several numerical examples.

[0]Received September 4, 2022. Revised October 16, 2022. Accepted November 19, 2022.
[0]2020 Mathematics Subject Classification: 15A86, 26A33, 34A08, 34K37.
[0]Keywords: Fractional-order derivative operator, fractional-order integral operator, Adomian decomposition method.
[0]Corresponding author: S. Alshorm(alshormanshams@gmail.com).

1. Introduction

Many authors and researchers have investigated the fractional calculus operations as partial differential equations [4, 7, 10, 12], fractional integrodifferential equations, and dynamic systems using a variety of methods [13], including variational iteration method, spectral method, Adomian decomposition method [3, 16, 18], homotopy perturbation method [19], homotopy analysis method, and others [9, 15].

Kilicman and Al-Zhou investigated several procedures related to fractional calculus for dealing with some problems, including matrices. Then they extended the convolution product to the Riemann–Liouville fractional integral [14]. They also offered general systems of matrices of fractional partial differential equations for diagonal unknown matrices, and to demonstrate this, they presented a theory of non-homogeneous matrix of fractional partial differential equations with examples. For dealing with the numerical integration of fractional differential equations, Saadatmandi and Dehghan generalized the Legendre operational matrix to fractional differential equations of two types, linear and nonlinear, and used the Legendre series in conjunction with the Legendre operational matrix of fractional derivatives in the Caputo sense [17].

Bhrawy and Alofi introduced a newly shifted Chebyshev operational matrix of fractional integration in the Riemann–Liouville sense. They obtained a satisfactory result with a small number of shifted Chebyshev polynomials. Agarwal et al. discovered fractional integration formulas for generalized multi-index systems of the Marichev–Saigo–Maeda type [1]. Also, a new nonlinear duffing system with sequential fractional derivatives was developed in 2021 in [11]. Bhrawy et al. presented operational matrices for several polynomials on bounded domains, including the Legendre, Chebyshev, Jacobi, and Bernstein polynomials, and used them with various spectral techniques to solve the aforementioned equations on bounded domains. They also used numerical approaches for solving fractional differential equations on a semi-infinite interval with operational matrices for orthogonal Laguerre and modified generalized Laguerre polynomials.

To get the exact solution of the matrix fractional differential equation for diagonal unknown matrices in the Caputo sense, Al-Zuhiri et al. used vector extraction operations and the Hadamard product. Agarwal and Choi developed image formulas for several fractional integral operators using various forms of generalized hypergeometric functions, which are primarily defined in terms of Hadamard product [2]. It is commonly known that most fractional differential equations lack accurate analytic solutions, necessitating approximation and numerical techniques. Bhrawy published numerical solutions for fractional differential equations based on finite difference methods and numerous spectral algorithms. For one- and two-dimensional non-linear fractional subdiffusion equations, Bhrawy adopted an operational matrix version of the collocation method. To illustrate approximate solutions of one and two-dimensional instances, they used both double and triple-shifted Jacobi polynomials as basis functions. The Jacobi operational matrices express the space-time fractional derivatives in the highlighted problems. It aids in investigating spectral collocation techniques for temporal and spatial discretizations. For $(1+1)$- and $(2+1)$-fractional percolation equations, the first step relies on the shifted Legendre Gauss–Lobatto collocation method for spatial discretization, and the second step is to propose the shifted Chebyshev Gauss–Radau collocation scheme for temporal discretization, to reduce such a system to a system of algebraic equations for more see [8, 22].

2. System formulation

In this section, we list some basic definitions and theorems that we mainly use.

Definition 2.1. ([6]) The fractional integral of a function $f(Y,t) \in C_\delta$ $(\delta \leqslant -1)$ and of order $\mu > 0$ initially defined by Riemann-Liouville, is presented as:

$$J^\mu f(Y,t) = \frac{1}{\Gamma(\mu)} \int_0^t f(Y,\tau)(t-\tau)^{\mu-1}\, d\tau \ , t > 0, \mu > 0,$$

where $Y = (x_1, x_2, x_3, ..., x_n)$ is a vector of n variables.

Some of the properties of Riemann-Liouville:

(1) $J^0 f(Y,t) = f(Y,t)$.
(2) $J^\mu t^\gamma = \frac{\Gamma(\gamma+1)t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}, \gamma \geq -1$.
(3) $J^\mu J^\beta f(Y,t) = J^\beta J^\mu f(Y,t), \alpha, \beta \geq 0$.
(4) $J^\mu J^\beta f(Y,t) = J^{\mu+\beta} f(Y,t), \alpha, \beta \geq 0$.

Definition 2.2. For m is a smaller number greater than α, the Caputo time fractional derivative of order $\alpha > 0$ is defined as:

$$D_t^\alpha f(Y,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{-\alpha+n-1} \frac{\partial^m f(Y,\tau)}{\partial \tau^m}\, d\tau, \ m-1 < \alpha < m, \\ \frac{\partial^m f(Y,t)}{\partial t^m}, \ \alpha = m, \end{cases}$$

where $Y = (x_1, x_2, x_3, ..., x_n)$ is a vector of n variables and for $n-1 < \alpha \leq n, n \in \mathbf{N}$, $t > 0$, $f(Y,t)$ is a real valued-function while the parameter α is the order of the derivative.

Some of the characteristics of the Caputo's are listed below:

(1) $D_t^\alpha c = 0$, where c is constant.

(2)
$$D_t^{\alpha} t^{\rho} = \begin{cases} \frac{\Gamma(\rho+1)}{\Gamma(\rho-\alpha+1)} t^{\rho-\alpha}, & \rho > \alpha - 1, \\ 0, & \text{otherwise.} \end{cases}$$

(3) A linear operation is also true for Caputo's:
$$D_t^{\alpha}(\mu f(Y,t) + \omega k(Y,t)) = \mu D_t^{\alpha}(f(Y,t)) + \omega D_t^{\alpha}(k(t)),$$
where μ and ω are constant.

Also, we need to the following two of its basic properties, that is, if $n-1 < \alpha \leq n$, $n \in N$, then
$$D^{\alpha} J^{\alpha} f(x) = f(x), \;\; J^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{i=1}^{n} f^{i}(0^{+}) \frac{x^{i}}{i!}, \;\; x > 0.$$

Definition 2.3. The fractional derivative can be defined using the definition of the fractional integral. To this end, suppose that $v = n-u$, where $0 < v \leq 1$ and n is the smallest integral greater than u. Then, the fractional derivative of $f(x)$ of the order u is
$$D^{u} f(x) = D^{n}[D^{-v} f(x)].$$

Throughout this work, we will utilize the Adomian Decomposition Method (ADM) to solve a 2×2-system of linear incommensurate fractional-order differential equations. For this purpose, we will consider the general form of this system as follows:
$$\begin{aligned} D_*^{\delta} u(t) &= b_{11} u(t) + b_{12} v(t) + g_1(t), \\ D_*^{\gamma} v(t) &= b_{21} u(t) + b_{22} v(t) + g_2(t) \end{aligned} \tag{2.1}$$
with initial conditions:
$$u(0) = w_1, \;\; v(0) = w_2,$$
where g_1, g_2 are continuous functions, $U = [u\ v]^T$ is a vector-valued function of two variables u, v, and $b_{11}, b_{12}, b_{21}, b_{22}$ are real numbers, and where D_*^{δ}, D_*^{γ} are the Caputo operators of order δ, γ, respectively. We will attempt in this work to find an approximate solution for this system using the ADM. To this aim, we need first to convert system (2.1) to other equivalent forms, which can be illustrated with the help of using the next lemmas.

Lemma 2.4. *Suppose that $g_1(t) = g_2(t) = 0$ in system (2.1), that is, we have:*
$$\begin{aligned} D_*^{\delta} u(t) &= b_{11} u(t) + b_{12} v(t), \\ D_*^{\gamma} v(t) &= b_{21} u(t) + b_{22} v(t). \end{aligned} \tag{2.2}$$
Then, this system can be converted into the following two equations:
$$u(t) = \frac{1}{b_{21}} \left(D_*^{\gamma} v(t) - b_{22} v(t) \right) \tag{2.3}$$

and

$$D_*^{\delta+\gamma}v(t) = b_{11}D_*^{\gamma}v(t) + b_{22}D_*^{\delta}v(t) - det(B)v(t), \tag{2.4}$$

where

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Proof. By the second equation of system (2.2), we can directly obtain:

$$u(t) = \frac{1}{b_{21}}\left(D_*^{\gamma}v(t) - b_{22}v(t)\right). \tag{2.5}$$

Now, by operating D_*^{δ} to both sides of the second equation of system (2.2), we can have:

$$D_*^{\delta+\gamma}v(t) = b_{21}D_*^{\delta}u(t) + b_{22}D_*^{\delta}v(t). \tag{2.6}$$

Setting the first equation of system (2.2) in (2.6) yields the following assertion:

$$D_*^{\delta+\gamma}v(t) = b_{21}\left(b_{11}u(t) + b_{12}v(t)\right) + b_{22}D_*^{\delta}v(t),$$

that is,

$$D_*^{\delta+\gamma}v(t) = b_{11}\left(b_{21}u(t)\right) + b_{21}b_{12}v(t) + b_{22}D_*^{\delta}v(t). \tag{2.7}$$

By substituting (2.5) in (2.7), we obtain:

$$D_*^{\delta+\gamma}v(t) = b_{11}\left(D_*^{\gamma}v(t) - b_{22}v(t)\right) + b_{21}b_{12}v(t) + b_{22}D_*^{\delta}v(t)$$

or

$$D_*^{\delta+\gamma}v(t) = b_{11}D_*^{\gamma}v(t) - \left(b_{11}b_{22} - b_{21}b_{12}\right)v(t) + b_{22}D_*^{\delta}v(t),$$

which immediately gives the desired result. □

Corollary 2.5. *If* $\delta = \gamma$ *in the system* (2.2), *then such system will be converted into the following two equations:*

$$u(t) = \frac{1}{b_{21}}\left(D_*^{\delta}v(t) - b_{22}v(t)\right) \tag{2.8}$$

and

$$D_*^{2\delta}v(t) = tr(A)D_*^{\delta}v(t) - det(B)v(t). \tag{2.9}$$

Lemma 2.6. *Suppose that* $g_1(t), g_2(t) \neq 0$ *in system* (2.1), *then this system can be converted into the following two equations:*

$$u(t) = \frac{1}{b_{21}}\left(D_*^{\gamma}v(t) - b_{22}v(t) - g_2(t)\right) \tag{2.10}$$

and

$$D_*^{\delta+\gamma}v(t) = b_{11}D_*^{\gamma}v(t) + b_{22}D_*^{\delta}v(t) - det(B)v(t) + D_*^{\delta}g_2(t) + b_{12}g_1(t) - b_{11}g_2(t), \tag{2.11}$$

where

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Proof. We can get (2.10) directly from the second equation of the system (2.1). Now, if one operates D_*^δ to both sides of the second equation of system (2.1), we obtain:

$$D_*^{\delta+\gamma} v(t) = b_{21} D_*^\delta u(t) + b_{22} D_*^\delta v(t) + D_*^\delta g_2(t). \tag{2.12}$$

By substituting the first equation of system (2.1) into (2.12), we get:

$$D_*^{\delta+\gamma} v(t) = b_{21} \left(b_{11} u(t) + b_{12} v(t) + g_1(t)\right) + b_{22} D_*^\delta v(t) + D_*^\delta g_2(t),$$

that is,

$$D_*^{\delta+\gamma} v(t) = b_{11} \left(b_{21} u(t)\right) + b_{21} b_{12} v(t) + b_{21} g_1(t) + b_{22} D_*^\delta v(t) + D_*^\delta g_2(t). \tag{2.13}$$

In this regard, if one substitute (2.10) in (2.13), we get:

$$\begin{aligned} D_*^{\delta+\gamma} v(t) = b_{11} &\left(D_*^\delta v(t) - b_{22} v(t) - g_2(t)\right) \\ &+ b_{21} b_{12} v(t) + b_{22} D_*^\delta v(t) + D_*^\delta g_2(t) + b_{21} g_1(t). \end{aligned}$$

This yields:

$$\begin{aligned} D_*^{\delta+\gamma} v(t) = b_{11} &D_*^\gamma v(t) - \left(b_{11} b_{22} - b_{21} b_{12}\right) v(t) \\ &+ b_{22} D_*^\delta v(t) + D_*^\delta g_2(t) + b_{21} g_1(t) - b_{11} g_2(t), \end{aligned}$$

which immediately gives the desired result. □

Corollary 2.7. *If $\delta = \gamma$ in system (2.1), then this system will be equivalent to the following two equations:*

$$u(t) = \frac{1}{b_{21}} \left(D_*^\delta v(t) - b_{22} v(t) - g_2(t)\right) \tag{2.14}$$

and

$$D_*^{2\delta} v(t) = tr(B) D_*^\delta v(t) - det(B) v(t) + D_*^\delta g_2(t) + b_{21} g_1(t) - a_{11} g_2(t). \tag{2.15}$$

3. Main results

In this section, a new analytical scheme is developed to obtain the solution of 2×2-the system of linear homogeneous incommensurate fractional-order differential equations.

3.1. Dealing with the homogeneous incommensurate system by ADM.

In this part, we intend to use the ADM to solve the homogeneous system reported in (2.2). To carry out this task, we introduce next a new result that would illustrate the general solution of system (2.2).

Lemma 3.1. *The homogeneous system:*

$$\begin{aligned} D_*^\delta u(t) &= b_{11} u(t) + b_{12} v(t), \\ D_*^\gamma v(t) &= b_{21} u(t) + b_{22} v(t) \end{aligned} \tag{3.1}$$

with initial conditions:

$$u(0) = w_1, \; v(0) = w_2 \tag{3.2}$$

has the solution of the form:

$$\begin{aligned} v(t) = w_2 &+ \frac{(c - b_{22}w_2)t^{\gamma}}{\Gamma(\gamma+1)} + \frac{b_{11}(c - b_{22}w_2)t^{\gamma+\delta}}{\Gamma(\gamma+\delta+1)} - \frac{b_{11}^2 w_2 t^{2\delta}}{\Gamma(2\delta+1)} \\ &+ \frac{b_{22}w_2t^{\gamma}}{\Gamma(\gamma+1)} + \frac{b_{22}(c - b_{22}w_2)t^{2\gamma}}{\Gamma(2\gamma+1)} - \frac{b_{22}b_{11}w_2t^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} - \frac{det(B)w_2t^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} \\ &- \frac{det(B)(c - b_{22}w_2)t^{2\gamma+\delta}}{\Gamma(2\gamma+\delta+1)} + \frac{b_{11}det(B)w_2t^{2\delta+\gamma}}{\Gamma(2\delta+\gamma+1)} + \ldots \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} u(t) = \frac{1}{b_{21}}\Bigg(&\frac{t^{\delta}}{\Gamma(\delta+1)}(b_{11}(c - b_{22}w_2) - b_{22}b_{11}w_2 - det(B)w_2) \\ &+ \frac{b_{22}t^{\delta+\gamma}}{\Gamma(\delta+\Gamma+1)}(b_{22}b_{11}w_2 + det(B)w_2 - b_{11}(c - b_{22}w_2) - det(B)(c - b_{22}w_2)) \\ &+ \frac{b_{22}det(B)t^{2\gamma+\delta}}{\Gamma(2\gamma+\delta+1)}((c - b_{22}w_2) - b_{11}w_2) - \frac{b_{11}^2w_2t^{2\delta-\gamma}}{\Gamma(2\delta-\gamma+1)} - \frac{b_{22}^2w_2t^{\gamma}}{\Gamma(\gamma+1)} \\ &+ \frac{b_{11}w_2t^{2\delta}}{\Gamma(2\delta+1)}(b_{22}b_{11} + det(B)) - \frac{b_{22}^2(c - b_{22}w_2)t^{2\gamma}}{\Gamma(2\gamma+1)} + (c - b_{22}w_2) + \ldots\Bigg). \end{aligned} \tag{3.4}$$

Proof. In view of Lemma 2.4, system (2.2) can be converted to the following form:

$$u(t) = \frac{1}{b_{21}}\left(D_*^{\gamma}v(t) - b_{22}v(t)\right) \tag{3.5}$$

and

$$D_*^{\delta+\gamma}v(t) = b_{11}D_*^{\gamma}v(t) + b_{22}D_*^{\delta}v(t) - det(B)v(t). \tag{3.6}$$

By operating J_0^{δ} to both sides of (3.6), we obtain:

$$J_0^{\delta}D_*^{\delta+\gamma}v(t) = b_{11}J_0^{\delta}D_*^{\gamma}v(t) + b_{22}J_0^{\delta}D_*^{\delta}v(t) - det(B)J_0^{\delta}v(t),$$

Now, again by operating J_0^{γ} to the both sides and by letting $v'(0) = c$, we get:

$$v(t) = J_0^{\gamma}(c - b_{22}w_2) + b_{11}J_0^{\delta}J_0^{\gamma}D_*^{\gamma}v(t) + b_{22}J_0^{\gamma}v(t) - det(B)J_0^{\gamma+\delta}v(t) + v(0). \tag{3.7}$$

This immediately yields:

$$v(t) = w_2 + \frac{(c - b_{22}w_2)t^{\gamma}}{\Gamma(\gamma+1)} - \frac{b_{11}w_2t^{\delta}}{\Gamma(\delta+1)} + b_{11}J_0^{\delta}v(t) + b_{22}J^{\gamma}v(t) - det(B)J_0^{\gamma+\delta}v(t).$$

In the sense of ADM, we can assume $v(t) = \sum_{n=0}^{\infty} v_n(t)$ to obtain:

$$\sum_{n=0}^{\infty} v_n(t) = w_2 + \frac{(c - b_{22}w_2)t^{\gamma}}{\Gamma(\gamma+1)} - \frac{b_{11}w_2t^{\delta}}{\Gamma(\delta+1)} + b_{11}J_0^{\delta}\left(\sum_{n=0}^{\infty} v_n(t)\right) + b_{22}J^{\gamma}\left(\sum_{n=0}^{\infty} v_n(t)\right) - det(B)J_0^{\gamma+\delta}\left(\sum_{n=0}^{\infty} v_n(t)\right).$$

This consequently implies:

$$v_0 = w_2 + \frac{(c - b_{22}w_2)t^{\gamma}}{\Gamma(\gamma+1)} - \frac{b_{11}w_2t^{\delta}}{\Gamma(\delta+1)} \tag{3.8}$$

and

$$v_n = b_{11}J_0^{\delta}v_{n-1}(t) + b_{22}J_0^{\gamma}v_{n-1}(t) - det(B)J_0^{\gamma+\delta}v_{n-1}(t),\ n \geq 1. \tag{3.9}$$

With the help of using the above relations, we can have:

$$v_1 = b_{11}J_0^{\delta}v_0(t) + b_{22}J^{\gamma}v_0(t) - det(B)J_0^{\gamma+\delta}v_0(t).$$

This means that

$$v_1 = \frac{b_{11}w_2t^{\delta}}{\Gamma(\delta+1)} + \frac{b_{11}(c - b_{22}w_2)t^{\gamma+\delta}}{\Gamma(\gamma+\delta+1)} - \frac{b_{11}^2w_2t^{2\delta}}{\Gamma(2\delta+1)} + \frac{b_{22}w_2t^{\gamma}}{\Gamma(\gamma+1)} + \frac{b_{22}(c - b_{22}w_2)t^{2\gamma}}{\Gamma(2\gamma+1)} - \frac{b_{22}b_{11}w_2t^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} - \frac{det(B)w_2t^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} - \frac{det(B)(c - b_{22}w_2)t^{2\gamma+\delta}}{\Gamma(2\gamma+\delta+1)} + \frac{b_{11}det(B)w_2t^{2\delta+\gamma}}{\Gamma(2\delta+\gamma+1)}.$$

Due to it is difficult to find the other terms $v_2, v_3, v_4, ...$ of the solution $v(t)$, we will suffice here to present the first two terms of such solution, that is,

$$v(t) = w_2 + \frac{(c - b_{22}w_2)t^{\gamma}}{\Gamma(\gamma+1)} + \frac{b_{11}(c - b_{22}w_2)t^{\gamma+\delta}}{\Gamma(\gamma+\delta+1)} - \frac{b_{11}^2w_2t^{2\delta}}{\Gamma(2\delta+1)} + \frac{b_{22}w_2t^{\gamma}}{\Gamma(\gamma+1)} + \frac{b_{22}(c - b_{22}w_2)t^{2\gamma}}{\Gamma(2\gamma+1)} - \frac{b_{22}b_{11}w_2t^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} - \frac{det(B)w_2t^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} - \frac{det(B)(c - b_{22}w_2)t^{2\gamma+\delta}}{\Gamma(2\gamma+\delta+1)} + \frac{b_{11}det(B)w_2t^{2\delta+\gamma}}{\Gamma(2\delta+\gamma+1)} + ...$$

Now, to obtain the solution $u(t)$ of the system (3.1), we use (3.5) as follows:

$$\begin{aligned} u(t) = \frac{1}{b_{21}}\Bigg(& \frac{t^{\delta}}{\Gamma(\delta+1)}(b_{11}(c-b_{22}w_2) - b_{22}b_{11}w_2 - det(B)w_2) \\ & + \frac{b_{22}t^{\delta+\gamma}}{\Gamma(\delta+\Gamma+1)}(b_{22}b_{11}w_2 + det(B)w_2 - b_{11}(c-b_{22}w_2) - det(B)(c-b_{22}w_2)) \\ & + \frac{b_{22}det(B)t^{2\gamma+\delta}}{\Gamma(2\gamma+\delta+1)}((c-b_{22}w_2)-b_{11}w_2) - \frac{b_{11}^2 w_2 t^{2\delta-\gamma}}{\Gamma(2\delta-\gamma+1)} - \frac{b_{22}^2 w_2 t^{\gamma}}{\Gamma(\gamma+1)} \\ & + \frac{b_{11}w_2 t^{2\delta}}{\Gamma(2\delta+1)}(b_{22}b_{11} + det(B)) - \frac{b_{22}^2(c-b_{22}w_2)t^{2\gamma}}{\Gamma(2\gamma+1)} + (c-b_{22}w_2) + \ldots\Bigg). \end{aligned}$$

□

Remark 3.2. It should be noted that we have previously assumed in the proof of Lemma 2.6 that $v'(0) = c$, but we did not outline a certain expression of this assumption, which is considered very important to find the forms of $v(t)$ and $u(t)$. In this regard, we can confirm that this assumption has the following form:

$$c = v'(0) = b_{21}w_1 + b_{22}w_2.$$

To see this, we consider (3.5) by letting $u(0) = w_1$ to obtain:

$$w_1 = \frac{1}{b_{21}}\left(D_*^{\gamma}w_2 - b_{22}w_2\right),$$

which consequently yields the desired result.

3.2. **Dealing with the nonhomogeneous incommensurate system by ADM.** In what follows, we aim to implement the ADM to solve the nonhomogeneous system (2.1). To carry out this task, we propose another result that would illustrate the general solution of that system.

Lemma 3.3. *The nonhomogeneous incommensurate system:*

$$\begin{aligned} D_*^{\delta}u(t) &= b_{11}u(t) + b_{12}v(t) + g_1(t), \\ D_*^{\gamma}v(t) &= b_{21}u(t) + b_{22}v(t) + g_2(t) \end{aligned} \tag{3.10}$$

with initial conditions: $u(0) = c,\ v(0) = w_2$, *has the solution of the form:*

$$\begin{aligned}
v(t) &= w_2 + \frac{(c - b_{22}w_2 - g_2(0))t^{\gamma}}{\Gamma(\gamma+1)} + J_0^{\gamma} g_2(t) + b_{12}J_0^{\gamma+\delta} g_1(t) - b_{11}J_0^{\gamma+\delta} g_2(t) \\
&+ \frac{b_{11}(c - b_{22}w_2 - g_2(0))t^{\gamma+\delta}}{\Gamma(\gamma+\delta+1)} - \frac{b_{11}^2 w_2 t^{2\delta}}{\Gamma(2\delta+1)} + b_{11}J_0^{\gamma+\delta} g_2(t) - b_{11}^2 J_0^{\gamma+2\delta} g_2(t) \\
&+ b_{11}b_{12}J_0^{\gamma+2\delta} g_1(t) + \frac{b_{22}w_2 t^{\gamma}}{\Gamma(\gamma+1)} - \frac{b_{22}b_{11}w_2 t^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} + \frac{b_{22}(c - b_{22}w_2 - g_2(0))t^{2\gamma}}{\Gamma(2\gamma+1)} \\
&+ b_{22}J_0^{2\gamma} g_2(t) + b_{22}b_{12}J_0^{2\gamma+\delta} g_1(t) - b_{22}b_{11}J_0^{2\gamma+\delta} g_2(t) - \frac{det(B)w_2 t^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} \\
&- \frac{det(B)(c - b_{22}w_2 - g_2(0))t^{2\gamma+\delta}}{\Gamma(2\gamma+\delta+1)} + \frac{b_{11}det(B)w_2 t^{2\delta+\gamma}}{\Gamma(2\delta+\gamma+1)} - det(B)J_0^{2\gamma+\delta} g_2(t) \\
&- det(B)b_{12}J_0^{2\gamma+2\delta} g_1(t) + det(B)b_{11}J_0^{2\gamma+2\delta} g_2(t) + ...,
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
u(t) &= \frac{1}{b_{21}}\Bigg(\frac{t^{\delta}}{\Gamma(\delta+1)}\big(b_{11}(c - b_{22}w_2 - g_2(0)) - b_{22}b_{11}w_2 - det(B)w_2\big) \\
&+ \frac{t^{\delta+\gamma}}{\Gamma(\delta+\Gamma+1)}\big(b_{22}^2 b_{11}w_2 + det(B)b_{22}w_2 - b_{11}b_{22}(c - b_{22}w_2 - g_2(0)) \\
&- det(B)(c - b_{22}w_2 - g_2(0))\big) + \frac{b_{22}det(B)t^{2\gamma+\delta}}{\Gamma(2\gamma+\delta+1)}\big((c - b_{22}w_2 - g_2(0)) - b_{11}w_2\big) \\
&- \frac{b_{11}^2 w_2 t^{2\delta-\gamma}}{\Gamma(2\delta-\gamma+1)} - \frac{b_{22}^2(c - b_{22}w_2 - g_2(0))t^{2\gamma}}{\Gamma(2\gamma+1)} + (c - b_{22}w_2 - g_2(0)) \\
&- \frac{b_{22}^2 w_2 t^{\gamma}}{\Gamma(\gamma+1)} + \frac{b_{11}w_2 t^{2\gamma}}{\Gamma(2\delta+1)}\big(det(B) + b_{22}b_{11}\big) + D^{\gamma}\Big(J_0^{\gamma} g_2(t) + b_{12}J_0^{\gamma+\delta} g_1(t) \\
&- b_{11}J_0^{\gamma+\delta} g_2(t) + b_{11}J_0^{\gamma+\delta} g_2(t) + b_{11}b_{12}J_0^{\gamma+2\delta} g_1(t) - b_{11}^2 J_0^{\gamma+2\delta} g_2(t) \\
&+ b_{22}J_0^{2\gamma} g_2(t) + b_{22}b_{12}J_0^{2\gamma+\delta} g_1(t) - b_{22}b_{11}J_0^{2\gamma+\delta} g_2(t) \\
&- det(B)J_0^{2\gamma+\delta} g_2(t) - det(B)b_{12}J_0^{2\gamma+2\delta} g_1(t) + det(B)b_{11}J_0^{2\gamma+2\delta} g_2(t)\Big) \\
&- b_{22}\Big(J_0^{\gamma} g_2(t) + b_{12}J_0^{\gamma+\delta} g_1(t) - b_{11}J_0^{\gamma+\delta} g_2(t) + b_{11}J_0^{\gamma+\delta} g_2(t) \\
&+ b_{11}b_{12}J_0^{\gamma+2\delta} g_1(t) - b_{11}^2 J_0^{\gamma+2\delta} g_2(t) + b_{22}J_0^{2\gamma} g_2(t) + b_{22}b_{12}J_0^{2\gamma+\delta} g_1(t) \\
&- b_{22}b_{11}J_0^{2\gamma+\delta} g_2(t) - det(B)J_0^{2\gamma+\delta} g_2(t) - det(B)b_{12}J_0^{2\gamma+2\delta} g_1(t) \\
&+ det(B)b_{11}J_0^{2\gamma+2\delta} g_2(t)\Big) - g_2(t) + ...\Bigg).
\end{aligned} \tag{3.12}$$

Proof. By Lemma 2.4, system (2.1) can be converted into the following form:

$$u(t) = \frac{1}{b_{21}} \left(D_*^{\gamma} v(t) - b_{22} v(t) - g_2(t) \right) \tag{3.13}$$

and

$$D_*^{\delta+\gamma} v(t) = b_{11} D_*^{\gamma} v(t) + b_{22} D_*^{\delta} v(t) - det(B) v(t) + D_*^{\delta} g_2(t) + b_{12} g_1(t) - b_{11} g_2(t). \tag{3.14}$$

By operating J_0^{δ} to both sides of (3.14), we obtain

$$\begin{aligned} D_*^{\gamma} v(t) - v'(0) = b_{11} J_0^{\delta} D_*^{\gamma} v(t) + b_{22}(v(t) - v(0)) - det(B) J_0^{\delta} v(t) \\ + (g_2(t) - g_2(0)) + b_{12} J_0^{\delta} g_1(t) - b_{11} J_0^{\delta} g_2(t). \end{aligned}$$

By operating J_0^{γ} to both sides of the above equality and by letting $v'(0) = c$, we get

$$\begin{aligned} v(t) - v(0) = J_0^{\gamma}(c - b_{22} w_2 - g_2(0)) + b_{11} J_0^{\delta}(v(t) - v(0)) + b_{22} J_0^{\gamma} v(t) \\ - det(B) J_0^{\gamma+\delta} v(t) + J_0^{\gamma} g_2(t) + b_{12} J_0^{\gamma+\delta} g_1(t) - b_{11} J_0^{\gamma+\delta} g_2(t). \end{aligned}$$

This immediately yields

$$\begin{aligned} v(t) = w_2 - \frac{b_{11} w_2 t^{\delta}}{\Gamma(\delta+1)} + \frac{(c - b_{22} w_2 - g_2(0)) t^{\gamma}}{\Gamma(\gamma+1)} + b_{11} J_0^{\delta} v(t) + b_{22} J_0^{\gamma} v(t) \\ - det(B) J_0^{\gamma+\delta} v(t) + J_0^{\gamma} g_2(t) + b_{12} J_0^{\gamma+\delta} g_1(t) - b_{11} J_0^{\gamma+\delta} g_2(t). \end{aligned}$$

In the sense of ADM, we assume $v(t) = \sum_{n=0}^{\infty} v_n(t)$ to obtain

$$\begin{aligned} \sum_{n=0}^{\infty} v_n(t) = w_2 - \frac{b_{11} w_2 t^{\delta}}{\Gamma(\delta+1)} + \frac{(c - b_{22} w_2 - g_2(0)) t^{\gamma}}{\Gamma(\gamma+1)} \\ + b_{11} J_0^{\delta} \left(\sum_{n=0}^{\infty} v_n(t) \right) + b_{22} J_0^{\gamma} \left(\sum_{n=0}^{\infty} v_n(t) \right) \\ - det(B) J_0^{\gamma+\delta} \left(\sum_{n=0}^{\infty} v_n(t) \right) \\ + J_0^{\gamma} g_2(t) + b_{12} J_0^{\gamma+\delta} g_1(t) - b_{11} J_0^{\gamma+\delta} g_2(t). \end{aligned}$$

This consequently implies

$$\begin{aligned} v_0 = w_2 - \frac{b_{11} w_2 t^{\delta}}{\Gamma(\delta+1)} + \frac{(c - b_{22} w_2 - g_2(0)) t^{\gamma}}{\Gamma(\gamma+1)} + J_0^{\gamma} g_2(t) \\ + b_{12} J_0^{\gamma+\delta} g_1(t) - b_{11} J_0^{\gamma+\delta} g_2(t) \end{aligned} \tag{3.15}$$

and

$$v_n = b_{11} J_0^{\delta} v_{n-1}(t) + b_{22} J_0^{\gamma} v_{n-1}(t) - det(B) J_0^{\gamma+\delta} v_{n-1}(t),\ n \geq 1. \tag{3.16}$$

With the help of using the above relations, we can have

$$v_1 = b_{11} J_0^{\delta} v_0(t) + b_{22} J^{\gamma} v_0(t) - det(B) J_0^{\gamma+\delta} v_0(t).$$

This means that

$$\begin{aligned} v_1 = & \frac{b_{11} w_2 t^{\delta}}{\Gamma(\delta+1)} + \frac{b_{11}(c - b_{22} w_2 - g_2(0)) t^{\gamma+\delta}}{\Gamma(\gamma+\delta+1)} - \frac{b_{11}^2 w_2 t^{2\delta}}{\Gamma(2\delta+1)} \\ & + b_{11} J_0^{\gamma+\delta} g_2(t) + b_{11} b_{12} J_0^{\gamma+2\delta} g_1(t) - b_{11}^2 J_0^{\gamma+2\delta} g_2(t) \\ & + \frac{b_{22} w_2 t^{\gamma}}{\Gamma(\gamma+1)} + \frac{b_{22}(c - b_{22} w_2 - g_2(0)) t^{2\gamma}}{\Gamma(2\gamma+1)} - \frac{b_{22} b_{11} w_2 t^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} \\ & + b_{22} J_0^{2\gamma} g_2(t) + b_{22} b_{12} J_0^{2\gamma+\delta} g_1(t) - b_{22} b_{11} J_0^{2\gamma+\delta} g_2(t) \\ & - \frac{det(B) w_2 t^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} - \frac{det(B)(c - b_{22} w_2 - g_2(0)) t^{2\gamma+\delta}}{\Gamma(2\gamma+\delta+1)} + \frac{b_{11} det(B) w_2 t^{2\delta+\gamma}}{\Gamma(2\delta+\gamma+1)} \\ & - det(B) J_0^{2\gamma+\delta} g_2(t) - det(B) b_{12} J_0^{2\gamma+2\delta} g_1(t) + det(B) b_{11} J_0^{2\gamma+2\delta} g_2(t). \end{aligned}$$

In a similar manner, we can find $v_2, v_3, v_4, \cdots$. In general, due to $v = \sum_{n=0}^{\infty} v_n$ then we can write here the first few terms of the above solution as follows:

$$\begin{aligned} v(t) = & w_2 + \frac{(c - b_{22} w_2 - g_2(0)) t^{\gamma}}{\Gamma(\gamma+1)} + J_0^{\gamma} g_2(t) + b_{12} J_0^{\gamma+\delta} g_1(t) - b_{11} J_0^{\gamma+\delta} g_2(t) \\ & + \frac{b_{11}(c - b_{22} w_2 - g_2(0)) t^{\gamma+\delta}}{\Gamma(\gamma+\delta+1)} - \frac{b_{11}^2 w_2 t^{2\delta}}{\Gamma(2\delta+1)} + b_{11} J_0^{\gamma+\delta} g_2(t) \\ & + b_{11} b_{12} J_0^{\gamma+2\delta} g_1(t) - b_{11}^2 J_0^{\gamma+2\delta} g_2(t) + \frac{b_{22} w_2 t^{\gamma}}{\Gamma(\gamma+1)} \\ & + \frac{b_{22}(c - b_{22} w_2 - g_2(0)) t^{2\gamma}}{\Gamma(2\gamma+1)} - \frac{b_{22} b_{11} w_2 t^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} \\ & + b_{22} J_0^{2\gamma} g_2(t) + b_{22} b_{12} J_0^{2\gamma+\delta} g_1(t) - b_{22} b_{11} J_0^{2\gamma+\delta} g_2(t) \\ & - \frac{det(B) w_2 t^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} - \frac{det(B)(c - b_{22} w_2 - g_2(0)) t^{2\gamma+\delta}}{\Gamma(2\gamma+\delta+1)} + \frac{b_{11} det(B) w_2 t^{2\delta+\gamma}}{\Gamma(2\delta+\gamma+1)} \\ & - det(B) J_0^{2\gamma+\delta} g_2(t) - det(B) b_{12} J_0^{2\gamma+2\delta} g_1(t) + det(B) b_{11} J_0^{2\gamma+2\delta} g_2(t) + \cdots . \end{aligned}$$

In order to get $u(t)$, we use (3.13) as follows:

$$\begin{aligned}
u(t) = \frac{1}{b_{21}}\Bigg(& \frac{t^{\delta}}{\Gamma(\delta+1)}\big(b_{11}(c - b_{22}w_2 - g_2(0)) - b_{22}b_{11}w_2 - det(B)w_2\big) \\
& + \frac{t^{\delta+\gamma}}{\Gamma(\delta+\Gamma+1)}\big(b_{22}^2 b_{11}w_2 + det(B)b_{22}w_2 - b_{11}b_{22}(c - b_{22}w_2 - g_2(0)) \\
& - det(B)(c - b_{22}w_2 - g_2(0))\big) \\
& + \frac{b_{22}det(B)t^{2\gamma+\delta}}{\Gamma(2\gamma+\delta+1)}\big((c - b_{22}w_2 - g_2(0)) - b_{11}w_2\big) \\
& - \frac{b_{11}^2 w_2 t^{2\delta-\gamma}}{\Gamma(2\delta-\gamma+1)} - \frac{b_{22}^2(c - b_{22}w_2 - g_2(0))t^{2\gamma}}{\Gamma(2\gamma+1)} + (c - b_{22}w_2 - g_2(0)) \\
& - \frac{b_{22}^2 w_2 t^{\gamma}}{\Gamma(\gamma+1)} + \frac{b_{11}w_2 t^{2\gamma}}{\Gamma(2\delta+1)}\big(det(B) + b_{22}b_{11}\big) \\
& + D^{\gamma}\Big(J_0^{\gamma}g_2(t) + b_{12}J_0^{\gamma+\delta}g_1(t) - b_{11}J_0^{\gamma+\delta}g_2(t) + b_{11}J_0^{\gamma+\delta}g_2(t) \\
& + b_{11}b_{12}J_0^{\gamma+2\delta}g_1(t) - b_{11}^2 J_0^{\gamma+2\delta}g_2(t) + b_{22}J_0^{2\gamma}g_2(t) \\
& + b_{22}b_{12}J_0^{2\gamma+\delta}g_1(t) - b_{22}b_{11}J_0^{2\gamma+\delta}g_2(t) \\
& - det(B)J_0^{2\gamma+\delta}g_2(t) - det(B)b_{12}J_0^{2\gamma+2\delta}g_1(t) + det(B)b_{11}J_0^{2\gamma+2\delta}g_2(t)\Big) \\
& - b_{22}\Big(J_0^{\gamma}g_2(t) + b_{12}J_0^{\gamma+\delta}g_1(t) - b_{11}J_0^{\gamma+\delta}g_2(t) + b_{11}J_0^{\gamma+\delta}g_2(t) \\
& + b_{11}b_{12}J_0^{\gamma+2\delta}g_1(t) - b_{11}^2 J_0^{\gamma+2\delta}g_2(t) + b_{22}J_0^{2\gamma}g_2(t) \\
& + b_{22}b_{12}J_0^{2\gamma+\delta}g_1(t) - b_{22}b_{11}J_0^{2\gamma+\delta}g_2(t) - det(B)J_0^{2\gamma+\delta}g_2(t) \\
& - det(B)b_{12}J_0^{2\gamma+2\delta}g_1(t) + det(B)b_{11}J_0^{2\gamma+2\delta}g_2(t)\Big) - g_2(t) + \cdots \Bigg).
\end{aligned}$$

□

Remark 3.4. It should be noted that we have previously assumed in the proof of Lemma 2.6 that $v'(0) = c$, but we have not yet determined the expression of this constant. In this regard, we can confirm the following:

$$c = v'(0) = b_{21}w_1 + b_{22}w_2 + g_2(0).$$

To see this, we consider (3.13) by letting $u(0) = w_1$ to obtain

$$w_1 = \frac{1}{b_{21}}\left(D_*^{\gamma}w_2 - b_{22}w_2 - g_2(0)\right),$$

which leads to the desired result.

3.3. **Dealing with commensurate systems.** In this subsection, we will present two significant results that address the commensurate case of the homogenuous system (2.2) and the nonhomogenuous system (2.1), respectively. In other words, we will assume that $\delta = \gamma$ in what follows to obtain two further results.

Lemma 3.5. *If $\delta = \gamma$ in system (2.2), then the solution of such system will be as*

$$\begin{aligned} v(t) &= w_2 + \frac{(c - b_{22}w_2)t^{\delta}}{\Gamma(\delta+1)} + \frac{b_{11}(c - b_{22}w_2)t^{2\delta}}{\Gamma(2\delta+1)} - \frac{b_{11}^2 w_2 t^{2\delta}}{\Gamma(2\delta+1)} \\ &\quad + \frac{b_{22}w_2t^{\delta}}{\Gamma(\delta+1)} + \frac{b_{22}(c - b_{22}w_2)t^{2\delta}}{\Gamma(2\delta+1)} - \frac{b_{22}b_{11}w_2t^{2\delta}}{\Gamma(2\delta+1)} \\ &\quad - \frac{det(B)w_2t^{2\delta}}{\Gamma(2\delta+1)} - \frac{det(B)(c - b_{22}w_2)t^{3\delta}}{\Gamma(3\delta+1)} + \frac{b_{11}det(B)w_2t^{3\delta}}{\Gamma(3\delta+1)} + \cdots \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} u(t) = \frac{1}{b_{21}}\Bigg(& \frac{t^{\delta}}{\Gamma(\delta+1)}(b_{11}(c - b_{22}w_2) - b_{22}b_{11}w_2 - det(B)w_2) \\ & + \frac{b_{22}t^{2\delta}}{\Gamma(2\delta+1)}(b_{22}b_{11}w_2 + det(B)w_2 - b_{11}(c - b_{22}w_2) - det(B)(c - b_{22}w_2)) \\ & + \frac{b_{22}det(B)t^{3\delta}}{\Gamma(3\delta+1)}((c - b_{22}w_2) - b_{11}w_2) - \frac{b_{11}^2w_2t^{\delta}}{\Gamma(\delta+1)} - \frac{b_{22}^2w_2t^{\delta}}{\Gamma(\delta+1)} \\ & + \frac{b_{11}w_2t^{2\delta}}{\Gamma(2\delta+1)}(b_{22}b_{11} + det(B)) - \frac{b_{22}^2(c - b_{22}w_2)t^{2\delta}}{\Gamma(2\delta+1)} + (c - b_{22}w_2) + \cdots \Bigg). \end{aligned} \tag{3.18}$$

Lemma 3.6. *If $\delta = \gamma$ in system (2.1), then the solution of such system will be as*

$$\begin{aligned} v(t) &= w_2 + \frac{(c - b_{22}w_2 - g_2(0))t^{\delta}}{\Gamma(\delta+1)} + J_0^{\delta}g_2(t) + b_{12}J_0^{2\delta}g_1(t) - b_{11}J_0^{2\delta}g_2(t) \\ &\quad + \frac{b_{11}(c - b_{22}w_2 - g_2(0))t^{2\delta}}{\Gamma(2\delta+1)} - \frac{b_{11}^2w_2t^{2\delta}}{\Gamma(2\delta+1)} + b_{11}J_0^{2\delta}g_2(t) \\ &\quad + b_{11}b_{12}J_0^{3\delta}g_1(t) - b_{11}^2J_0^{3\delta}g_2(t) + \frac{b_{22}w_2t^{\delta}}{\Gamma(\delta+1)} + \frac{b_{22}(c - b_{22}w_2 - g_2(0))t^{2\delta}}{\Gamma(2\delta+1)} \\ &\quad - \frac{b_{22}b_{11}w_2t^{2\delta}}{\Gamma(2\delta+1)} + b_{22}J_0^{2\delta}g_2(t) + b_{22}b_{12}J_0^{3\delta}g_1(t) - b_{22}b_{11}J_0^{3\delta}g_2(t) \end{aligned}$$

$$
\begin{aligned}
&-\frac{det(B)w_2t^{2\delta}}{\Gamma(2\delta+1)}-\frac{det(B)(c-b_{22}w_2-g_2(0))t^{3\delta}}{\Gamma(3\delta+1)}+\frac{b_{11}det(B)w_2t^{3\delta}}{\Gamma(3\delta+1)} \\
&-det(B)J_0^{3\delta}g_2(t)-det(B)b_{12}J_0^{4\delta}g_1(t)+det(B)b_{11}J_0^{4\delta}g_2(t)+...
\end{aligned}
\tag{3.19}
$$

and

$$
\begin{aligned}
u(t)=\frac{1}{b_{21}}\Bigg(&\frac{t^{\delta}}{\Gamma(\delta+1)}\big(b_{11}(c-b_{22}w_2-g_2(0))-b_{22}b_{11}w_2-det(B)w_2\big)\\
&+\frac{t^{2\delta}}{\Gamma(2\delta+1)}\big(b_{22}^2b_{11}w_2+det(B)b_{22}w_2-b_{11}b_{22}(c-b_{22}w_2-g_2(0))\\
&-det(B)(c-b_{22}w_2-g_2(0))\big)+\frac{b_{22}det(B)t^{3\delta}}{\Gamma(3\delta+1)}\big((c-b_{22}w_2-g_2(0))-b_{11}w_2\big)\\
&-\frac{b_{11}^2w_2t^{\delta}}{\Gamma(\delta+1)}-\frac{b_{22}^2(c-b_{22}w_2-g_2(0))t^{2\delta}}{\Gamma(2\delta+1)}+(c-b_{22}w_2-g_2(0))\\
&-\frac{b_{22}^2w_2t^{\delta}}{\Gamma(\delta+1)}+\frac{b_{11}w_2t^{2\delta}}{\Gamma(2\delta+1)}\big(det(B)+b_{22}b_{11}\big)\\
&+D^{\delta}\Big(J_0^{\delta}g_2(t)+b_{12}J_0^{2\delta}g_1(t)-b_{11}J_0^{2\delta}g_2(t)+b_{11}J_0^{2\delta}g_2(t)+b_{11}b_{12}J_0^{3\delta}g_1(t)\\
&-b_{11}^2J_0^{3\delta}g_2(t)+b_{22}J_0^{2\delta}g_2(t)+b_{22}b_{12}J_0^{3\delta}g_1(t)-b_{22}b_{11}J_0^{3\delta}g_2(t)\\
&-det(B)J_0^{3\delta}g_2(t)-det(B)b_{12}J_0^{4\delta}g_1(t)+det(B)b_{11}J_0^{4\delta}g_2(t)\Big)\\
&-b_{22}\Big(J_0^{\delta}g_2(t)+b_{12}J_0^{2\delta}g_1(t)-b_{11}J_0^{2\delta}g_2(t)+b_{11}J_0^{2\delta}g_2(t)+b_{11}b_{12}J_0^{3\delta}g_1(t)\\
&-b_{11}^2J_0^{3\delta}g_2(t)+b_{22}J_0^{2\delta}g_2(t)+b_{22}b_{12}J_0^{3\delta}g_1(t)-b_{22}b_{11}J_0^{3\delta}g_2(t)\\
&-det(B)J_0^{3\delta}g_2(t)-det(B)b_{12}J_0^{4\delta}g_1(t)+det(B)b_{11}J_0^{4\delta}g_2(t)\Big)-g_2(t)+\cdots\Bigg).
\end{aligned}
\tag{3.20}
$$

4. Illustrative examples

In this section, we will introduce some illustrative examples of how we can apply our proposed scheme to find approximate solutions for some given linear systems of fractional differential equations. This would confirm the validity of our proposed scheme.

Example 4.1. Consider the following fractional-order system:

$$
\begin{aligned}
D_*^{\delta}u(t)&=u(t)+v(t),\\
D_*^{\gamma}v(t)&=-u(t)+v(t)
\end{aligned}
\tag{4.1}
$$

with initial conditions:

$$u(0)=0, v(0)=1.$$

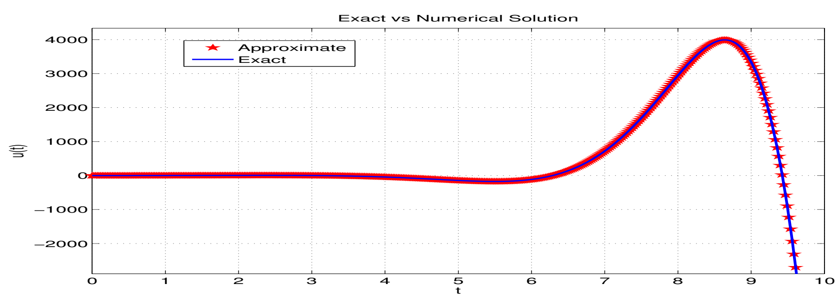
Solution: Note that this system has an exact solution when $\delta = \gamma = 1$ in the form $u(t) = e^t \sin t$, $v(t) = e^t \cos t$. To solve this system using our proposed scheme, we should observe that

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

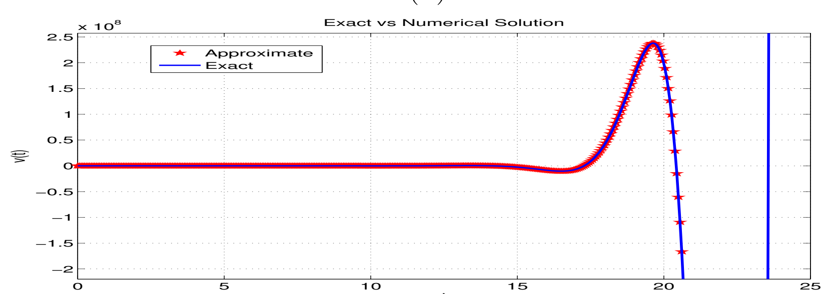
with $tr(B) = 2$ and $det(B) = 2$. By using the two formulas (3.3) and (3.4) provided in Lemma 3.1, we can respectively outline the first few terms of our approximate solutions as follows:

$$v(t) = 1 + t - 2t^2 + \frac{t^3}{3} + \cdots \text{ and } u(t) = 5t - 3t^2 + t^3 + \cdots .$$

To verify the above solution $(u(t), v(t))$ of system (4.1), we plot Figure 1(A-B), which are generated with the help of using a prepared MATLAB code. For more clarification, we also plot Figure 2(A-B), which represent the absolute value of the errors gained from $u(t)$ and $v(t)$, respectively.

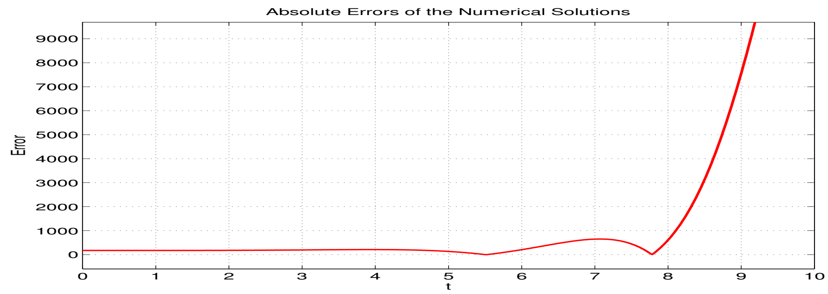


(A)

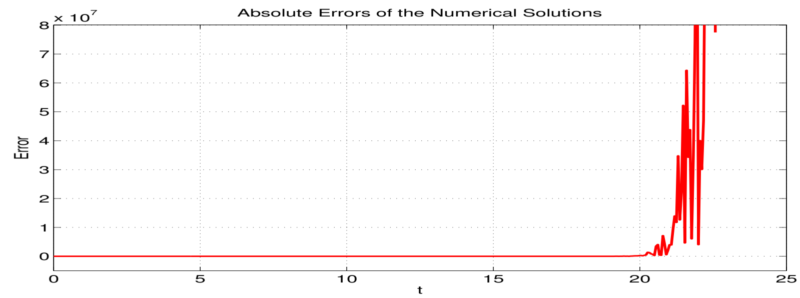


(B)

FIGURE 1. **Comparison results between the exact solutions and semi-analytical solutions (A) $u(t)$, (B) $v(t)$.**



(A)



(B)

FIGURE 2. **Absolute errors related to (A) $u(t)$, (B) $v(t)$.**

In light of the previous simulations, it can be obviously noted that our established approach is valid. In other words, all theoretical results derived previously are true. In this regard, it should be mentioned that our constructed solution $(u(t), v(t))$ can be further improved to completely coincide with the exact solution $(e^t \sin t, e^t \cos t)$ by easily adding some further new terms to their established solutions. This would certainly improve our graphical comparisons, but it needs, at the same time, more time to run.

Example 4.2. Consider the following fractional-order linear system:

$$\begin{aligned} D_*^{\delta} u(t) &= 3u(t) - 2v(t), \\ D_*^{\gamma} v(t) &= 2u(t) - 2v(t) \end{aligned} \tag{4.2}$$

with initial conditions:

$$u(0) = 1, v(0) = 5.$$

Solution: Herein, the exact solutions of the above system when $\delta = \gamma = 1$ is of the form:

$$u(t) = -2e^t + 3e^{-t} \text{ and } v(t) = -e^{2t} + 6e^{-4t}.$$

In this regard, we can obtain:

$$B = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

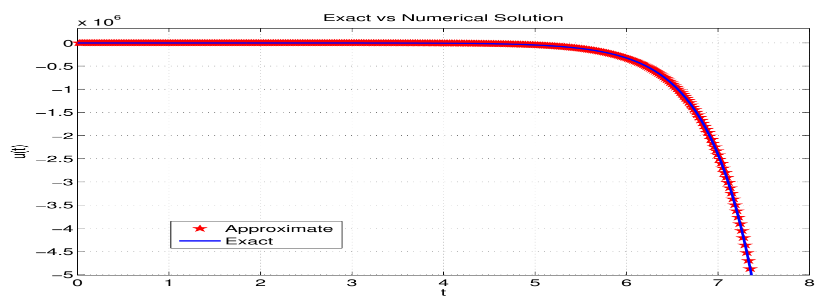
with $tr(B) = 1$ and $det(B) = -2$. Similarly to the previous example, we use the two formulas (3.3) and (3.4) provided in Lemma 3.1 This would respectively outline the first few terms of our approximate solutions as follows:

$$v(t) = 5 - 8t - \frac{3t^2}{2} - \frac{13t^3}{3} + \cdots$$

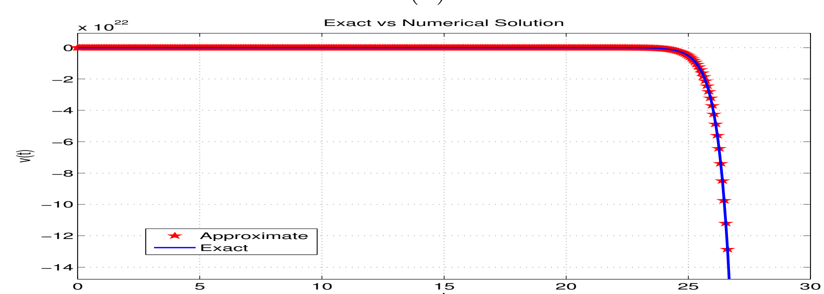
and

$$u(t) = 1 - \frac{19t}{2} - 11t^2 - \frac{26t^3}{3} + \cdots .$$

For more illustration, we plot the above solution $(u(t), v(t))$ in Figure 3(A-B) by using a prepared MATLAB code. This solution is compared with the exact one. In addition, we plot in Figure 4(A-B) the absolute value of the errors gained from $u(t)$ and $v(t)$, respectively.

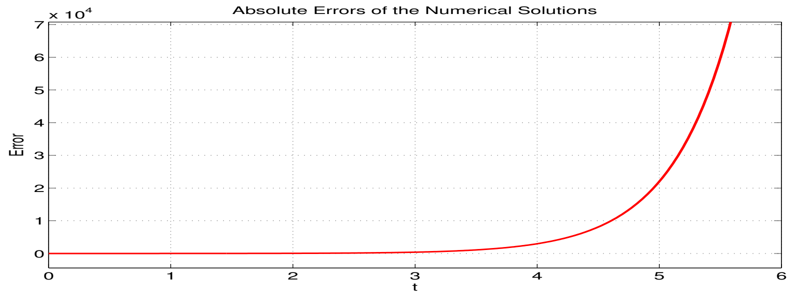


(A)

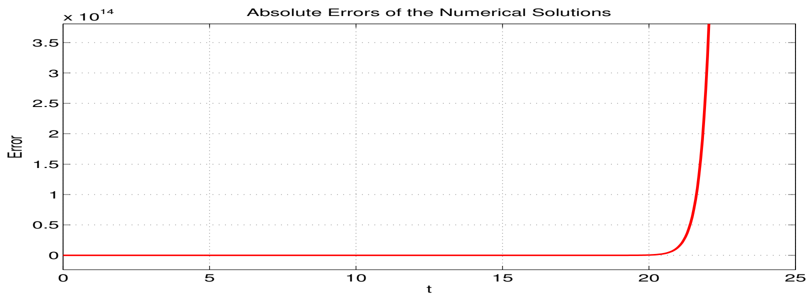


(B)

FIGURE 3. **Comparison results between the exact solutions and semi-analytical solutions (A)** $u(t)$**, (B)** $v(t)$**.**



(A)



(B)

FIGURE 4. **Absolute errors related to (A)** $u(t)$**, (B)** $v(t)$**.**

Clearly, based on the previous comparison, one can note that the numerical solution of system (4.2) established by our scheme coincides approximately with the exact solution of the same system. This confirms the validity of our constructed solution.

Example 4.3. Consider the following fractional-order linear system:

$$\begin{aligned} D_*^{\delta} u(t) &= u(t) + 3v(t) + 2e^t, \\ D_*^{\gamma} v(t) &= 3u(t) + v(t) + 2t \end{aligned} \tag{4.3}$$

with initial conditions:

$$u(0) = \frac{3}{16}, v(0) = \frac{-5}{16}.$$

Solution: This system has an exact solution when $\delta = \gamma = 1$ in the form:

$$u(t) = \frac{1}{3}(e^{4t} - e^{-2t}) + \frac{1}{16}(3 - 12t) \text{ and } v(t) = \frac{1}{3}(e^{-2t} + e^{4t} + 2e^t) + \frac{1}{16}(4t - 5).$$

To solve this system using our proposed scheme, we should first observe that:

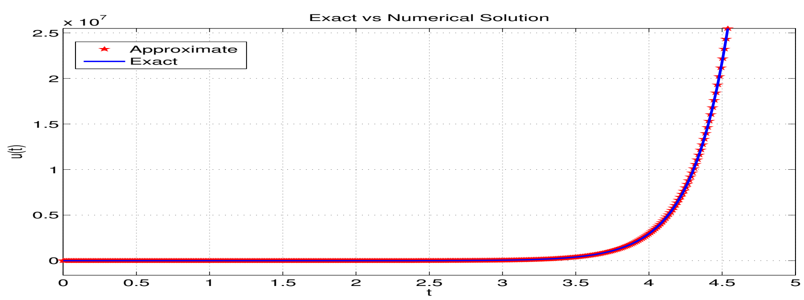
$$B = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

with $tr(B) = 2$ and $det(B) = -8$. Now, with the help of using (3.11) and (3.12), we can respectively outline the first few terms of our approximate solution as follows:

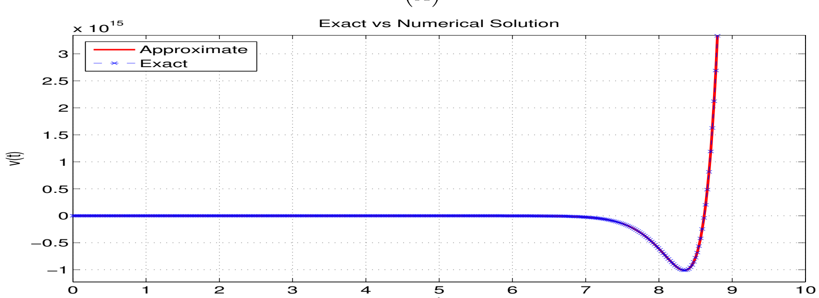
$$v(t) = \frac{-5}{16} + 66e - 66 + \frac{t}{4} - \frac{12t^2}{32} + \frac{29t^3}{48} + \cdots ,$$

$$u(t) = \frac{11}{24} + \frac{10t}{14} + \frac{13t^2}{24} - \frac{7t^3}{18} + \cdots .$$

To see what the above solution looks like compared to the exact solution, we plot Figure 5(A-B). In addition, we also plot in Figure 6(A-B) the absolute value of the errors generated by $u(t)$ and $v(t)$, respectively.

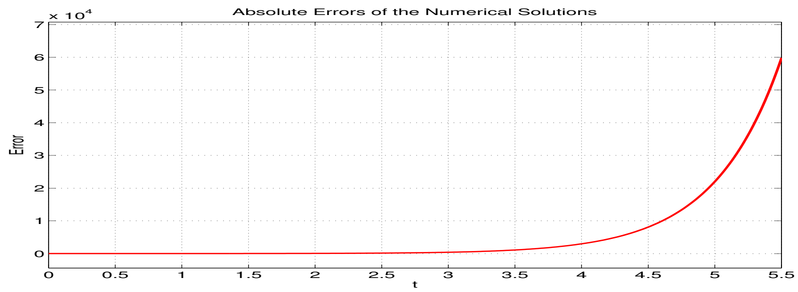


(A)

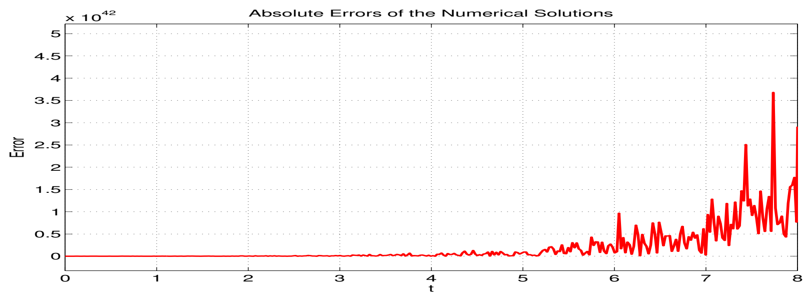


(B)

FIGURE 5. **Comparison results between the exact solutions and semi-analytical solutions (A) $u(t)$, (B) $v(t)$.**



(A)



(B)

FIGURE 6. **Absolute errors related to (A)** $u(t)$**, (B)** $v(t)$**.**

Given the simulations performed above, we can clearly see that the approximate solutions of system (4.3) gained from using our proposed approach completely coincide with the exact for the first few terms. Thus, it could be said that if we continue in the same manner, we will obtain an extension of this solution for more time which will be coincided with the remaining exact solution.

5. Conclusion

In this paper, a linear system of homogeneous commensurate/ incommensurate fractional-order differential equations have been studied by developing a new scheme. In particular, by decoupling the system into two fractional-order differential equations, so that the first equation of order $(\delta + \gamma)$, while the second equation depends on the solution for the first equation, we have solved the under consideration system, where $0 < \delta, \gamma \leq 1$. The Adomian decomposition method (ADM) has been used to obtain the general solution. The efficiency of this method has been verified by solving several numerical examples.

References

[1] P. Agarwal, J. Choi, and R.B. Paris, *Extended Riemann-Liouville fractional derivative operator and its applications*, J. Nonlinear Sci. Appl., **8**(5) (2015), 451-466.

[2] P. Agarwal and J. Choi, *Fractional calculus operators and their image formulas*, J. Korean Math. Soc., **53**(5) (2016), 1183-1210.

[3] R.B. Albadarneh, I.M. Batiha, A. Adwai, N. Tahat, and A.K. Alomari, *Numerical approach of Riemann-Liouville fractional derivative operator*, Int. J. Electr. Comput. Eng., **11**(6) (2021), 5367-5378.

[4] R.B. Albadarneh, I.M. Batiha and M. Zurigat, *Numerical solutions for linear fractional differential equations of order* $1 < \alpha < 2$ *using finite difference method*, Int. J. Math. Comput. Sci., **16**(1) (2016), 103-111.

[5] I.M. Batiha, S. Alshorm, A. Al-Husban, R. Saadeh, G. Gharib, and S. Momani, *The n-Point Composite Fractional Formula for Approximating Riemann-Liouville Integrator*, Symmetry, **15**(4) (2023), 938.

[6] I.M. Batiha, S. Alshorm, I.H. Jebril and M. Abu Hammad, *A Brief Review about Fractional Calculus*, Int. J. Open Probl. Compt. Math., **15**(4) (2022).

[7] I.M. Batiha, S. Alshorm, I. Jebril, A. Zraiqat, Z. Momani and S. Momani, *Modified 5-point fractional formula with Richardson extrapolation*, AIMS Math, **8** (2023), 9520-9534.

[8] I.M. Batiha, L. Ben Aoua, T.-E. Oussaeif, A. Ouannas, S. Alshorman, I.H. Jebril and S. Momani, *Common Fixed Point Theorem in Non-Archimedean Menger PM-Spaces Using CLR Property with Application to Functional Equations*, IAENG Int. J. Appl. Math., **53**(1) (2023), 1-9.

[9] I.M. Batiha, S. Alshorm, A. Ouannas, S. Momani, O. Ababneh and M. Albdareen. *Modified Three-Point Fractional Formulas with Richardson Extrapolation,* Mathematics., **10**(19) (2022), 3489.

[10] I.M. Batiha, R. El-Khazali, A. AlSaedi and S. Momani, *The general solution of singular fractional-order linear time-invariant continuous systems with regular pencils*, Entropy, **20**(6) (2018), 400.

[11] M. Bezziou, I. Jebril and Z. Dahmani, *A new nonlinear Duffing system with sequential fractional derivatives*, Chaos, Solitons & Fractals, **151** (2021), 111247.

[12] A. Carpinteri and F. Mainardi, *Fractals and fractional calculus in continuum mechanics*, Springer, Vienna, New York, (2014).

[13] V.S. Ertürk and S. Momani, *Solving systems of fractional differential equations using differential transform method*, J. Comput. Appl. Math., **215**(1) (2008), 142-151.

[14] G. Jumarie, *Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results*, Comput. Math. Appl., **51**(9-10) (2006), 1367-1376.

[15] A. Kilicman and Z.A.A. Zhour, *Kronecker operational matrices for fractional calculus and some applications*, Appl. Math. Comput., **187**(1) (2007), 250-265.

[16] S.S. Ray and R.K. Bera, *An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method*, Appl. Math. Comput., **167**(1) (2005), 561-571.

[17] A. Saadatmandi and M. Dehghan, *A new operational matrix for solving fractional-order differential equations*, Comput. Math. Appl., **59**(3) (2010), 1326-1336.

[18] I. Talbi, A. Ouannas, A.-A. Khennaoui, A. Berkane, I.M. Batiha, G. Grassi and V.-T. Pham, *Different dimensional fractional-order discrete chaotic systems based on the*

Caputo h-difference discrete operator: dynamics, control, and synchronization, Adv. Differ. Equ., **2020**(1) (2020), 1-15.

[19] M. Zurigat, S. Momani and A. Al-Masarwah, *Homotopy analysis method for solving fractional IVPs*, Commun. Nonlinear Sci. Numer. Simul., **15**(9) (2010), 2293-2302.

[20] M. Zurigat and S. Momani, *A new algorithm for calculating Adomian polynomials for nonlinear operators*, Commun. Nonlinear Sci. Numer. Simul., **17**(1) (2012), 2-7.

[21] M. Zurigat, S. Momani and A.-M. Wazwaz, *Solving a generalized Hirota-Satsuma coupled KdV equation using homotopy analysis method*, J. Math. Anal. Appl., **414**(2) (2014), 705-715.

[22] A. Zraiqat, S.K. Paikray and H. Dutta, *A certain class of deferred weighted statistical B-summability involving (p,q)-integers and analogous approximation theorems*, Filomat, **33**(5) (2019), 1425-1444.