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A FIXED POINT THEOREM FOR CARISTI-MAPPINGS ON ORDERED T-METRIC SPACES

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Abstract. In this paper, we give a fixed point theorem for Caristi-mapping on ordered T-metric spaces.

1. Introduction

It is well known that the Banach contraction principle is a fundamental result in fixed point theory. After this classical result, many authors have extended, generalized and improved this theorem in different ways. For example, ultra metric spaces [4], fuzzy metric spaces [1] and uniform spaces [3]. Recently in [2], the authors have introduced the concept of $T-metric$ spaces and utilized it to prove a common fixed point theorem for single valued oprators in terms of a ω-T-distance. This paper is begun by some definitions and lemmas from [2].

2. Preliminaries

In what follows, $\mathbb{N}$ is the set of all natural numbers and $\mathbb{R}^+$ is the set of all nonnegative real numbers.

A binary operation is a mapping $\diamond : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ which satisfy the following conditions:

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(i) $\diamond$ is associative and commutative,
(ii) $\diamond$ is continuous,
(iii) $a \diamond 0 = a$ for all $a \in [0, \infty)$,
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, \infty)$.

Let $a, b \in [0, \infty)$. Five typical examples of $\diamond$ are:
(a) $a \diamond_1 b = \max\{a, b\}$,
(b) $a \diamond_2 b = \sqrt{a^2 + b^2}$,
(c) $a \diamond_3 b = a + b$,
(d) $a \diamond_4 b = ab + a + b$,
(e) $a \diamond_5 b = (\sqrt{a} + \sqrt{b})^2$.

For $a, b \in \mathbb{R}^+$, straight forward calculations lead to the following relations among normed binary operations giving above

$$a \diamond_1 b \leq a \diamond_2 b \leq a \diamond_3 b \leq a \diamond_4 b$$

and

$$a \diamond_3 b \leq a \diamond_5 b.$$

Lemma 2.1. ([2]) *Let $f : [0, \infty) \longrightarrow [0, \infty)$ be a continuous, onto, and increasing map. If define $a \diamond b = f^{-1}(f(a) + f(b))$ for each $a, b \in [0, \infty)$, then $\diamond$ is a binary operation.*

Example 2.2. ([2]) *$f : [0, \infty) \longrightarrow [0, \infty)$ defined by $f(x) = e^x - 1$. Then f is a continuous, onto and increasing map and $a \diamond b = Ln(e^a + e^b - 1)$ for $a, b \in [0, \infty)$ is a binary operation.*

We have the following simple lemma.

Lemma 2.3. ([2]) (i) *If $r, r' \geq 0$, then $r \leq r \diamond r'$.*
(ii) *If $\delta \in (0, r)$, there exist $\delta' \in (0, r)$ such that $\delta' \diamond \delta < r$.*
(iii) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\delta \diamond \delta < \varepsilon$.*

Definition 2.4. ([2]) Let X be a nonempty set. A $T-metric$ on X is a function $T : X^2 \longrightarrow \mathbb{R}$ that satisfies the following conditions: for $x, y, z \in X$
(i) $T(x, y) \geq 0$ and $T(x, y) = 0$ if and only if $x = y$,
(ii) $T(x, y) = T(y, x)$,
(iii) $T(x, y) \leq T(x, z) \diamond T(z, y)$.

The $3-tuple$ $(X, T, \diamond)$ is called a $T-metric$ space.

Example 2.5. ([2]) (i) *Every ordinary metric d is a $T-metric$ with $a \diamond b = a + b$.*

(ii) *Let* $X = \mathbb{R}$ *and* $T(x, y) = \sqrt{|x - y|}$ *for every* $x, y \in \mathbb{R}$. *if we take* $a \diamond b = \sqrt{a^2 + b^2}$, *then the function* T *is a* $T - metric$ *on* X.
(iii) *Let* $X = \mathbb{R}$ *and* $T(x, y) = (x - y)^2$ *for* $x, y \in \mathbb{R}$. *If we take* $a \diamond b = (\sqrt{a} + \sqrt{b})^2$, *then the function* T *is a* $T - metric$ *on* X.

Remark 2.6. ([2]) For fixed point $0 \leq \alpha \leq \frac{\pi}{4}$ if there exist β, γ such that

$$0 \leq \alpha \leq \beta + \gamma < \frac{\pi}{2},$$

then

$$\tan \alpha \leq \tan \beta + \tan \gamma + \tan \beta \cdot \tan \gamma.$$

Example 2.7. ([2]) *Let* $X = [0, 1]$ *and* $T(x, y) = \tan(\frac{\pi}{4} |x - y|)$ *for every* $x, y \in X$. *If we take* $a \diamond b = a + b + ab$, *then by Remark* 2.6 *we have*

$$T(x, y) \leq T(x, z) \diamond T(z, y).$$

Therefore the function T *is a* $T - metric$ *on* X.

Definition 2.8. ([2]) Let $(X, T, \diamond)$ be a $T - metric$ space, $r > 0$ and $A \subset X$.
(1) The set $B_T(x, r) = \{y \in X : T(x, y) < r\}$ is called a ball, centered at x and radius r.
(2) If for every $x \in A$ there exists $r > 0$ such that $B_T(x, r) \subset A$, then the set A is called open subset of X.
(3) A sequence $\{x_n\}$ in X converges to x if $T(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$ and write $\lim_{n \longrightarrow \infty} x_n = x$. That is for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $T(x_n, x) < \varepsilon$ for all $n \geq n_0$, then $\{x_n\}$ converges to x.
(4) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $T(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$.
(5) The $T - metric$ space $(X, T, \diamond)$ is said to be complete if every Cauchy sequence in X is convergent in X.

Let τ be the set of all open subsets of X, then τ is a topology on X (induced by the $T - metric$ T).

Lemma 2.9. ([2]) *Let* $(X, T, \diamond)$ *be a* $T - metric$ *space. If* $r > 0$, *then the ball* $B_T(x, r)$ *is an open set.*

Lemma 2.10. ([2]) *Let* $(X, T, \diamond)$ *be a* $T - metric$ *space. If a sequence* $\{x_n\}$ *in* X *converges to* x, *then* x *is unique.*

Lemma 2.11. ([2]) *Let $(X, T, \diamond)$ be a $T-$metric space. Then every convergent sequence $\{x_n\}$ in X is a Cauchy sequence.*

Definition 2.12. ([2]) Let $(X, T, \diamond)$ be a $T - metric$ space. T is said to be continuous if $\lim_{n \longrightarrow \infty} T(x_n, y_n) = T(x, y)$, whenever

$$\lim_{n \longrightarrow \infty} T(x_n, x) = \lim_{n \longrightarrow \infty} T(y_n, y) = 0.$$

Lemma 2.13. ([2]) *Let $(X, T, \diamond)$ be a $T - metric$ space. Then T is a continuous function.*

3. Main Results

In this section, we establish a common fixed point theorem in ordered $T-$ *metric* space.

Definition 3.1. Let $(X, T, \diamond)$ be a $T - metric$ space and ϕ be a mapping from X to $\mathbb{R}$.
(i) Let $S : X \longrightarrow X$ be an arbitrary self-mapping on X such that

$$T(x, Sx) \leq \phi(x) - \phi(Sx) \text{ for all } x \in X,$$

where S is called a Caristi map on $(X, T, \diamond)$.
(ii) Let $S, R : X \longrightarrow X$ be two self-mappings on X such that

$$T(Sx, Rx) \leq \phi(Sx) - \phi(Rx) \quad \text{for all } x \in X,$$

where R is called a S-Caristi map on $(X, T, \diamond)$.
(iii) Define the relation $\preceq$ on X (induced by ϕ) as follows:

$$x \preceq y \Longleftrightarrow T(x, y) \leq \phi(x) - \phi(y).$$

In the sequel, the operation $\diamond : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfies the property:

$$a \diamond b \leq a + b.$$

Lemma 3.2. *Let $(X, T, \diamond)$ be a $T - metric$ space and ϕ be a mapping from X to $\mathbb{R}$. The $\preceq$ induced by ϕ is a (partial) order on X.*

Proof. (i) Obviously $x \preceq x$.
(ii) It is easy to see that if $x \preceq y$ and $y \preceq x$ then $x = y$.
(iii) Let $x \preceq y$ then $T(x, y) \leq \phi(x) - \phi(y)$. Also, if $y \preceq z$ then $T(y, z) \leq \phi(y) - \phi(z)$. From Definition 2.4 (iii), we have

$$\begin{aligned} T(x,z) &\leq T(x,y) \diamond T(y,z) \\ &\leq T(x,y) + T(y,z) \\ &\leq \phi(x) - \phi(y) + \phi(y) - \phi(z) \\ &= \phi(x) - \phi(z). \end{aligned}$$

So, $x \preceq z$. □

Now, let us present our main results.

Theorem 3.3. *Let $(X, T, \diamond)$ be a complete $T-$metric space and $\phi : X \longrightarrow \mathbb{R}$ be a lower semicontinuous function which is bounded below and $\preceq$ be the order induced by ϕ. Let $S, R : X \longrightarrow X$ be two selfmappings such that R is a $S-$Caristi map on (X, T). If $S(X)$ be a closed subspace of X, then there exists $z \in X$ such that $Sz = Rz$.*

Proof. For each $x \in X$, define

$$H(x) = \{z \in X : Sx \preceq z\}$$

and

$$\alpha(x) = \inf\{\phi(Sx) : Sx \in H(x)\}. \tag{3.1}$$

Since $Sx \in H(x)$, then $H(x) \neq \emptyset$. From (3.1) we have $\alpha(x) \leq \phi(Sx)$. Take $x \in X$. We construct a sequence $\{x_n\}$ in the following way:

$$x_1 := Sx,$$

$$Sx_{n+1} \in H(x_n) \text{ such that } \phi(Sx_{n+1}) \leq \alpha(x_n) + \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

It is easy to see that

$$T(Sx_n, Sx_{n+1}) \leq \phi(Sx_n) - \phi(Sx_{n+1}) \tag{3.2}$$

and

$$\alpha(x_n) \leq \phi(Sx_{n+1}) \leq \alpha(x_n) + \frac{1}{n}, \text{ for every } n \in \mathbb{N}. \tag{3.3}$$

Note that (3.2) implies that $\{\phi(Sx_n)\}$ is a decreasing sequence of real numbers, and it is bounded. Therefore, the sequence $\{\phi(Sx_n)\}$ is convergent to some positive real number, say L. Thus regarding (3.3), we have

$$L = \lim_{n \longrightarrow \infty} \phi(Sx_n) = \lim_{n \longrightarrow \infty} \alpha(x_n). \tag{3.4}$$

From (3.3) and (3.4), for each $k \in \mathbb{N}$, there exists $\mathbb{N}_k \in \mathbb{N}$ such that

$$\phi(Sx_n) \leq L + \frac{1}{k}, \text{ for all } n \geq \mathbb{N}_k. \tag{3.5}$$

Regarding the monotonicity of $\{\phi(Sx_n)\}$, for $m \geq n \geq \mathbb{N}_k$, we have

$$L \leq \phi(Sx_m) \leq \phi(Sx_n) \leq L + \frac{1}{k}. \tag{3.6}$$

Thus, we get that

$$\phi(Sx_n) - \phi(Sx_m) < \frac{1}{k}, \text{ for all } m \geq n \geq \mathbb{N}_k. \tag{3.7}$$

On the other hand, taking (3.2) into account, together with the triangle inequality, we observe that

$$\begin{aligned} T(Sx_n, Sx_m) &\leq T(Sx_n, Sx_{n+1}) \diamond T(Sx_{n+1}, Sx_{n+2}) \diamond \cdots \diamond T(Sx_{m-1}, Sx_m) \\ &\leq T(Sx_n, Sx_m) + T(Sx_{n+1}, Sx_{n+2}) + \cdots + T(Sx_{m-1}, Sx_m) \\ &\leq \phi(Sx_n) - \phi(Sx_{n+1}) + \phi(Sx_{n+1}) - \phi(Sx_{n+2}) + \cdots \\ &\quad + \phi(Sx_{m-1}) - \phi(Sx_m) \\ &= \phi(Sx_n) - \phi(Sx_m) \end{aligned} \tag{3.8}$$

for all $m \geq n$. That is

$$T(Sx_n, Sx_m) \leq \phi(Sx_n) - \phi(Sx_m)$$

and taking (3.7) into account, (3.8) turns into

$$T(Sx_n, Sx_m) \leq \phi(Sx_n) - \phi(Sx_m) < \frac{1}{k}, \text{ for all } m \geq n \geq \mathbb{N}_k. \tag{3.9}$$

Since the sequence $\{\phi(Sx_n)\}$ is convergent, the right hand side of (3.9) tends to zero. Which means that $\{Sx_n\}$ is a Cauchy sequence in $(X, T, \diamond)$. Since $(X, T, \diamond)$ is complete, the sequence $\{Sx_n\}$ converges in $(X, T, \diamond)$, say $\lim_{n\to\infty} T(Sx_n, x^*) = 0$. Since $S(X)$ is a closed subspace of X, there exists $z \in X$ such that $\lim_{n\longrightarrow\infty} Rx_n = x^* = Sz$.

On the other hand, with the triangle inequality, we observe that

$$\begin{aligned} T(Sx_n, Rz) &\leq T(Sx_n, Sz) \diamond T(Sz, Rz) \\ &\leq T(Sx_n, Sz) + T(Sz, Rz) \\ &\leq \phi(Sx_n) - \phi(Sz) + \phi(Sz) - \phi(Rz). \end{aligned}$$

That is

$$T(Sx_n, Rz) \leq \phi(Sx_n) - \phi(Rz).$$

Hence, $Rz \in H(x_n)$ for all $n \in \mathbb{N}$ which yields that $\alpha(x_n) \leq \phi(Rz)$ for all $n \in \mathbb{N}$. From(3.4), the inequality $L \leq \phi(Rz)$ is obtained. Moreover, by lower semi-continuity of ϕ, we have

$$\phi(Sz) \leq \liminf_{n\longrightarrow\infty} \phi(Sx_n) = L \leq \phi(Rz).$$

Sinse R is a $S-$Caristi map for each $x \in X$, then we have

$$T(Sz, Rz) \leq \phi(Sz) - \phi(Rz).$$

Thus $\phi(Rz) \leq \phi(Sz)$ and we get that $\phi(Sz) = \phi(Rz)$. Hence $T(Sz, Rz) \leq 0$ which implies that $Sz = Rz$. □

Corollary 3.4. *Let $(X, T, \diamond)$ be a complete $T-$metric space and $\phi : X \longrightarrow \mathbb{R}$ be a lower semicontinuous function which is bounded below and $\preceq$ be the order induced by ϕ. Let $R : X \longrightarrow X$ be a Caristi selfmapping. Then R has a unique fixed point in X.*

Proof. If we take $S = I$ (Identity map) in Theorem 3.3, we get that $Rz = z$. For the uniqueness, Suppose z' is another common fixed point of R. We have

$$T(z, z') = T(z, Rz') \leq \phi(z) - \phi(Rz') = \phi(z) - \phi(z').$$

Also

$$T(z', z) = T(z', Rz) \leq \phi(z') - \phi(Rz) = \phi(z') - \phi(z).$$

Therefore, $2T(z', z) = 0$ which implies that $z = z'$. □

Example 3.5. *Let $X = [0, 1]$ and T be the usual metric $d(x, y) = |x - y|$ for every $x, y \in [0, 1]$. Define $\phi : X \longrightarrow \mathbb{R}$ by $\phi(x) = -2x$. Let $\leq$ be usual order on X and define*

$$x \leq y \Longleftrightarrow T(x, y) \leq \phi(x) - \phi(y).$$

It is easy to see that $\leq$ is a partial order induced by ϕ on X. Let $R(x) = \dfrac{2x+1}{3}$, then obviously $T(x, Rx) \leq \phi(x) - \phi(Rx)$ and all conditions of Corollary 3.4 *are holds, also $x = 1$ is the unique fixed point of R.*

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